Stochastic Local Search Algorithms for Graph Set T-Colouring and Frequency Assignment

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Abstract

The graph set T-colouring problem (GSTCP) generalises the classical graph colouring problem; it asks for the assignment of sets of integers to the vertices of a graph such that constraints on the separation of any two numbers assigned to a single vertex or to adjacent vertices are satisfied and some objective function is optimised. Among the objective functions of interest is the minimisation of the difference between the largest and the smallest integers used (the span).

In this article, we present an experimental study of local search algorithms for solving general and large size instances of the GSTCP. We compare the performance of previously known as well as new algorithms covering both simple construction heuristics and elaborated stochastic local search algorithms. We investigate systematically different models and search strategies in the algorithms and determine the best choices for different types of instance. The study is an example of design of effective local search for constraint optimisation problems.

1 Introduction

The *graph set T-colouring problem* (GSTCP) is a generalisation of the graph (vertex) colouring problem (GCP) that was introduced to model the assignment of frequencies to radio transmitters [22]. In the GSTCP, one is given a graph, a number of colours required at each vertex and separation constraints for each vertex and for each pair of connected vertices. Colours correspond to nonnegative integers and the separation constraints represent the disallowed differences between the integers assigned to each vertex. The decision version of the GSTCP asks, given a number of colours, for an assignment of sets of colours to the vertices of the graph such that all constraints are satisfied. Such an assignment is called a *set T-colouring*. There are various optimisation versions of the GSTCP, which differ in the objective function to be minimised [22, 49]. One version asks to determine the *T-order* (or T-chromatic number), that is, the minimal number of integers, such that a set T-colouring exists. A second optimisation version asks to determine the *T-span*, that is, the minimal difference between the largest and the smallest integers such that a set T-colouring exists. Another alternative, sets a bound on the number of available colours and asks for a set T-colouring that minimises the number of constraint violations.

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The GSTCP is at the core of a number of real-world applications. Examples are traffic phasing, fleet maintenance [44] and task assignment. In this latter case, a large task is divided into incompatible subtasks (for example, due to resource conflicts) and the problem is to assign a set of time periods to each subtask such that incompatible subtasks are treated in different time periods [49]. The most important GSTCP application is in the assignment of frequencies to radio transmitters. In this case, vertices represent transmitters and colours the frequencies to be assigned to the transmitters subject to certain interference constraints, the T-constraints [22]. These T-constraints are expressed in the form of restrictions on the possible separations of the frequencies at each vertex and between the vertices.

In the frequency assignment application, there are two possible scenarios of interest: the design scenario, in which it has to be decided which frequency band to acquire in order to satisfy a forecast network utilisation; and the planning scenario, in which the frequency band available at the provider is fixed and it must be decided which frequencies to assign to each cell such that the service is maximised.[1] The first scenario corresponds to minimising the span while the second scenario to minimising the number of violated constraints. In this latter case, penalties can then be associated to disallowed separations and acceptable interferences are weighted differently from forbidden interferences. This gives rise to the *minimal interference problem*, where the total penalty has to be minimised [1, 17]. As we will see, most of the solution methods discussed in this article are valid for both scenarios. Indeed, following a known practice in the constraint satisfaction community [36], the *minimal span problem* can be approached by solving a series of *minimal interference problems*.

The GSTCP has received significant attention from both the theoretical and the algorithmic perspective. The theoretical contributions often focus on special types of instances and provide results that might be roughly grouped into three classes: computational complexity [44, 49, 20, 21], lower bounds [2, 28, 47, 37] and approximation schemes [45, 27].[2]

The algorithmic contributions on the GSTCP are mostly linked to the research on frequency assignment and, more recently, to the "Computational Challenge on Graph Colouring and its Generalisations", which was organised by Johnson, Mehrotra and Trick.[3] The problem is also relevant to the constraint satisfaction community [15] although little experimental research has been conducted on constraint programming models [52]. Mathematical programming approaches are instead described in [17, 1]. However, complete search algorithms have proven to be inefficient when applied to general and large scale instances, which are more of interest for practical applications.

Stochastic local search (SLS) algorithms, although incomplete, obtain typically much better quality solutions that the complete algorithms in practicable times and, hence, have also been more studied. Costa is the first to address the GSTCP by means of heuristics and proposes a generalisation of the DSATUR heuristic for the GCP [12]. Dorne and Hao apply tabu search to very large, randomly generated GSTCP instances [16]. In the context of the COLOR02/03/04 DIMACS challenge, Phan and Skiena devised a simu-

[1] Both cases of frequency assignment give rise to what is called, more precisely, the fixed channel assignment problem, whose name is used to emphasise that the model is static, *i.e.*, the set of connections remains stable over time.

[2] Theoretically driven research on the GSTCP was also the target of the *DIMACS/DIMATIA/Renyi Working Group on Graph Colourings and their Generalizations*; see `<http://dimacs.rutgers.edu/Workshops/GraphColor/main.html>` (Last update: January 10, 2005) for more details.

[3] M. Trick. "Computational Series: Graph Coloring and its Generalizations.", 2002, `<http://mat.tepper.cmu.edu/COLOR04>`. In this challenge, the GSTCP is called *bandwidth multi-colouring problem*.

lated annealing algorithm using their general-purpose platform *Discropt* [39]. Prestwich presented a randomised backtracking algorithm [42] while Lim et al. designed a squeaky wheel algorithm [31, 32]. This latter approach also gives the best results for the benchmark instances that were proposed in the COLOR02/03/04 DIMACS challenge. However, from the published results it is unclear how this algorithm compares with the earlier proposed tabu search algorithm by Dorne and Hao [16]. In general, we noted a tendency in this area to publish horse-race type papers that present new best performing algorithms but rarely compare them directly or study the algorithm components that determine the success.

The main contribution of this article is a systematic study of previously known and new construction heuristics as well as advanced SLS algorithms for the GSTCP, with the goal to bring instructive insights into the design and implementation of efficient local search algorithms for this constraint optimisation problem. We have decided to (re)-implement all the algorithms using the same framework, data structures, platform and implementer's ability to increase the fairness of the experimental comparison. Our benchmark collection comprises several sets of randomly generated instances including random graphs and geometric graphs as well as instances derived from the frequency assignment literature [3]. As a side product, we also extend the set of previously available benchmark instances to allow for a more systematic study of the influence of instance features on performance. The experiments are designed in a way to bring light on the sources of differences rather than only indicating which is the best algorithm. Our analysis, supported by statistical methods, allows us to investigate (i) which are the key algorithmic features that are responsible for the good performance of the construction heuristics and SLS algorithms, (ii) which are the features of the instances that have an influence on the results and, (iii) which is the current state-of-the-art in GSTCP solving.

We draw two kinds of conclusions. The first is of general character and consists of indications about modelling constraints in local search. Perhaps the most remarkable is the fact that we found it helpful to use constraints to reduce the search space size. Although this might be intuitively clear in the constraint satisfaction community, previous results have shown that some sort of constraints, like those for symmetry breaking, may have a negative effect on the performance of local search algorithms [43]. Another indication worth mentioning is that we found convenient solving constrained optimisation problems as a series of constraint satisfaction problems with bounded objective is more convenient than using a weighted sum of optimisation objective and infeasibility. The second kind of conclusions concern specifically the GSTCP. We show that the best construction heuristic for this problem is a generalisation of DSATUR that chooses vertices in a different order to that proposed in previous researches and that the best performing SLS algorithms are a squeaky wheel algorithm and a new tabu search algorithm with a novel randomised exact search neighbourhood.

The paper is organised as follows. Section 2 introduces the notation and provides some basic results. Section 3 defines the benchmark instances. Section 4 describes construction heuristics which are empirically assessed in Section 5. Section 6 is dedicated to the description of the SLS algorithms. Section 7 reports the experimental analysis on SLS algorithms. Finally, Section 8 resumes the results of this work.

2 Definitions, Notation and Basic Results

The GSTCP is defined by

(i) an undirected graph $G = (V, E)$, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges, with uv representing an edge in E;

(ii) a set of integers (called colours) Γ;

(iii) a number $r(v)$ of required colours for each vertex $v \in V$; and

(iv) a collection $\mathcal{T}$ of sets (called T-sets), such that there is a set T_{uv} for each edge $uv \in E$ and a set T_u for each vertex $u \in V$. The T-sets are sets of nonnegative integers and represent the disallowed separations of colours between and within vertices, respectively.

The task in the decision version of the GSTCP is to find, for a given number of $k = |\Gamma|$ colours, a mapping $\varphi : V \mapsto \mathcal{P}(\Gamma)$ such that the three following groups of constraints

$$|\varphi(v)| = r(v) \qquad \forall\, v \in V \tag{1}$$

$$|x - y| \notin T_u \qquad \forall u \in V,\ \forall x, y \in \varphi(u),\ x \neq y \tag{2}$$

$$|x - y| \notin T_{uv} \qquad \forall uv \in E,\ \forall x \in \varphi(v),\ \forall y \in \varphi(u) \tag{3}$$

are satisfied. The three groups of constraints are called *requirement constraints*, *vertex constraints*, and *edge constraints*, respectively. We call a mapping φ a *proper set T-colouring* if all constraints are satisfied and *improper*, otherwise.

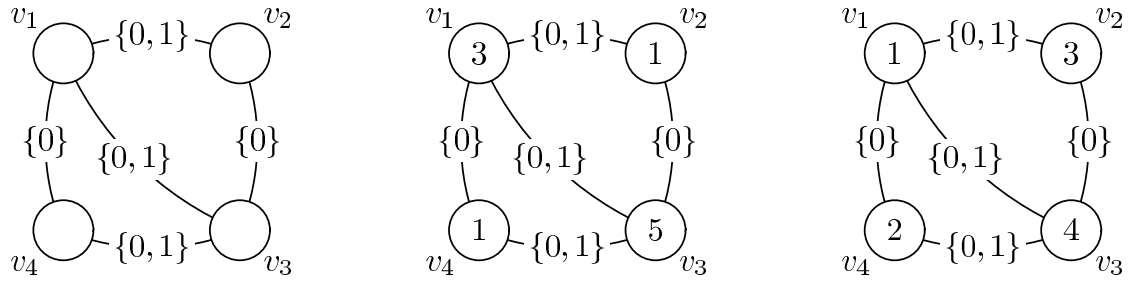
The *graph T-colouring problem* (GTCP) is a special case of the GSTCP, with $r(v) = 1$ for all v in V and only the edge constraints are present. Both, the decision versions of the GSTCP and the GTCP are obviously in $\mathcal{NP}$ and, hence, also $\mathcal{NP}$-complete, since they are generalisations of the *k-colouring problem* [19].

Any GSTCP instance defined by a graph $G = (V, E)$, a collection $\mathcal{T}$ and vertex requirements $r(v)$ for each $v \in V$ has an equivalent GTCP instance $G^S = (V^S, E^S)$ such that the existence of a proper T-colouring in the GTCP instance implies that there exists a proper set T-colouring in the GSTCP instance (and vice versa). The equivalence is shown by the construction of the GTCP instance. G^S is obtained from G by creating a vertex u for each requirement of a vertex $v \in V(G)$ such that $|V^S| = \sum_{v \in V} r(v)$. In G^S, the vertices $u \in V^S$ derived from a vertex $v \in V$ form a clique of order $r(v)$ in which each edge receives the set of constraints $T_v \in \mathcal{T}$. Every such vertex is then connected with each vertex of the clique induced by another vertex $w \in V$, if $vw \in E$. The colour separation associated with these edges is T_{vw}. The graph G^S is called *split graph*.

Two properties of a proper (set) T-colouring φ are of interest for its quality: the *order*, *i.e.*, the number of different colours effectively used by φ and the *span*, *i.e.*, the maximal difference between the colours used: $\max_{u,v \in V}\{|x-y| : x \in \varphi(v), y \in \varphi(u)\}$. The minimum order for which a proper T-colouring exists is called the *(set) T-chromatic number* $\chi_T(G)$; the minimum span is the *(set) T-span* $sp_T(G)$. Minimising the order, the span or both is the objective in the optimisation versions of the GSTCP. For all graphs G and T-sets $\mathcal{T}$ it holds that $\chi_T(G) \leq sp_T(G)$ [22]. Note that $\chi_T(G) = \chi_T(G^S)$ and $sp_T(G) = sp_T(G^S)$ and, hence, the solution to a GSTCP can be found by solving the GTCP on the split graph.

For the GTCP, it was proved that the T-chromatic number of a graph G, $\chi_T(G)$, is equal to the chromatic number $\chi(G)$ [13]. However, Hale (1980) [22] pointed out that minimising the span of a T-colouring is different from minimising the order, or the span of a colouring without the T-constraints. There are cases, indeed, where no optimal span T-colouring gives an optimal order and vice-versa. For example, if the graph G and the T-set $\mathcal{T}$ are those of Figure 1, then $\chi_T(G) = 3$ and $sp_T(G) = 3$ but a T-colouring with order three has span at least four, whereas a T-colouring with span three has order at least four.

For the frequency assignment application, more relevant is the case in which all $T_u, T_{uv} \in \mathcal{T}$ consist of consecutive integers $\{0, 1, \ldots, t_{uv}-1\}$ for all $uv \in E$ and $\{0, 1, \ldots, t_u-$



(a) The original instance.

(b) A solution with 3 colours and span 4.

(c) A solution with 4 colours and span 3.

Figure 1: A GTCP instance with $\chi_T(G) = 3$ and $sp_T(G) = 3$ (a). Every solution that uses 3 colours has span at least 4 (b), whereas every solution with span 3 uses 4 colours (c).

1} for all $u \in V$. This particular version of the problem is known as *separation distance GSTCP* and the values t_u, t_{uv} are called *colour distances* [17]. The edge and vertex constraints then have the form $|x - y| \geq t$, where x and y are two assigned integers.

In the rest of the article we assume the GSTCP to be in the separation distance form and study the minimisation of the set T-span. In each set T-colouring instance we assume $\min_{u \in V}\{x : x \in \varphi(u)\} = 1$ and, hence, the set T-span to be $k - 1$, where k is the largest colour used. In addition, we denote by $A_G(v)$ the set of vertices adjacent to a vertex $v \in V$, *i.e.*, $A_G(v) = \{u | u \in V, uv \in E\}$, and by $d(v)$ the degree of a vertex v, *i.e.*, $|A_G(v)|$.

3 Benchmark instances

We use a benchmark suite composed of three sets of random instances that differ in the structure of the graphs and of the constraints, and a set of instances derived from the frequency assignment application.

Random geometric instances. This set was proposed by M. Trick in the context of the "Computational Challenge on graph colouring and its generalisations". Vertices correspond to points that are uniformly distributed in a square $[10,000 \times 10,000]$. Edges connect vertices whose corresponding points are closer than some pre-defined value. The separation distances associated to edges are inversely proportional to the distances between the points. Vertex requirements are chosen uniformly at random from the set $\{1, \ldots, r\}$ and vertex separation distances are fixed to 10. The size of these instances ranges from 20 to 120 vertices. We denote this set by `GEOM`, its class of sparse instances by `GEOM`n and its classes of denser instances by `GEOM`n`a` and `GEOM`n`b`. The instances `GEOM`n`b` have higher requirements per node than `GEOM`n`a`. Statistics of this instance set are given in Table 1a.

Random uniform instances. This set was introduced by Dorne and Hao for testing their tabu search algorithm [16]. The graphs are uniform, random graphs, which are typically identified as G_{np}, where n is the number of vertices and each of the $\binom{n}{2}$ possible edges is included in the graph with a probability p. Vertex requirements, vertex-distances and edge-distances are generated according to a uniform distribution in the interval $\{1, \ldots, t\}$. Statistics on these instances are in Table 1b. We call this set `HD-RU` and maintain the original nomenclature for the single instances: `essai`.$n.R.p$ where $R = \sum r(v) = |V^S|$.

$\|V\|$	$\bar{\rho}$	r	t_u	$\overline{t_{uv}}$	# instances
$20, 30, \ldots, 120$	0.1	3	10	4.5	–
		10	10	4.5	11 (GEOMn)
	0.2	3	10	4.5	11 (GEOMna)
		10	10	4.5	11 (GEOMnb)

(a) Random geometric instances

$\|V\|$	p	$r = t_u = t_{uv}$	# instances
$30, 100, 300, 500, 1000$	0.1	5	5
	0.5	5	5
	0.9	5	5

(b) Random uniform instances [16]

$\|V\|$	p	r	t	# instances
60	0.1	5	5	10
			10	10
		10	5	10
	0.5	5	5	10
			10	10
		10	5	10
	0.9	5	5	10
			10	10
		10	5	10

(c) New random uniform instances

Inst	$\bar{\rho}$	$[r^{min}; r^{max}]$	t_u	$[t_{uv}^{min}; t_{uv}^{max}]$
P1	0.73	[8;77]	5	[1; 2]
P2	0.49	[8; 77]	5	[1; 2]
P3	0.73	[5; 45]	5	[1; 2]
P4	0.49	[5; 45]	5	[1; 2]
P5	0.73	[20; 20]	5	[1; 2]
P6	0.49	[20; 20]	5	[1; 2]
P7	0.73	[16; 154]	5	[1; 2]
P8	0.73	[8; 77]	5	[1; 2]
P9	0.73	[32; 308]	5	[1; 2]

(d) Philadelphia instances

Table 1: Statistics on the benchmark instances. t_u and t_{uv} are used explicitely if the range of values differs among vertex and edge constraints. $\bar{\rho}$ gives the measured edge density of the graphs.

Random uniform instances (new). Since we experienced that some of the HD-RU instances require very high computational times before it becomes unlikely for the algorithms here studied to further improve their solutions, we generated a new set of instances that are similar to those of HD-RU. The only difference is that vertex requirements are chosen uniformly at random from the interval $\{1, \ldots, r\}$, thus, possibly, $r \neq t$. The values considered for the parameters and the number of instances generated are reported in Table 1c. The smaller instance size with respect to the set HD-RU allows us to run extensive computational tests. We denote this set of instances NRU and each single class as TG-n-p-r.t.

Philadelphia instances. This set comprises the well-known Philadelphia instances from the frequency assignment literature [3]. Vertices correspond to 21 hexagons representing the cells of a cellular phone network. Edges and separation distances are assigned according to the interferences between sites such that adjacent or close cells have higher separation constraints than distant cells. For details on the generation of these instances we refer to the FAP web repository.[4] In conformity with the frequency assignment literature, we denote these instances by P1–P9.

4 Construction Heuristics

Construction heuristics are the fastest methods to generate good quality solutions for optimisation problems. Another main use is to provide good initial solutions to SLS

[4] A. Eisenblätter and A. Koster. "FAP web – A website devoted to frequency assignment". June 2000. <http://fap.zib.de>. (Last update: January 2007.)

algorithms. For the GSTCP, various construction heuristics have been presented. The most comprehensive study so far by Hurley et al. comprises a number of different sequential heuristics [26]. They conclude that it is difficult to predict which variant will perform best on a specific instance; however, their comparison is limited to small size instances of around 20 vertices and no statistical analysis is applied to support the results.

We enlarge the study of construction heuristics for the GSTCP to include in addition the assignment heuristics proposed in [46] and heuristics that were adapted from the GCP. Some of the proposed heuristics are new, like the adaption of RLF, or are enhancements of previously published ones, like generalised DSATUR.

4.1 Split Graph (T-Colouring) Approach

Sequential heuristics. Sequential heuristics iteratively assign colours to vertices until a complete colouring is reached. Each assignment decision consists of two steps: first, the vertex that is to be coloured next is chosen, then a colour is assigned to this vertex.

A basic heuristic to assign the colour is the *greedy algorithm* [13]. This algorithm, which we denote T-greedy, takes as input a permutation π of the vertices. At each iteration i, it assigns to the vertex $\pi(i)$ the colour corresponding to the smallest possible integer such that the partial T-colouring obtained after colouring the vertex $\pi(i)$ remains proper. After $|V^S|$ iterations, T-greedy returns a proper complete T-colouring for any graph and T-sets. An efficient implementation of T-greedy maintains a set of forbidden colours for each vertex that is still to be coloured and updates it at each iteration. The complexity is $\mathcal{O}(k|V^S|)$; note that k is not bounded by the number of vertices.

The permutation π can be determined in a *static* manner before assigning the colours. As in the GCP, the vertex degree is the information which is used to order the vertices. We examine the following orders: Random (RO algorithm), Largest First (LF algorithm) and Smallest Last (SL algorithm). The LF algorithm sorts the vertices in non-increasing order of their degree. The SL algorithm arranges the vertices such that the vertex at position i, $v_{\pi(i)}$ has the smallest degree in the subgraph $G' \subset G$ induced by $V' = \{v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(i)}\}$. This sequence is determined starting at the final vertex and proceeding in reverse order. We denote the three heuristics G-ROS, G-LFS, and G-SLS.

Various authors have tested an *adjusted vertex degree* that uses the additional information of the separation distance constraints between adjacent vertices, which is defined as $d^T(v) = \sum_{u \in A_G(v)} t_{uv}$. We therefore include in our analysis for all the sequential heuristics two versions, one using the vertex degree and another using the adjusted vertex degree.

Generalised DSATUR heuristics. The DSATUR heuristics are also based on T-greedy but they generate the order of the vertices in a *dynamic* manner at construction time depending on the current partial colouring. The information used is the *vertex saturation degree*, *i.e.*, the number of different colours forbidden at the current vertex by the colours assigned to the adjacent vertices. We denote this generalisation of the DSATUR scheme to the GTCP as G-DSATUR and study the instantiation of its components.

We examine two sorting strategies: largest saturation first (LSF) and smallest saturation first (SSF). In both cases, ties are broken by the largest vertex degree. As for the sequential heuristics, we also examine the use of an adjusted vertex degree to break ties. We observed, however, that ties occur very rarely.

In addition, we consider a modification of the T-greedy algorithm that selects, whenever more than one previously used colour is feasible, the colour that is feasible for the smallest number of uncoloured vertices. The idea is to favour the use of colours with less chances to be used later in the process. We denote this algorithm as T-greedy$^{\#}$. Its worst case

complexity when applied to a static vertex order is $\mathcal{O}(k^2|V^S|^2)$. Empirical observations showed that even for high density graphs with $\rho(G) = 0.9$, where vertices are less likely to be colourable with previously used colours, this rule applies in about 50% of the decisions taken.

Generalised RLF heuristic. For the GCP, one of the best performing construction heuristics, the *Recursive Largest First* (RLF) heuristic, is based on a partitioning approach. It iteratively fills colour classes by colouring as many vertices as feasible with the same colour before using a new colour. When applied to the GTCP, the concept of independent set is still valid but vertices are also subject to distance constraints and, hence, their insertion in a set must also guarantee the satisfaction of the separation constraints between the colours that will be assigned to such sets. We denote this adapted RLF heuristic by G-RLF. For G-RLF, we also study the effect of an adjusted vertex degree.

4.2 Original Graph (Set *T*-Colouring) Approach

Sequential heuristics. The T-greedy algorithm can be modified to assign at each construction step $r(v)$ colours to a vertex v. The colour assignment policy remains the same: priority is given to the smallest possible colours. The order of the vertices can be determined using the same rules (RO, LF, SL) as before. We implemented these heuristics using both the vertex degree and the adjusted vertex degree.

Assignment heuristics. The heuristics proposed by Sivarajan et al. [46] look at the problem from an assignment perspective. These heuristics were only studied on instances with 21 vertices and never compared with the heuristics introduced above. They work at the level of single vertex requirements. First the requirements are disposed in a matrix of size $|V| \times \max_{v \in V} r(v)$ whose rows correspond to vertices and the elements of each row to cyclically shifted versions of the list $\{1, \ldots, r(v)\}$ and zeros. The cylic shifts are organised in such a way that the number of requirements is almost equally distributed among the columns. Then an order in the examination of the matrix is decided and the colours are assigned to the requirements in the order they are visited. Three factors influence the heuristic procedure: the *vertex order* (*i.e.*, the order of the rows in the matrix) determined by the smallest last (C) or the largest first (D) order of an adjusted vertex degree; the *requirement order* (*i.e.*, the order of examination in the matrix) determined by row-wise ordering (R) or column-wise ordering (C); and the *assignment policy* which may be colour exhaustive (F) if we assign the smallest colour that does not introduce any violation to the next requirement, or requirement exhaustive (R) if for each colour in order from 1 to k we scan all yet uncoloured requirements and assign the current colour to all those for which it is feasible.

We identify these assignment heuristics by the letter abbreviations as defined above. For example, CRR refers to a heuristic that uses smallest last vertex order, row-wise requirement order and an requirement exhaustive assignment policy.

5 Experimental Assessment of Construction Heuristics

The experimental study on construction heuristics aims at identifying the algorithmic features that have an impact on the heuristics' performance and the best performing construction heuristics. In fact, the attempt to distinguish and separate the effects of the single choices that compose the heuristics is a novelty of this analysis which is made

	T-colouring approach	set T-colouring approach
Sequential	order={RO,LF,SL} adjust={yes,no}	order={RO,LF,SL} adjust={yes,no}
DSATUR	adjust={yes,no},order={LSF,SSF}, assign={T-greedy,T-greedy$^{\#}$}	–
RLF	adjust={yes,no}	–
Assignment	–	order={C, D},requ.={R, C}, assign={F, R}

Table 2: Implementation choices for the construction heuristics for GSTCP.

possible by a careful organisation and design of experiments. Some of these choices are, for example, the use of the GSTCP formulation or the GTCP transformation or the use of an adjusted vertex degree.

Concerning performance, we focus uniquely on the solution quality returned by the heuristics since for all heuristics, the computational times are negligible being always below 0.15 seconds on our computer, a 2 GHz AMD Athlon MP 2400+ processor with 256 KB cache and 1GB RAM.[5]

The different algorithmic choices, whose effects we want to examine, are described in Table 2. We need to split the analysis into four separate experimental designs, one for each type of approach, that is, one for each row of the table. We want to distinguish main and interaction effects of the algorithmic factors and of other factors, such as the instance type. This latter effect tells us how the relative performance of the algorithms depends on some features of the instance that can be recognised *a-priori*. We use the classes of instances taken from the `GEOM` and `NRU` benchmark sets. In addition, we treat the single instances as blocking factor. Since randomised decisions may occur due to random tie breaking, we collect for each heuristic 10 runs per instance and measure the span returned in each run. Then we rank these results for each instance to avoid problems due to different instance scales.

Sequential Heuristics. We considered four factors: problem representation {GTCP, GSTCP}, adjusted degree {Y, N}, vertex order {RO, LF, SL} and instance class {`GEOM`, `GEOMa`, `GEOMb`, `TG-120-0.1-5.5`, `TG-120-0.5-5.5`, `TG-120-0.9-5.5`}. An analysis of variance (ANOVA) on ranks [38, Section 3-10.2] indicates that only the main effect of instance class is not signficant. This is due to the transformation in ranks that removes differences in scale between instances. All other factors' main effects and interactions are instead significant and indicate the impossibility to state an algorithmic feature as the most important without defining first the instance class and the choices for the other features. Even trying to simplify the presentation of results through regression trees [7, 4] is not helpful to gain a synthetic view of results. The only choice that seems to be preferable among the different instance classes is the use of an adjusted vertex degree. However, even the random vertex order resulted sometimes to be the best heuristic on the classes `GEOM` and `GEOMa`.

DSATUR. We considered the factors: order $\{LSF, SSF\}$, adjusted degree {Y, N}, greedy assignment {T-greedy, T-greedy$^{\#}$}, and instance class (same levels as above). The ANOVA indicates as significant the main effect of the factors order and adjusted degree and the interaction of this last one with the instance classes. A closer examination of the computational results showed that the adjusted vertex degree is preferable for all instance

[5]The output of the benchmark code available from the DIMACS "Computational Challenge" web site is: `DFMAX(r500.5.b) 8.64 (user)`. The machine runs Debian Linux, the code was implemented in `C++` and compiled with `gcc` and flags `-O3`.

classes, except in the class TG-120-0.1-5.5, where adjusted and non-adjusted vertex degree result in similar performance; this explains why ANOVA indicated as significant the interaction effect. Hence, we can conclude that G-DSATUR performs better with an adjusted vertex degree. Note that this is a novel feature with respect to previous implementations of G-DSATUR. Moreover, a closer examination on the order reveals another novelty: the preferable order is smallest saturation first. This result contrasts with what is used in DSATUR in the GCP and with the implementations of generalised DSATUR in [6, 12, 16, 26]. No significant difference is instead detected between T-greedy and T-greedy$^{\#}$. Hence, given the worse computational complexity of T-greedy$^{\#}$, T-greedy is to be preferred. Finally, we additionally tested the suggestion in [26] to colour first vertices in the largest clique but we could not detect any improvement.

RLF. We study the factor adjusted vertex degree within the different instance classes. Supported by the Wilcoxon sum rank matched pairs test, we conclude that an adjusted vertex degree worsens the solution quality and therefore should not be used.

Assignment heuristics. We study the factors vertex order {C, D}, requirement order {R, C}, assignment policy {F, R} and instance class. The ANOVA indicates as significant all main effects but not the interactions. However, differences among the heuristics arise in the instance classes and the only choice that remains significantly the best across all instance classes is the use of a requirement exhaustive assignment policy in the heuristics.

Selection of the best heuristic. The previous analysis indicated the best configuration for G-DSATUR, (smallest saturation first with adjusted vertex degree and T-greedy), and RLF and it allowed us to discard all the assignment heuristics that do not use a requirement exhaustive assignment policy. No general conclusions were instead possible for the sequential heuristics. In order to identify an overall best heuristic, we apply F-race, a racing algorithm described in [5]. Racing algorithms sequentially evaluate candidate algorithms and discard them as soon as statistically significant evidence arises against them. We run one race for each instance class {GEOM, GEOMa, GEOMb, TG-120-0.1-5.5, TG-120-0.5-5.5, TG-120-0.9-5.5}. We use a Friedman test for replicated designs [11] to test the statistical significance of the differences among the algorithms and we do not discard candidates in the first three stages of the race; one stage of the race corresponds to the collection of one run of each heuristic on all the instances of a class. The F-race stops if only one winner remains or after a maximum of 20 stages. A total of 16 candidate construction heuristics took part in the race. The results are reported in Figure 2.

We conclude that the overall best algorithm is G-DSATUR. Only on the instance class TG-120-0.1-5.5 it is not the best heuristic but it clearly dominates on all other classes. Its computational times on these instances are never above 70 msec. We remark that the results on the simple construction heuristics also have implications on the choice of the heuristics for guiding a tree search algorithm, resulting in possibly better complete algorithms for this or related problems.

6 Stochastic Local Search Algorithms

Local search algorithms iteratively move in the search space of (complete) candidate solutions $\mathcal{C}$, where the possible set of successor solutions is defined by a *neighbourhood structure* $\mathcal{N} : \mathcal{C} \rightarrow 2^{\mathcal{C}}$. Their extensions through general-purpose stochastic local search methods [25] are among the top performering algorithms for the GCP and previous experience suggests that this is also the case for the GSTCP. The new algorithms we developed and the previously known SLS algorithms for the GSTCP essentially follow three different schemes

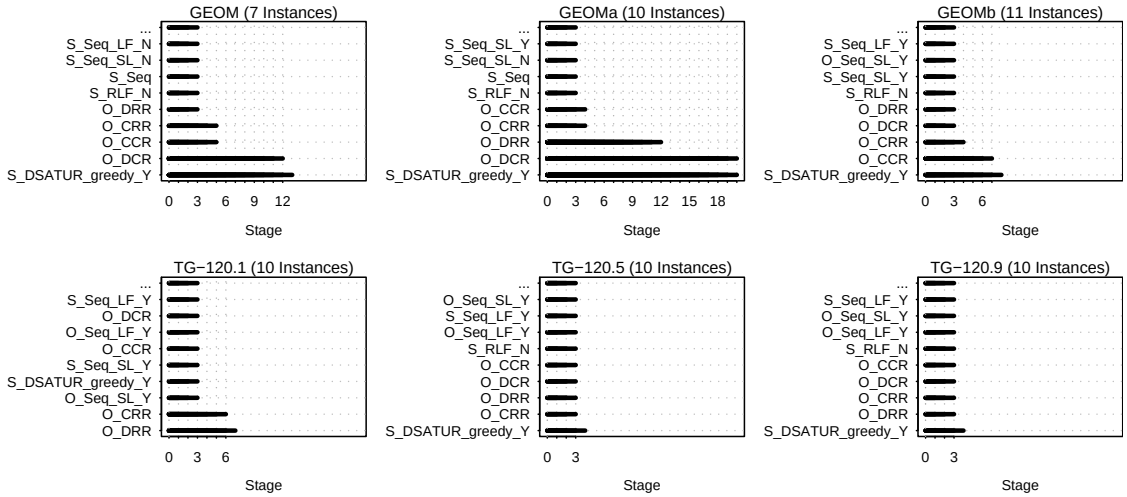


Figure 2: The outcome of the F-races for the construction heuristics on the 6 instance classes. A stage corresponds to the collection of one run of each heuristic on all the instances of a class. On the y-axis, algorithms are ordered according to average rank, the best being closer to the origin. The line indicates the life of the heuristic in the race; if only one heuristic survives, the race is stopped. Only the best 9 algorithms out of 16 are depicted. In the labels the first letter refers to the split graph S or original graph O approach.

for tackling the problem. These schemes depend on the choices taken for the application of the underlying local search.

6.1 Schemes for applying SLS algorithms to the GSTCP

For defining how to tackle the GSTCP, there are two main decisions to take: how to combine the constraint satisfaction and the optimisation part of the problem and which constraints to enforce in the definition of candidate solutions.

The first aspect is tackled usually in two possible ways in graph colouring problems. Either one reduces the optimisation problem to a series of constraint satisfaction problems that allow for a decreasing number of colours k. In this case, each constraint satisfaction problem is tackled as an optimisation problem where the candidate solutions are infeasible and the local search tries to minimise the number of constraint violations. (This is a typical procedure in local search for constraint satisfaction.) Or one defines an objective function that is the weighted sum of the number of colours and the violated constraints and that is to be minimised. In the rest of the article we denote the former choice "k fixed" and the latter "k variable", to emphasise the presence or not of k in the objective function.

The second decision affects the search space of the local search algorithm in two ways, by determining its size and by influencing its landscape. If no constraints are enforced in the solution representation, the search space might be very large but the landscape highly connected. On the contrary, the enforcement of many constraints in the solution representation might reduce the search space size but it might also make impossible to move from a given solution to another or it might introduce basins of attractions from which it could be hard to escape. Clearly, a good trade-off must be found. In the GSTCP, we decide to maintain the requirement constraints always satisfied. (However, the alternative choice would also be worth investigation and corresponds to a solution representation that allows also partial solutions.) The possibility of the transformation of GSTCP instances into GTCP instances suggests two alternative choices that we examine: (i) enforcing the ver-

tex constraints and minimising the violation of edge constraints only (GSTCP formulation on the original graph) or (ii) maintaining on the same level edge and vertex constraints and minimising the violation of both (GTCP formulation on the split graph).

The combination of these choices gives rise to different local search schemes. We have implemented the following three.

Scheme 1: k fixed, split graph. Candidate solutions are represented as complete assignments, *i.e.*, one colour for each vertex. The evaluation function counts the number of vertex and edge constraint violations and the goal is to find an assignment with zero violations. The basic neighbourhood is the one-exchange neighbourhood, where for a given candidate solution all those candidate solutions are neighbours that differ in the colour assignment of exactly one vertex. We further restrict this neighbourhood by allowing only the vertices that are involved in a constraint violation to change their colour. We call this restricted neighbourhood N_E.

Scheme 2: k fixed, original graph. This scheme works on the original GSTCP formulation and, hence, candidate solutions are represented as sets of $r(v)$ colours for each vertex $v \in V$. In this representation, we also enforce that the colours within each set satisfy the vertex separation constraints, thus, reducing significantly the search space size when compared to the previous scheme. The evaluation function counts only the number of violated edge constraints and is to be minimised. The basic one-exchange neighbourhood is analogous to N_E with the further restriction that the new colours must also satisfy the vertex constraints. We denote this neighbourhood by N'_E.

In this scheme, we additionally defined a new vertex colour reassignment neighbourhood [10]. Given a candidate solution, the set of its neighbours comprises all candidate solutions where all colours at a single vertex have been reassigned. Since this set would be of exponential size, we restrict it to comprise only those candidate solutions obtained by vertex colour reassignments that satisfy the requirement, vertex and edge constraints acting on the vertex. We denote this neighbourhood by N_R. N_R requires the exact solution of a sub-problem, which is finding a feasible assignment of $r(v)$ colours among the k available ones to a vertex v under the condition that none of the other vertices change their colour assignment. Clearly, such a reassignment of colours may not exist, and, hence, this may lead to the case that N_R is empty.

Scheme 3: k variable, split graph. In this scheme, the candidate solutions are again complete colour assignments but now the colour assignments can be both proper and improper. The number of colours k is left variable but it is, for practical reasons, limited to a value k_I that we determine by a construction heuristic. An *evaluation function* to guide the search towards proper colourings and towards colourings with small span was proposed by Hurley et al. [26]. Similarly, for a given candidate solution s, we define

$$f(s) = k_{max} + k_I \cdot \sum_{uv \in E} I_E(uv) + (k_{max} - k_{min}) + \sum_{i=1}^{k_I} I_C(i), \tag{4}$$

where k_{max} is the largest colour used by s, $I_E(uv)$ is an indicator function that returns one if the corresponding edge or vertex constraint is broken, $k_{max} - k_{min}$ is the span in s, and $I_C(i)$ is an indicator function that returns one if colour i is used in s (thus, this last term computes the order). Note that the edge conflicts are weighted by k_I so that a solution

that reduces the number of violations will always be preferred to those that modify other terms of the sum. The inclusion of the order in the sum contributes to break ties. The term k_{max} is the least important and contributes only to use the first colours, avoiding to move with the same span over and over through the interval $[1, k_I]$. Finally, it is again straightforward to use N_E as the neighbourhood.

6.2 Neighbourhood Examination

One-exchange neighbourhood. A crucial aspect of local search algorithms is the examination of the neighbourhood. The size of N_E is $k \cdot |V^c|$, where V^c is the set of vertices that are involved in a conflict. Additional data structures can be used to make the evaluation of a neighbour fast. This is especially important in efficient implementations of a best-improvement strategy, where the best neighbouring solution replaces the current one. To this aim, one can define, for example on the split graph, the matrix $\mathbf{\Delta}$ of size $|V^S| \times k$ where each element $\mathbf{\Delta}(v, c)$ indicates the contribution to the evaluation function of the assignment of the colour c to vertex $v \in V^S$. This matrix can be initialised and updated taking into account the usual speed-up techniques for the GCP; however, due to the separation distance constraints, the update of $\mathbf{\Delta}$ is by a factor t more complex than in the GCP case, where t is the maximum occurring separation distance. In the local search for the GSTCP formulation, an additional matrix $\mathbf{\Delta}_2$ of size $|V| \times k$ is maintained to forbid the assignment of colours that break vertex constraints.

Exact vertex colour reassignment neighbourhood. In our implementation, we search N_R by considering in random order the vertices currently involved in at least one conflict. For each chosen vertex, a set F is determined that comprises the colours that are proper, that is, not in conflict with any colour assigned to the adjacent vertices. The construction of this set can be done in $\mathcal{O}(|V|k)$ using the auxiliary data structures introduced in the previous paragraph. If one simply sought for $r(v)$ colours from F such that the vertex constraints are satisfied, this would be easy: it suffices to order the values in F and scan the set once, skipping the values that are not sufficiently distant from the previous ones. However, this procedure is deterministic and, when visiting a vertex a next time, the search would use the same colour reassignment if nothing had changed in the adjacent vertices. To avoid this cycling behaviour, randomisation is introduced in the reassignment. This requires to find all subsets of F of size $r(v)$ that satisfy the vertex constraints and pick one at random.[6] We solve this problem in a dynamic programming fashion [10]. Given the ordered sequence of integers in F, a proper colouring is an ordered subsequence of integers composed by other subsequences, each allowing a number of proper solutions, corresponding to the different ways the subsequence can be extended to a sequence of length $r(v)$ by adding elements from F. The total number of such solutions can be defined recursively and computed in a bottom-up fashion. Afterwards it is possible to choose one solution uniformly at random.

More specifically, let s be the ordered vector of integers in F, $L = |F|$, $D = t_v$ and $H = r(v)$. Then for each position i of the vector s we define $next(i) = \min_j\{j \mid j > i,\ s[j] - s[i] \geq D\}$. For each subsequence l of s, the number $N_h[i]$ of proper subsequences of s of length $h \in \{1, \ldots, H\}$ containing $l = \{s[1], \ldots, s[i]\}$, can be determined by the

[6] This problem corresponds to generating all subsequences of length L of a set of H integers, $H > L$, such that all pair-wise distances between the integers of the subsequence are larger than a constant D.

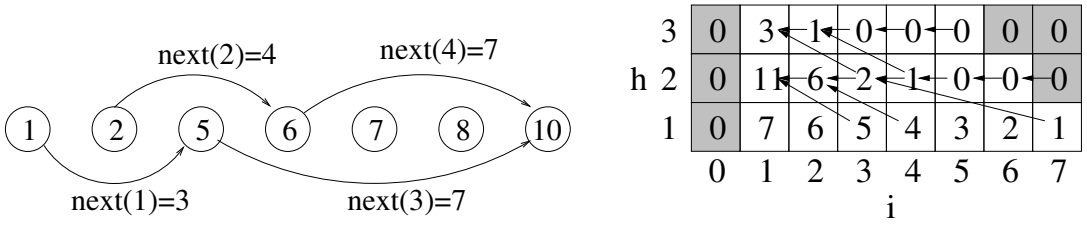


Figure 3: An example of vertex exact colour reassignment for a case in which $F = \{1, 2, 5, 6, 7, 8, 10\}$, $L = |F| = 7$, $D = t_v = 4$ and $H = r(v) = 3$. On the left the vector s of 7 integers. For each integer in the sequence the pointer $next()$ is computed, where not indicated it is set to 0. On the right the table of $N_h[i]$ values. Its construction starts from the low right corner. Arrows indicate the stored values that are used to compute the entries. The grey cells indicate the values without a proper meaning and hence assigned by convention.

recursion

$$N_h[i] = \begin{cases} L - i + 1 & \text{if } h = 1 \\ N_{h-1}[next(i)] + N_h[i+1] & \text{if } L - i - 1 > h \geq 2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

if $i \geq 1$ and $N_h[0] = 0$ by convention.

The total number of proper subsequences of length $r(v)$ corresponds to $N_H[1]$. To select one solution at random among the $N_h[i]$ solutions one has then simply to scan the sequence s and select each element with probability $N_{h-1}[next(i)]/N_h[i]$, since this is the fraction of the extensions, which contain the element in question. Whenever an element is chosen the scanning moves to $next(i)$. Scanning the sequence of numbers takes linear time, but computing each time the recursion 5 takes exponential time. However, this can be done much more efficiently if all values $N_h[i]$ are computed at the beginning and recorded in a table. Referring to Figure 3, right, if the table is filled from bottom to top and from right to left within each row, each new entry needs the values of $N_{h-1}[next(i)]$ and $N_h[i+1]$, which are already determined and stored. The *next* function and the table together can then be computed in $\mathcal{O}(\max(D, H, \log L)L)$ while to chose a subsequence randomly a further scan of the sequence s is needed.

6.3 SLS Algorithms

The local search schemes in the previous section perform poorly if applied in a simple iterative improvement algorithm. They become really effective, once they are used inside general-purpose SLS methods. We have therefore implemented various SLS algorithms which include re-implementations of previously proposed algorithms for the GSTCP or their extension, new adaptations of high-performing SLS algorithms for the GCP to solve the GSTCP, and newly developed algorithms. In all cases, we solve the GSTCP by starting from a k and a colouring determined by the G-DSATUR heuristic.[7] For algorithms that solve the problem as a series of constraint satisfaction problems with k fixed, some vertices have to be recoloured once a proper colouring with k colours is found. This is done by

[7]It might be objected that the best initial solution is not necessarily the most favourable for the local search. For the GCP, we did not note any difference in final solution quality by using one or another construction heuristic but there is instead a reduction in CPU time if the initial k is the best possible. Moreover, in the scheme that holds k fixed, a new initial solution for local search is determined every time k is decreased by one. In this case we found that there is no difference in using a random reassignment of colours to uncoloured vertices or any other choice. For our choice we assumed these same results to be valid also on the GSTCP.

choosing for those vertices that had assigned colour k a colour among the $k-1$ remaining ones uniformly at random. The best solution returned is the smallest value for k at which a proper set T-colouring is found.

6.3.1 SLS algorithms for Scheme 1: k fixed, split graph

Tabu search. We implemented a standard tabu search procedure that chooses at each iteration a best non-tabu move or a tabu but "aspired" neighbouring solution from N_E. The tabu list forbids to reverse a move and the tabu tenure is chosen as $tt = \mathit{random}(10) + 2\delta|V^c|$, where V^c is the set of vertices which are involved in at least one conflict, δ is a parameter, and $\mathit{random}(10)$ is an integer random number uniformly distributed in $[0, 10]$; this choice follows a successful tabu search algorithm for the GCP by Galinier and Hao [18]. We denote this algorithm SF-TS.

SF-TS is very similar to the tabu search algorithms proposed in [12], [8], and [26]. In those papers, tabu search was reported to perform better than simulated annealing and genetic algorithms. Our tests with a simulated annealing algorithm of the FASoft system [26] for the GEOM benchmark set confirmed these results.

Min-conflicts. The min-conflicts heuristic [36] is based on N_E but implements a particular two-stage process for examining the neighbourhood. In a first stage, a vertex is chosen uniformly at random from the set V^c; in a second stage, this vertex is assigned a colour such that the number of conflicts is minimised, breaking ties randomly. As shown in [48], the min-conflicts heuristic can be profitably enhanced by tabu search, where the reversal of a move is forbidden. If all colours of a vertex are tabu, one is chosen randomly. One advantage of this neighbourhood examination strategy is that it allows for an easier implementation, since it does not require the usage of auxiliary data structures such as the $\mathbf{\Delta}$ matrix described in the previous section. In addition, the chances of cycling are reduced due to the random choices in the first stage of the selection process, allowing for shorter tabu lists. The tabu tenure is set constantly to an integer tt during the whole search. We denote this algorithm SF-MC.

Guided local search. Guided local search (GLS) is an SLS strategy that modifies the evaluation function in order to escape from local optima [51]. An application of GLS to the GCP has shown good performance on geometric graphs [9]. GLS uses a penalty weight w_e that is associated to each edge e and that is initialised to one. On the GTCP, the algorithm derived from this strategy, SF-GLS, uses an augmented evaluation function g' defined as

$$g'(s) = f(s) + \lambda \cdot \sum_{e \in E^C} w_e \cdot I_E(e),$$

where $f(s)$ is the usual evaluation function, λ a parameter that determines the influence of the penalties on the augmented cost function, and $I_E(e)$ an indicator function that takes the value 1 if the end points of edge e are in conflict in the candidate solution s and 0 otherwise. The penalties are initialised to 0 and are updated each time an iterative improvement algorithm reaches a local optimum of g'. The modification of the penalty weights is done by first computing a utility u_e for each violated edge, $u_e = I_E(e)/(1 + w_e)$, and then incrementing the penalties of all edges with maximal utility by one. The underlying local search is a best-improvement algorithm on N_E. When a local optimum

is reached, the search continues for a maximum number of sw plateau moves before the evaluation function g' is updated.

GLS was already applied in the context of frequency assignment [50]. There, the local search was more complex than ours since it includes the use of *don't look bits* to avoid the repetition of recent exchanges. However, by restricting the neighbourhood to only vertices involved in conflicts we practically achieve a similar effect.

Hybrid evolutionary algorithm. A hybrid evolutionary algorithm was shown to be a top performer for the GCP [18]. This algorithm uses a tabu search procedure as the underlying local search. We denote the hybrid algorithm on the GTCP as SF-HEA. SF-HEA starts with a population P of candidate solutions and then iteratively generates new candidate solutions by re-combining two members of the current population. The recombination operator, which is deemed responsible for the high performance of this algorithm on the GCP, is the greedy partition crossover (GPX) [18]. The new candidate partition returned by GPX is then improved by SF-TS, run for l_{LS} iterations, and is inserted in the population P replacing the worse parent. Beside the parameter δ of SF-TS, SF-HEA requires to set a value for l_{LS}.

6.3.2 Scheme 2: k fixed, original graph

Tabu search. An application of tabu search using the one-exchange neighbourhood N'_E on the original graph was designed by Dorne and Hao (1998) [16]. Other versions of this algorithm for frequency assignment [23] differ only in the management of the tabu length or are more rudimentary [24]. Our version uses the same tabu tenure definition as SF-TS and is in this equal to the version in [16]. We denote this algorithm OF-TS.

We also include two enhanced versions of OF-TS, which make use of the vertex reassignment neighbourhood N_R. Since the exploration of the union of N'_E and N_R would be computationally expensive, we adopt a heuristic rule for choosing the next move to apply. First the best non tabu move in the neighbourhood N'_E is determined. If it improves on the current solution, it is accepted. If it leaves the evaluation function value unchanged or worsens it, a move is searched in the neighbourhood N_R, restricted to vertices involved in at least one conflict. If a proper reassignment is found, it is applied, otherwise the best non-tabu move in N'_E is applied. We call the overall algorithm OF-TS+R.

A variant of OF-TS+R tries the colour reassignment to a random vertex from V in case no move is found in N_R restricted to conflicting vertices. The motivation for this is that a random reassignment of colours to vertices, where no conflict is present, may produce a change that can propagate profitably. We denote this variant OF-TS+R*.

The tabu search mechanism applied to moves in N'_E is the same used in OF-TS and [16] (aspiration criterion included). No tabu search mechanism is instead applied to moves in N_R. In this case, repetitions in the search are avoided by the randomisation of the reassignment. Preliminary experiments clearly indicated that the use of a randomised reassignment instead of a deterministic one yields better results.

Squeaky wheel optimisation. The iterated greedy (IG) algorithm by Culberson for the GCP [14] consists in applying the greedy colouring repeatedly each time to a new permutation of the vertices. Each new permutation keeps vertices with the same colour in the previous colouring in consecutive positions. A nice theorem shows that by this definition of a permutation as input to a greedy colouring procedure, results in a solution that does not use more colours than the previous colouring, but possibly less. For the

GSTCP, due to the distance separation constraints, the result of the theorem is not true anymore. In fact, only permutations in which the distance between colour classes remains the same will guarantee such a result. One such permutation is the reverse order of the colour classes.

The IG algorithm has inspired the application of squeaky wheel optimisation (SWO) [30] to the GSTCP by Lim et al. [31]. The method works in three phases: construction of a solution, analysis and identification of trouble makers, and prioritisation of the trouble makers that are thus handled earlier by the constructor in the next iteration. In the description of the first SWO algorithm for the GSTCP [31], a few details are missing, making impossible a precise re-implementation of the algorithm. Instead, we implemented our own version. The algorithm works as follows. An initial solution is determined by a construction heuristic. Then, a colouring that uses $k-1$ colours is constructed applying the T-greedy algorithm to a permutation of the vertices obtained from the previous colouring. The permutation maintains vertices that have the same colour in consecutive positions but reverses the order of the colour classes and shuffles vertices in the classes. The rationale behind this procedure is that, if the reverse order of colour classes yields a colouring that uses the same or a lower number of colours, then it is possible that this order or small variations thereof lead to a feasible colouring with $k-1$ colours. Perturbations in the reverse order are introduced by inserting vertices that had colour k before vertices that had colour b, with $b \in \{k-1, \ldots, 1\}$. Each colouring generated by T-greedy with a proposed permutation of the vertices is then improved by OF-TS applied for $\mu \cdot V^c \cdot \sum_{v \in V} r(v)$ iterations. If no feasible solution is found and all possible values of b have been attempted, the search proceeds by OF-TS solely. We denote this algorithm as OF-SW. Note that the algorithm can be used both with OF-TS and with SF-TS. The preference for OF-TS is the result of preliminary experiments. In addition, we include another variant OF-SW+R, which uses OF-TS+R instead of OF-TS.

More recently, Lim et al. published an enhanced version of their algorithm [32]. This version differs from ours in the fact that 5% of colours are placed after b in the new permutation and b is kept fixed to 8. Moreover their algorithm uses a tabu search on the split graph based on a swap neighbourhood. Experiments, however, show that our version produces often results superior to those reported in [32].

6.3.3 Scheme 3: *k* variable, split graph

Tabu search. We implemented a tabu search algorithm in all equal to SF-TS except that it uses the evaluation function of Equation 4. We denote this algorithm as SV-TS.

7 Experimental Assessment of SLS Algorithms

In this section, we compare experimentally the 10 SLS algorithms described in Section 6.3. Additionally, we include the tree search algorithm with incomplete randomised dynamic backtracking by Prestwich [41, 42]. The implementation of this algorithm, which we denote FCNS, was kindly made available by the author.

The empirical analysis aims at determining (i) which local search scheme performs the best; (ii) the impact of the new colour reassignment neighbourhood; and (iii) the overall best algorithm. Moreover, by separating the analysis between the instance classes we aim at assessing differences in performances due to instance features.

Differently from construction heuristics, the SLS algorithms studied here do not have a natural stopping criterion. To make the comparison fair among the algorithms, we

	NRU, HD-RU	GEOM	Philadelphia
OF-TS	δ ={0.5; 1; **10**; 20; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; **20**; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; 20; 30; **40**; 50; 60; 70; 100
OF-TS+R	δ ={0.5; 1; **10**; 20; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; **20**; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; 20; 30; **40**; 50; 60; 70; 100
OF-TS+R*	δ ={**20**; 40}	δ ={**20**; 40}	δ =20; **40**}
SF-TS	δ ={0.5; 1; **10**; 20; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; **20**; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; 20; 30; **40**; 50; 60; 70; 100
SV-TS	δ ={0.5; 1; **10**; 20; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; **20**; 30; 40; 50; 60; 70; 100}	δ ={0.5; 1; 10; 20; 30; **40**; 50; 60; 70; 100
SF-HEA	μ = {1000, **10000**}; δ = {**20**,40}	μ = {1000, **10000**}; δ = {**20**,40}	μ = {1000, **10000**}; δ = {**20**,40}
SF-GLS	λ = 1; sw = {1, **20**, 100, 200}	λ = 1; sw = {1, **20**, 100, 200}	λ = 1; sw = {1, **20**, 100, 200}
SF-MC FCNS	tt = {2, 10, **20**, 30} B=1	tt = {2, 10, **20**, 30} B=1	tt = {2, 10, **20**, 30} B=1

Table 3: The parameter values tested and selected (in bold) for the SLS algorithms.

need to allocate the same amount of resources. Since we cannot use the number of steps per algorithms, because they entail different computational effort among the algorithms, we use for all algorithms a common CPU-time limit. This approach is feasible, since the algorithms are all implemented in a same framework, share data structures where reasonable and they were implemented by the same person.[8]

7.1 Experimental setup

CPU time limit. For defining the CPU-time limit, we use OF-TS as a reference algorithm. This algorithm is run for $I_{max} = 10^5 \times \sum_{v \in V} r(v)$ iterations and its computational time is measured.[9] The iteration limit is chosen such that on the one side the algorithm reaches a limiting behaviour and further improvements in solution quality are small, on the other side the experimental study remains feasible with respect to the overall computational time consumption. Note that I_{max} is set proportional to the number of vertices and to the requirements and, hence, the computational time varies among the instance classes. Moreover, the computational time within each class is stochastic because of the differences observed when running OF-TS on a same instance and to the presence of different instances. By using regression techniques [35], we fitted the computational times among all types of instances and expressed them as a function of the 5 variables $|V|, \rho, \bar{r}, \bar{t}_{vv}, \bar{t}_{uv}$. The details of this analysis are reported in [9]. We use the values predicted by the model as time limits in each class.[10]

Parameter tuning. All SLS algorithms in the comparison require some parameters to be tuned for the classes of problem instances and for the given computational time limit decided. In Table 3, we summarise the parameters presented in Section 6.3 and for each of them the values tested and the values selected.

The experimental design. We perform one analysis for each of the instance classes in Table 1. Within each class, the algorithms constitute the treatment factor and the

[8] Certainly, this does not remove any possible bias towards one algorithm; however, several factors that influence computational times like programming languages, computing environment, etc. are excluded from being the main responsible for the observed differences.

[9] Only on the HD-RU set we adjusted the iteration limit to $I_{max} = 10^4 \times \sum_{v \in V} r(v)$, because of the large size of these instances and the very high resulting computational times.

[10] To give an impression of the computational times involved, we mention that the longest computational time for any uniform random instances with 60 vertices resulted to be about 3 hours and 30 minutes. For numerical details, see the tables in Appendix B.

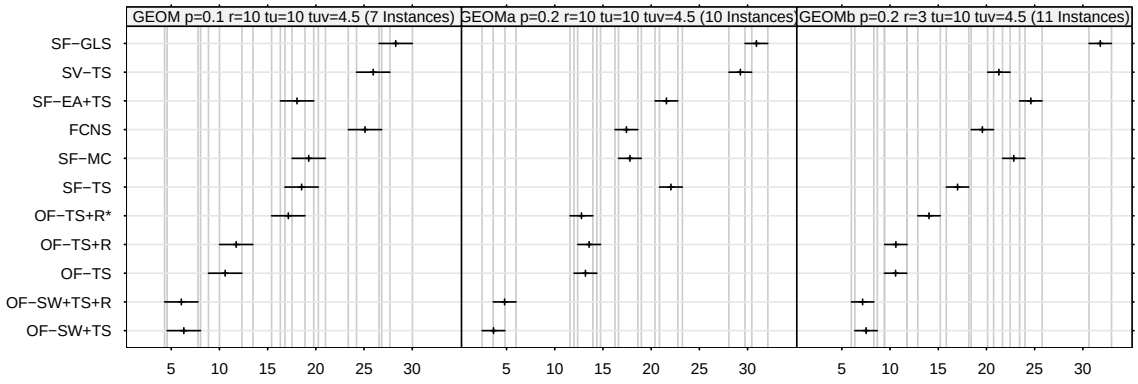


Figure 4: Simultaneous confidence intervals for SLS algorithms on the `GEOM` instance set of the GSTCP. The x-axis gives the average rank.

different instances the blocking factor. We collect 5 algorithmic runs per instance on the Philadelphia set and 3 runs per instance on the classes derived from the `GEOM`, `NRU` and `HD-RU` sets. We use the set `HD-RU` only for the validation of numerical results in Appendix A, as for other kinds of analyses the set `NRU` covers the same instance structures and it is better organised.

Statistical analysis. We use statistical methods to determine whether the data collected are enough to claim statistical significance of the observed differences. Checks on the assumptions for a parametric analysis indicated that these are violated and therefore we resort to non-parametric rank-based tests. Results are ranked within each instance among the 11 algorithms and across the number of runs collected. (That is, the ranks on each instance range from 1 to 33, if 3 runs per instance are collected.) We present the results by means of simultaneous confidence intervals as derived from the Friedman sum rank test [11]. The visualisation is given in Figures 4 to 6 for the three instance sets. On the y-axis, the algorithms are ordered according to their overall performance in the instance set, the best algorithms being closer to the origin. On the x-axis we report the average rank. Two algorithms are significantly different if the confidence intervals around their average ranks do not overlap. For a synthesis of the numerical results underlying these graphics, we refer to Appendix A.

7.2 Results

In general, there is a considerable improvement over the solution provided by G-DSATUR which indicates the importance of using SLS algorithms (see numerical results in Appendix A). Moreover the comparison with a randomised backtracking algorithm, FCNS, clearly indicates the superiority of approaching these large GSTCP instances by means of local search methods. However, no SLS algorithm is consistently the best in all instance classes and, hence, instance features influence the relative performances.

Local search scheme. For all instance classes solving the constraint optimisation problem as a series of constraint satisfaction problems results to be a better approach than weighting both infeasibility and span in a single objective. This can be appreciated by comparing the performance of SV-TS and SF-TS over all the classes. Although it is opportune to maintain a certain caution in commenting these results, because better algorithms

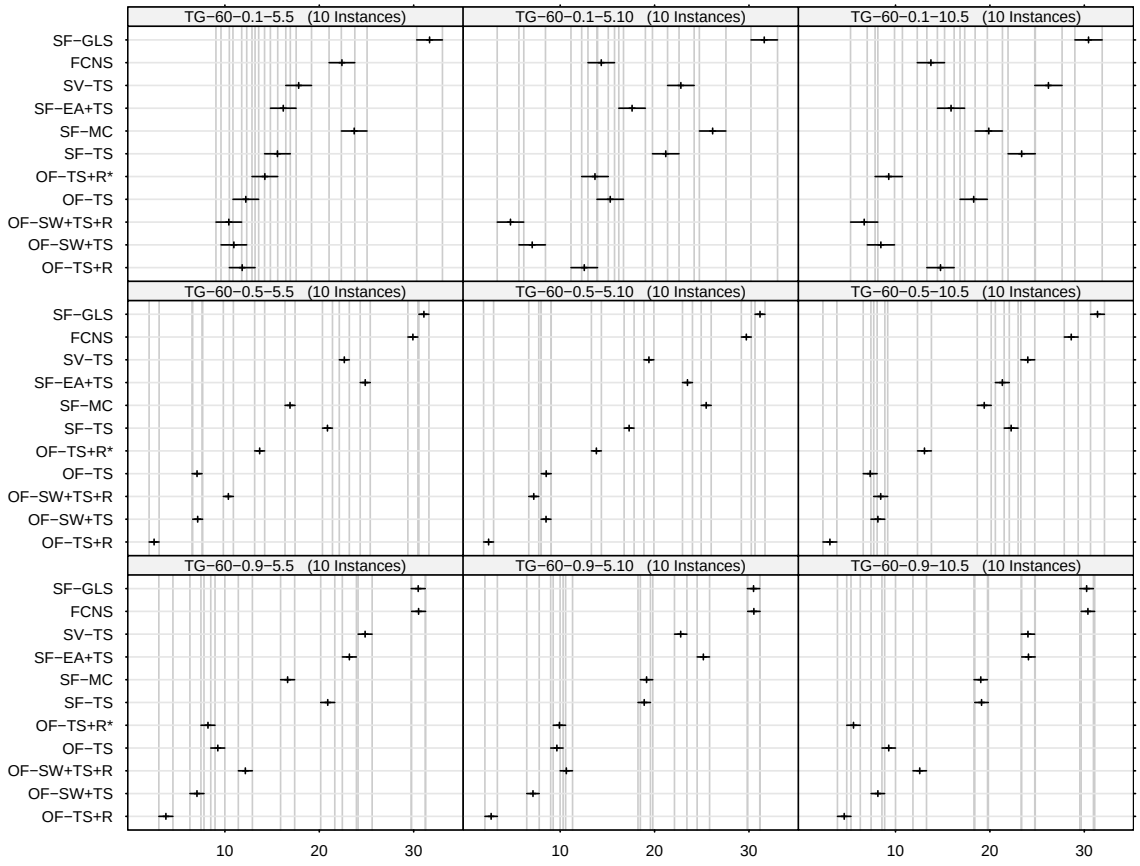


Figure 5: Simultaneous confidence intervals for SLS algorithms on the `NRU` instance set of the GSTCP. Each algorithm was run 3 times per instance with a time limit corresponding to I_{max} iterations of OF-TS. The three rows comprise the instances for values of p equal to 0.1, 0.5, and 0.9, respectively; the columns report different parameters for the requirements and separation distances. The x-axis represents the average rank.

than the simple SV-TS could be devised, we found that the approach that leaves k variable is more complex to implement and to tune than the one leaving k fixed.

The other clear result is the profitability of enforcing the vertex constraints to be always satisfied. In particular, we draw this conclusion by comparing OF-TS with SF-TS. This result may be somehow surprising and contrasts with the findings of [43] who show that symmetry breaking constraints have a negative effect on local search performance for SAT problems. The reasons for the improved performance by enforcing the vertex constraints are not absolutely clear. If they are not enforced, it may simply be that the resulting search space remains too large; while, if the constraints are enforced, the search landscape might exhibit barriers of increased height between local optima which should hinder the search. In this latter case, the search space remains however connected and tabu search might be sufficient to overcome the barriers. Moreover, the possibility of making larger steps in the vertex colour reassignment neighbourhood than in the one-exchange neighbourhood might also have helped to overcome some detrimental effects due to enforcing the vertex requirement constraints. A more detailed analysis to unveil the possible reasons for this behaviour is, however, beyond the scope of this article.

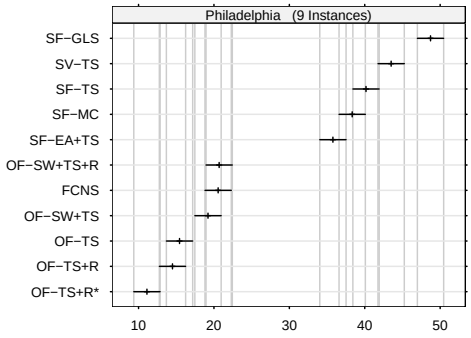


Figure 6: Simultaneous confidence intervals for SLS algorithms on the Philadelphia instance set of the GSTCP. Each algorithm was run 5 times per instance with a time limit corresponding to I_{max} iterations of OF-TS. The x-axis represents the average rank.

Vertex colour reassignment neighbourhood. The comparisons of OF-TS *vs.* OF-TS+R and OF-SW *vs.* OF-SW+R show that the usefulness of the colour reassignment neighbourhood N_R depends on the features of the instances. On the GEOM set, the usage of N_R does not yield any improvement but its usage at least does not worsen performance significantly. On the NRU set, however, the usage of N_R results in a statistically significant better performance on various instance classes. From a closer inspection of the results, we might argue that this neighbourhood becomes profitable with an increase of the edge density in the graph and an increase of the vertex requirements. Finally, on the Philadelphia set, OF-TS+R*, a variant of OF-TS+R, is the best performing algorithm. We also checked in more detail the contribution of the reassignment neighbourhood and found that it does, in fact, contribute to find a significant number of improvements, where a simple one-exchange does not. As an example, on instance type T-G.5.5-60.0.5 on 10,000 iterations of OF-TS+R we observed 776 improvements due to one exchanges and 194 due to vertex colour reassignments [9].

Best SLS algorithms. Tabu search is, as also for the GCP [9], a very competitive algorithm despite its simplicity. The use of squeaky wheel optimisation to improve its performance is not always effective. On the NRU set, the best results are typically found by OF-TS+R, *i.e.*, the tabu search with the colour reassignment neighbourhood. Squeaky wheel optimisation improves performance, however, on the low density NRU graphs and on the GEOM graphs. On the Philadelphia set, tabu search works again best as a stand-alone algorithm. Since the Philadelphia graphs are quite similar to the geometric ones, the reason for the observed difference in the contribution of squeaky wheel optimisation might be due to the higher vertex requirements of the Philadelphia instances.

Other methods, which are state-of-the-art for solving specific instance types of the GCP, perform surprisingly poorly on the GSTCP. These algorithms, SF-MC, SF-GLS and SF-HEA, perform poorer than SF-TS. Further analysis on SF-MC revealed, however, that its performance varies in relation to parameter tuning and some limited results indicate that it can become competitive for very large graphs [9]. Moreover, contemporaneously to our work, Malaguti and Toth [33] were developing an evolutionary algorithm with a recombination operator, tailored for this specific problem, that does seem to bring some advantages.

Comparison with the results in the literature. As a final step, it is worth comparing

the results of the algorithms on standard benchmarks. The detailed results of the algorithms on the instances of the sets GEOM and HD-RU and Philadelphia are given in Appendix A together with the best known values reported in the literature ([39, 40, 42, 31, 32] and the FAP website). We can make the following observations.

On the GEOM instance set, our version of squeaky wheel attains better results than the version in [32]. The results in the column best results are indeed almost all derived by that paper. Our best results found by OF-SW are better on 16 instances (with improvements of even 19 and 22 colours) and equal on 7, while they are worse only on 5 instances. Moreover, the results in [32] are the best out of 10 runs while in our case they are the best over only 3 runs per instance. Computational times seem to be longer in our case but a fair comparison is not possible because in [32] we find only the times to reach the best solution, while the whole run time should be compared.

On the HD-RU set we observe that our G-DSATUR performs better than the DSATUR presented in [16], thus indicating the importance of the novelties introduced into the algorithm. The results of OF-TS are, instead, inferior to those reported for the implementation of the same algorithm in [16] and they remain inferior even for our best algorithm in that class, OF-SW+R. This difference may be due to the different number of iterations allowed to the two tabu search algorithms. In fact, Dorne and Hao allow 10^7 or even $2 \cdot 10^7$ iterations for each constraint satisfaction problem in the series (that is, for each value of k examined) while in our experimental setup, we limit the number of iterations to be often lower than 10^7 across the whole series (that is, across all values of k). By comparing the computational times reported in [16] and adjusting the machine speed, we note that they used approximately 20 times more computation than we did in our experiments.

On the Philadelphia instance set, our G-DSATUR is in general better than the DSATUR implemented in the FASoft system [26]. If compared with the best solution returned by a portfolio of 64 sequential heuristics (DSATUR included) of FASoft, our G-DSATUR is able to improve their solution on 6 out of 9 instances. Hence, G-DSATUR would also be an excellent candidate for extending the algorithm portfolio of FASoft. Our best algorithm for this set of instances, OF-TS+R*, produces 7 times out of 9 the optimal solution. In the literature, only the algorithm by Matsui and Tokoro (2001) reached similar performance with 8 out of 9 optimal solutions found [34].

8 Conclusions

We presented an extensive study of stochastic local search algorithms for the GSTCP, a problem that finds practical application in frequency assignment. The study included construction heuristics and more advanced SLS algorithms and was organised with a rigorous separation of concepts so that the contribution of different modelling and search choices can be examined unambiguously. The assessment was conducted experimentally on three benchmark sets. A fourth benchmark set was created with the goal of better discriminating the influence of instance features on algorithm performance. The competitiveness of the results discussed have been shown on the random geometric instances proposed by M. Trick, the random uniform instances by [16] and on the Philadelphia instances from the frequency assignment literature.

The component-wise study on the construction heuristics allowed us to improve a generalised version of DSATUR and to show its superiority with respect to sequential, assignment and partitioning heuristics. The novelties of our G-DSATUR are the *smallest* first order of the vertices and the use of an adjusted vertex degree to break ties. Comparisons of the numerical results of our G-DSATUR algorithm on a number of benchmark instances

to the best construction heuristics in the literature confirm its very high performance. It should be also noted that the new heuristic functions used in G-DSATUR may lead to improvements in complete tree-search algorithms for the same problem.

Our study also has given a number of results that may be relevant for the design of SLS algorithms for other problems than the GSTCP. First, our analysis has confirmed that solving constraint optimisation problems as series of constraint satisfaction problems is more convenient than minimising a weighted sum of constraint violations and preference criteria (in this case the T-span). Second, we found that enforcing some constraints by the solution representation, in the GSTCP case the vertex constraints, has a positive effect on the local search. Finally, we proposed a novel exact examination of an enlarged neighbourhood, which uses randomisation to provide different solutions to the small subproblems being solved. The use of exact search in a complex neighbourhood can be seen as a successful example of hybridisation in local search, while the randomisation is inherent of stochastic local search as a basic way to avoid repeating actions and to diversify the search. Although these results are certainly problem domain dependent and their validity on other problems must be tested experimentally, they nevertheless may help to direct the design of local search for constraint optimisation problems.

Finally, we could identify the SLS algorithms that are the overall best. Besides tabu search, which showed to be a very useful metaheuristic in this specific problem domain, our re-implementation of a squeaky wheel optimisation algorithm performed the best on the Geometric instance set and we were able to produce 16 new best results on the benchmark instances. On the random uniform graphs, the best performing algorithm is instead our new tabu search with the randomised exact neighbourhood search. Its positive performance is confirmed and reinforced on the Philadelphia instances taken from the frequency assignment literature.

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Appendix A Numerical Results

In Tables 4 to 7 we report the numerical results of our experiments. In all tables, results are expressed in terms of the number of colours, k. Hence, the T-span may be derived by subtracting one. The time limit (Max time) at disposal for producing a solution in a single run is expressed in seconds. The lower bounds (Lwb) are determined through the TSP procedure described in [1] (we adjusted this value on the Philadelphia instances, as the Hamiltonian path derived from solving a related TSP is not always the shortest).

The results in Table 4 correspond to the set NRU. They are obtained across three runs per algorithm on the ten instances composing the class. For each class we aggregated the results and report the range. These results are used to obtain the visual analysis in Figure 5. The results in Table 5 correspond to the set HD-RU. DH-DSATUR and DH-TS are the DSATUR and tabu search algorithm proposed in [16]. In this case, we also report the number of maximal iterations that DH-TS was allowed to run for solving the resulting decision problem at each value of k [16]. These values are to be compared with the maximum total number of iterations used by our OF-TS algorithm. Our algorithms were run three times on each instance and we report best and average result as in [16]. The results in Table 6 correspond to the set GEOM. They are obtained by 3 runs per algorithm–instance combination and were visualised in Figure 4. We report the best and the median

result. Finally, the results in Table 7 correspond to the Philadelphia instances. We report the best and median results derived from 5 runs per algorithm–instance combination. These data constitute the numerical expression of the visual analysis of Figure 6.

Instance	Lwb		G-DSATUR		Max	OF-TS+R		OF-SW		OF-SW+R		OF-TS		OF-TS+R*	
	min	max	min	max	time	min	max	min	max	min	max	min	max	min	max
TG-60-0.1-5.5	30	41	41	62	10	36	46	36	46	36	46	36	46	36	46
TG-60-0.1-5.10	43	69	78	107	20	63	101	63	83	63	83	64	101	63	84
TG-60-0.1-10.5	56	92	79	116	710	70	97	70	93	69	92	69	98	71	92
TG-60-0.5-5.5	31	51	89	116	320	74	88	75	90	76	90	75	89	75	91
TG-60-0.5-5.10	62	106	168	231	3440	135	168	138	173	138	172	139	170	140	177
TG-60-0.5-10.5	67	101	148	222	6820	121	170	123	162	123	163	122	164	123	165
TG-60-0.9-5.5	72	97	160	194	1250	137	164	138	169	139	167	138	165	137	172
TG-60-0.9-5.10	92	126	304	370	6820	259	290	260	297	263	296	262	295	264	300
TG-60-0.9-10.5	142	174	288	353	12880	252	299	251	302	254	304	252	301	249	305

Table 4: Numerical results on the NRU instance set.

Instance	DH − DSATUR		G-DSATUR		DH-TS		Iter.	Max	OF-SW+R		OF-TS+R		OF-SW		OF-TS		Iter.
	min	avr	min	avr	min	avr	($\times 10^7$)	time	min	avr	min	avr	min	avr	min	avr	($\times 10^7$)
essai.100.275.10	63	63	61	61	46	46.5	1	10	47	47.3	47	52.3	47	47	**46**	47	0.28
essai.300.937.10	113	115.6	111	114.3	81	81.5	2	19	**82**	82.3	82	82.7	82	82.7	84	87.7	0.94
essai.500.1507.10	158	167	143	143.7	113	114	2	55	**111**	111.3	111	111.3	112	112.3	111	111.7	1.51
essai.1000.3049.10	282	286.7	224	228	179	179	3(?)	294	**182**	182.7	183	183.3	183	183	184	184	3.05
essai.30.95.50	75	77.1	68	68	62	62.5	1	10	**62**	62.3	62	62.3	62	62.3	63	63	0.10
essai.100.304.50	171	177.2	139	143.3	115	115.5	1	10	**116**	118.7	118	119.3	118	119.7	118	119	0.30
essai.300.905.50	432	444.9	330	331	268	269.5	2	78	**284**	285.7	284	286	284	287	284	287.7	0.91
essai.500.1484.50	678	678	479	479.7	403	403	2	253	**418**	422	420	422	423	425.7	425	426.3	1.48
essai.1000.3024.50	1221	1227.3	869	874.3	847	847	3(?)	1459	796	799.3	**792**	795.3	808	809	803	811	3.02
essai.30.90.90	146	154.7	119	120	103	103.33	1	10	**104**	104.3	104	104	104	104.7	105	105.3	0.09
essai.100.299.90	321	341.3	259	260	225	225.33	1	10	**234**	235.3	234	234.7	235	237	236	236.7	0.30
essai.300.940.90	903	922.2	661	671.3	573	574.33	2	168	**601**	606.7	607	608.7	610	611.3	608	610	0.94
essai.500.1536.90	1410	1433.3	1020	1026	932	932	2	523	**940**	950.3	950	955	955	958	950	957.7	1.54
essai.1000.2975.90	2523	2542	1804	1819	1724	1724	3(?)	2395	**1747**	1753.3	1748	1753.3	1754	1756.3	1760	1762	2.97

Table 5: Results on the HD-RU instance set.

Instance	Lwb	Best known	G-DSATUR min	G-DSATUR med	Max time	OF-SW min	OF-SW med	OF-SW+R min	OF-SW+R med	OF-TS min	OF-TS med	OF-TS+R min	OF-TS+R med	OF-TS+R* min	OF-TS+R* med
GEOM60	230	258	258	258	710	**258**	258	258	258	258	258	258	258	258	258
GEOM70	260	273	283	287.5	1060	**267**	267	268	269	269	270	270	271	271	272
GEOM80	365	383	392	395	1490	**382**	382	382	382	384	385	386	386	388	390
GEOM90	313	332	335	338.5	1810	334	334	333	333	**332**	333	332	332	335	337
GEOM100	378	404	412	416	2170	**404**	405	404	404	411	414	412	414	411	411
GEOM110	348	383	400	410	2510	378	378	**376**	377	383	383	381	382	387	389
GEOM120	343	402	412	419	2730	**397**	400	397	400	402	404	404	405	404	405
GEOM30a	182	209	238	238	380	**209**	211	210	211	212	213	212	212	212	212
GEOM40a	160	213	229	229	500	**214**	215	214	215	214	215	215	216	217	217
GEOM50a	199	318	335	345	1080	**315**	316	316	317	318	318	319	321	321	322
GEOM60a	290	358	369	373	1420	**356**	356	360	360	361	361	361	362	362	364
GEOM70a	425	469	487	487	1470	**478**	478	478	478	484	484	484	484	481	484
GEOM80a	241	379	388	396	1510	**360**	364	361	362	371	372	371	371	368	369
GEOM90a	285	377	398	405	1910	**377**	378	377	379	398	401	398	401	389	389
GEOM100a	302	459	462	471	2500	**437**	440	440	442	444	448	445	446	443	447
GEOM110a	385	494	523	523	3120	**490**	492	491	495	506	506	504	505	498	500
GEOM120a	514	556	571	578	3690	**549**	550	552	553	550	556	553	556	558	560
GEOM20b	39	44	45	45	30	**44**	44	44	44	44	44	44	44	44	44
GEOM30b	38	77	78	78	80	**77**	77	77	77	77	77	77	77	77	77
GEOM40b	74	74	79	86	140	**74**	74	74	74	74	74	74	74	74	74
GEOM50b	67	87	92	94	200	**84**	84	84	84	84	85	85	85	84	85
GEOM60b	79	116	123	132	300	117	119	**115**	118	120	120	118	119	119	119
GEOM70b	94	121	133	135	380	120	121	**119**	119	121	121	120	122	122	122
GEOM80b	110	141	148	149	490	**139**	139	139	139	140	140	139	140	141	142
GEOM90b	112	157	160	161	590	**147**	147	147	147	150	150	149	149	149	150
GEOM100b	133	170	173	179	690	**159**	160	161	161	162	163	164	164	162	165
GEOM110b	182	206	221	225.5	790	208	209	210	210	208	209	**206**	209	213	213
GEOM120b	172	199	206	219.5	910	**191**	196	193	197	195	198	197	198	201	201

Table 6: Results on the the GEOM instance set.

Instance	Lwb	OPT	Known heur.	G-DSATUR min	G-DSATUR med	Max time	OF-TS+R* min	OF-TS+R* med	OF-TS+R min	OF-TS+R med	OF-TS min	OF-TS med	OF-SW min	OF-SW med	FCNS min	FCNS med
P1	371	427	448	480	485	490	**427**	427	427	427	427	427	427	428	427	427
P2	428	427	476	458	464	712	**427**	427	427	448	428	459	427	427	427	427
P3	258	258	285	268	268	333	**258**	258	258	258	258	258	258	262	259	260
P4	254	253	269	260	266	225	**253**	254	253	253	253	253	254	257	255	255
P5	240	240	251	250	255	327	**240**	240	240	240	240	240	240	240	241	241
P6	179	180	231	195	199	279	183	184	185	185	185	186	**182**	183	193	194
P7	743	856	895	969	973	2439	**856**	856	871	877	860	871	874	877	857	859
P8	461	525	593	539	539	610	**525**	525	525	525	527	528	525	527	534	534
P9	1487	1714	1801	1938	1946	10381	**1715**	1719	1756	1758	1755	1759	1770	1775	1747	1751

Table 7: Numerical results on the Philadelphia instance set.

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