

DM811

Heuristics for Combinatorial Optimization

Compendium

Basic Concepts in Algorithmics

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Outline

1. Basic Concepts from Previous Courses

- Graphs

- Notation and runtime

- Machine model

- Pseudo-code

- Computational Complexity

- Analysis of Algorithms

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Graphs

Graphs are combinatorial structures useful to model several applications

Terminology:

- ▶ $G = (V, E)$, $E \subseteq V \times V$, vertices, edges, $n = |V|$, $m = |E|$, undirected graphs, subgraph, induced subgraph
- ▶ $e = (u, v) \in E$, e incident on u and v ; u, v adjacent, edge weight or cost
- ▶ particular cases often omitted: self-loops, multiple parallel edges
- ▶ degree, δ , Δ , outdegree, indegree
- ▶ path $P = \langle v_0, v_1, \dots, v_k \rangle$, $(v_0, v_1) \in E, \dots, (v_{k-1}, v_k) \in E$, $\langle v_0, v_1 \rangle$ has length 2, $\langle v_0, v_1, v_2, v_0 \rangle$ cycle, walk, path
- ▶ arcs, directed acyclic graph
- ▶ digraph strongly connected ($\forall u, v \exists (uv)$ -path), strongly connected components
- ▶ G is a tree ($\implies \exists$ path between any two vertices) $\iff G$ is connected and has $n - 1$ edges $\iff G$ is connected and contains no cycles.
- ▶ parent, children, sibling, height, depth

Representing Graphs

Operations:

- ▶ Access associated information (NodeArray, EdgeArray, Hashes)
- ▶ Navigation: access outgoing edges
- ▶ Edge queries: given u and v is there an edge?
- ▶ Update: add remove edges, vertices

Data Structures:

- ▶ Edge sequences
- ▶ Adjacency arrays
- ▶ Adjacency lists
- ▶ Adjacency matrix

How to choose?

- ▶ it depends on the graphs and the application
- ▶ if time and space not crucial no need to customize the structures
- ▶ use interfaces that make easy to change the data structure
- ▶ libraries offer different choices (Boost, lemon, Java `jds1.graph`)

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Motivations

Questions:

1. How good is the algorithm designed?
2. How hard, computationally, is a given a problem to solve using the most efficient algorithm for that problem?

1. Asymptotic notation, running time bounds
Approximation theory

2. Complexity theory

Asymptotic notation

$n \in \mathbf{N}$ instance size; $\pi \in \Pi_n$ instance π belonging to class Π_n

max time worst case $T(n) = \max\{T(\pi) : \pi \in \Pi_n\}$

average time average case $T(n) = \frac{1}{|\Pi_n|} \{\sum_{\pi} T(\pi) : \pi \in \Pi_n\}$

min time best case $T(n) = \min\{T(\pi) : \pi \in \Pi_n\}$

Growth rate or asymptotic analysis

$f(n)$ and $g(n)$ same growth rate if $c \leq \frac{f(n)}{g(n)} \leq d$ for n large

$f(n)$ grows faster than $g(n)$ if $f(n) \geq c \cdot g(n)$ for all c and n large

big O $O(f) = \{g(n) : \exists c > 0, \forall n > n_0 : g(n) \leq c \cdot f(n)\}$

big omega $\Omega(f) = \{g(n) : \exists c > 0, \forall n > n_0 : g(n) \geq c \cdot f(n)\}$

theta $\Theta(f) = O(f) \cap \Omega(f)$

(little o $o(f) = \{g : g \text{ grows strictly more slowly than } f\}$)

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Machine model

For asymptotic analysis we use RAM machine

- ▶ sequential, single processor unit
- ▶ all memory access take same amount of time

It is an **abstraction** from machine architecture: it ignores caches, memories hierarchies, parallel processing (SIMD, multi-threading), etc.

Total execution of a program = total number of instructions executed

We are not interested in constant and lower order terms

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Pseudo-code

We express algorithms in natural language and mathematical notation, and in **pseudo-code**, which is an abstraction from programming languages C, C++, Java, etc.

(In implementation you can choose your favorite language)

Programs must be correct.

Certifying algorithm: computes a certificate for a post condition (without increasing asymptotic running time)

Good Algorithms

We say that an algorithm A is

Efficient = good = polynomial time = polytime
iff
there exists polynomial $p(n)$ such that $T(A) = O(p(n))$

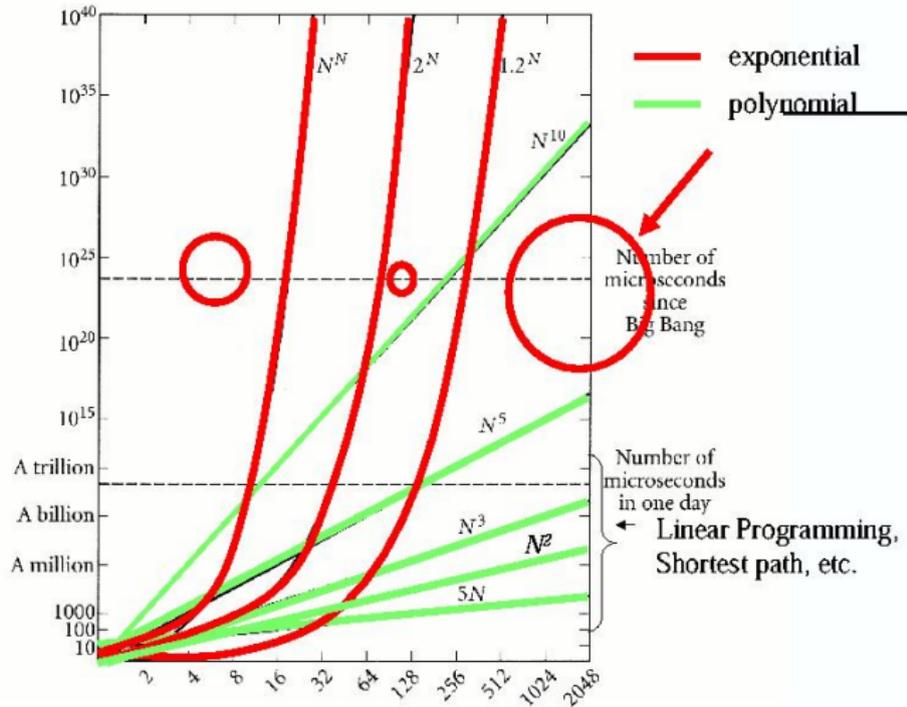
There are problems for which no polytime algorithm is known.
This course is about those problems.

Complexity theory classifies problems

Polynomial vs. exponential growth

(Harel 2000)

SATISFIABILITY



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Complexity Classes

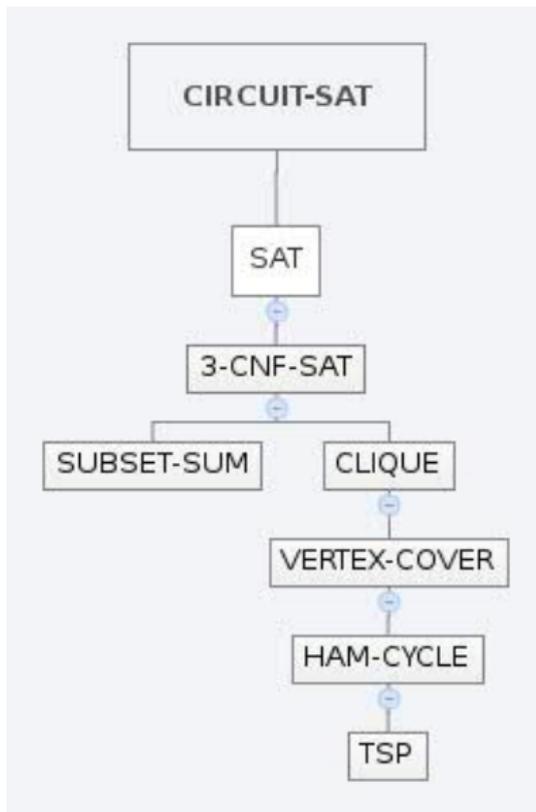
[Garey and Johnson, 1979]

Consider a Decision Search Problem Π :

- ▶ Π is in P if \exists algorithm \mathcal{A} that finds a solution in polynomial time.
- ▶ Π is in NP if \exists verification algorithm \mathcal{A} that verifies whether a binary certificate is a solution to the problem in polynomial time.
- ▶ a search problem Π' is (polynomially) reducible to Π ($\Pi' \rightarrow \Pi$) if there exists an algorithm \mathcal{A} that solves Π' by using a hypothetical subroutine \mathcal{S} for Π and except for \mathcal{S} everything runs in polynomial time.
- ▶ Π is NP -complete if
 1. it is in NP
 2. there exists some NP -complete problem Π' that reduces to Π ($\Pi' \rightarrow \Pi$)
- ▶ If Π satisfies property 2, but not necessarily property 1, we say that it is NP -hard:

- ▶ ***NP***: Class of problems that can be solved in polynomial time by a non-deterministic machine.
Note: non-deterministic \neq randomized;
non-deterministic machines are idealized models of computation that have the ability to make perfect guesses.
- ▶ ***NP-complete***: Among the most difficult problems in *NP*; believed to have at least exponential time-complexity for any realistic machine or programming model.
- ▶ ***NP-hard***: At least as difficult as the most difficult problems in *NP*, but possibly not in *NP-complete* (*i.e.*, may have even worse complexity than *NP-complete* problems).

NP-Completeness Proofs



Many combinatorial problems are hard
but some problems can be solved efficiently

- ▶ Longest path problem is *NP*-hard
but not shortest path problem
- ▶ SAT for 3-CNF is *NP*-complete
but not 2-CNF (linear time algorithm)
- ▶ Hamiltonian path is *NP*-complete
but not the Eulerian path problem
- ▶ TSP on Euclidean instances is *NP*-hard
but not where all vertices lie on a circle.

An online compendium on the computational complexity
of optimization problems:

<http://www.nada.kth.se/~viggo/problemelist/compendium.html>

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Theoretical Analysis

- ▶ Worst-case analysis (runtime and quality):
worst performance of algorithms over all possible instances
- ▶ Probabilistic analysis (runtime):
average-case performance over a given probability distribution of instances
- ▶ Average-case (runtime):
overall possible instances for randomized algorithms
- ▶ Asymptotic convergence results (quality)
- ▶ Approximation of optimal solutions:
sometimes possible in polynomial time (e.g., Euclidean TSP),
but in many cases also intractable (e.g., general TSP);
- ▶ Domination
- ▶ Algorithm invariance

Approximation Algorithms

Definition: Approximation Algorithms

An algorithm \mathcal{A} is said to be a δ -approximation algorithm if it runs in polynomial time and for every problem instance π with optimal solution value $OPT(\pi)$

$$\text{minimization: } \frac{\mathcal{A}(\pi)}{OPT(\pi)} \leq \delta \quad \delta \geq 1$$

$$\text{maximization: } \frac{\mathcal{A}(\pi)}{OPT(\pi)} \geq \delta \quad \delta \leq 1$$

(δ is called *worst case bound*, *worst case performance*, *approximation factor*, *approximation ratio*, *performance bound*, *performance ratio*, *error ratio*)

Approximation Algorithms

Definition: Polynomial approximation scheme

A family of approximation algorithms for a problem Π , $\{\mathcal{A}_\epsilon\}_\epsilon$, is called a **polynomial approximation scheme** (PAS), if algorithm \mathcal{A}_ϵ is a $(1 + \epsilon)$ -approximation algorithm and its running time is polynomial in the size of the input for each fixed ϵ

Definition: Fully polynomial approximation scheme

A family of approximation algorithms for a problem Π , $\{\mathcal{A}_\epsilon\}_\epsilon$, is called a **fully polynomial approximation scheme** (FPAS), if algorithm \mathcal{A}_ϵ is a $(1 + \epsilon)$ -approximation algorithm and its running time is polynomial in the size of the input and $1/\epsilon$

Useful Graph Algorithms

- ▶ Breadth first, depth first search, traversal
- ▶ Transitive closure
- ▶ Topological sorting
- ▶ (Strongly) connected components
- ▶ Shortest Path
- ▶ Minimum Spanning Tree
- ▶ Matching

Randomized Algorithms

Most often algorithms are randomized. Why?

- ▶ possibility of gains from re-runs
- ▶ adversary argument
- ▶ structural simplicity for comparable average performance,
- ▶ speed up,
- ▶ avoiding loops in the search
- ▶ ...

Randomized Algorithms

Definition: Randomized Algorithms

Their **running time** depends on the **random choices** made.
Hence, the running time is a **random variable**.

Las Vegas algorithm: it always gives the correct result but in random runtime (with finite expected value).

Monte Carlo algorithm: the result is not guaranteed correct. Typically halted due to bounded resources.

Randomized Heuristics

In the case of **randomized optimization heuristics** both **solution quality** and **runtime** are random variables.

We distinguish:

- ▶ **single-pass heuristics** (denoted \mathcal{A}^1): have an embedded termination, for example, upon reaching a certain state (generalized optimization Las Vegas algorithms [B2])
- ▶ **asymptotic heuristics** (denoted \mathcal{A}^∞): do not have an embedded termination and they might improve their solution asymptotically (both probabilistically approximately complete and essentially incomplete [B2])