DM63
HEURISTICS FOR
COMBINATORIAL OPTIMIZATION

Lecture 2

Combinatorial Problems and Local Search

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Outline

1. Example Problems
    - The Graph Colouring
    - The Satisfiability Problem

2. Local Search Methods

Example Problems

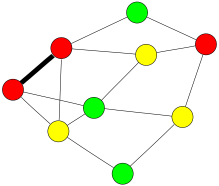
- They are chosen because conceptually concise, intended to illustrate the development, analysis and presentation of algorithms

- Although real-world problems tend to have much more complex formulations these problems capture their essence

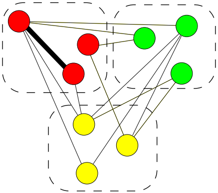
The Vertex Coloring Problem

Given: A graph G and a set of colors Γ.

A *proper coloring* is an assignment of one color to each vertex of the graph such that adjacent vertices receive different colors.



Assignment



Partition

Decision version (k-coloring)

Task: Find a proper coloring of G which uses at most k colors.

Optimization version (chromatic number)

Task: Find a proper coloring of G which uses the minimal number of colors.

The SAT Problem

General SAT Problem (search variant):

- *Given:* Formula F in propositional logic
- *Objective:* Find an assignment of truth values to variables in F that renders F true, or decide that no such assignment exists.

SAT: A simple example

- *Given:* Formula $F := (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
- *Objective:* Find an assignment of truth values to variables x_1, x_2 that renders F true, or decide that no such assignment exists.

MAX-SAT

Which is the maximal number of clauses satisfiable in a propositional logic formula F?

Definition:

- Formula in propositional logic: well-formed string that may contain
  - propositional variables $x_1, x_2, \dots, x_n$;
  - truth values $\top$ ('true'), $\bot$ ('false');
  - operators $\neg$ ('not'), $\wedge$ ('and'), $\vee$ ('or');
  - parentheses (for operator nesting).

- Model (or satisfying assignment) of a formula F: Assignment of truth values to the variables in F under which F becomes true (under the usual interpretation of the logical operators)

- Formula F is satisfiable iff there exists at least one model of F, unsatisfiable otherwise.

Definition:

- A formula is in conjunctive normal form (CNF) iff it is of the form

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{k(i)} l_{ij} = (l_{11} \vee \ldots \vee l_{1k(1)}) \ldots \wedge (l_{m1} \vee \ldots \vee l_{mk(m)})$$

  where each literal l_{ij} is a propositional variable or its negation. The disjunctions $(l_{i1} \vee \ldots \vee l_{ik(i)})$ are called clauses.

- A formula is in k-CNF iff it is in CNF and all clauses contain exactly k literals (*i.e.*, for all i, $k(i) = k$).

- In many cases, the restriction of SAT to CNF formulae is considered.
- The restriction of SAT to k-CNF formulae is called k-SAT.
- For every propositional formula, there is an equivalent formula in 3-CNF.

Example:

$$
\begin{aligned}
F := \quad & \wedge (\neg x_2 \vee x_1) \\
& \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\
& \wedge (x_1 \vee x_2) \\
& \wedge (\neg x_4 \vee x_3) \\
& \wedge (\neg x_5 \vee x_3)
\end{aligned}
$$

- F is in CNF.
- Is F satisfiable?
  Yes, *e.g.*, $x_1 := x_2 := \top$, $x_3 := x_4 := x_5 := \bot$ is a model of F.

Many combinatorial problems are hard
but some problems can be solved efficiently

- Longest path problem is $\mathcal{NP}$-hard
  but not shortest path problem
- SAT for 3-CNF is $\mathcal{NP}$-complete
  but not 2-CNF (linear time algorithm)
- TSP is $\mathcal{NP}$-hard, the associated decision problem (for any solution quality) is $\mathcal{NP}$-complete
  but not the Euler tour problem
- TSP on Euclidean instances is $\mathcal{NP}$-hard
  but not where all vertices lie on a circle.
- The Graph Coloring Problem is $\mathcal{NP}$-complete
  but not on interval graphs
- Many scheduling and timetabling problems are $\mathcal{NP}$-hard

Incomplete Search Paradigms

Construction Heuristics

An important class of Construction Heuristics are greedy algorithms.

- Strategy: always make the choice which is the best at the moment.
- They are not generally guaranteed to find globally optimal solutions (but sometimes they do: Minimum Spanning Tree, Single Source Shortest Path, etc.)

Possible extension of construction heuristics: beam search/pilot method

- maintains a set B of bw (beam width) partial candidate solutions
- at each level extend fw (filter width) candidate solutions and rank
- complete candidate solutions obtained by B are maintained in B_f

Incomplete Search Paradigms

Perturbative search

- search space = complete candidate solutions
- search step = modification of one or more solution components
- iteratively generate and evaluate candidate solutions
  - decision problems: evaluation = test if solution
  - optimization problems: evaluation = check objective function value
- evaluating candidate solutions is typically computationally much cheaper than finding (optimal) solutions

Iterative Improvement (II):

While s has better neighbors:

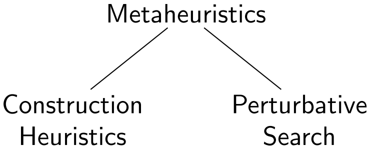
  choose a neighbor s' of s
    such that $g(s') < g(s)$
  $s := s'$

Local Search Methods

In order to obtain very high quality solutions
it is often successful to combine the two heuristic paradigms

Meta-heuristics

General guidance criteria over lower level heuristics.



Informed search based on *local* or incomplete knowledge as opposed to systematic search ⇒ **Local Search Methods**

Stochastic Local Search (SLS) algorithms use *randomized choices* in generating and modifying candidate solutions. They are introduced whenever it is unknown which deterministic rules are profitable for all the instances of interest.

Example: Uninformed random walk for k-coloring

procedure *URW-for-k-col*(*G*,*k*, *maxSteps*)
  input: *graph G, integer k, integer maxSteps*
  output: *feasible coloring of G* **or** $\emptyset$

  choose assignment φ of k colors to all vertices in G
    uniformly at random;
  steps $:= 0$;
  while not((φ is a proper coloring) **and** (*steps* $<$ *maxSteps*)) **do**
    randomly select vertex v in G;
    change value of $\varphi(v)$;
    steps $:=$ *steps*$+1$;
  end
  if φ is feasible**then**
    return φ
  else
    return $\emptyset$
  end
end *URW-for-k-col*

Systematic search is often better suited when ...

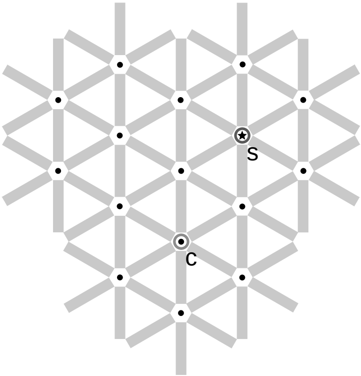
- proofs of insolubility or optimality are required;
- time constraints are not critical;
- problem-specific knowledge can be exploited.

Local search is often better suited when ...

- non linear constraints and non linear objective function;
- reasonably good solutions are required within a short time;
- problem-specific knowledge is rather limited.

Complementarity:
Local and systematic search can be fruitfully combined, *e.g.*, by using local search for finding solutions whose optimality is proved using systematic search.

Local search — global view



- vertices: candidate solutions (search positions)
- edges: connect neighboring positions
- s: (optimal) solution
- c: current search position

Definition: Local Search Algorithm (1)

For given problem instance π:

- search space $S(\pi)$
  (*e.g.*, for GCP: assignment of colors to each vertex)
- solution set $S'(\pi) \subseteq S(\pi)$
  (*e.g.*, for GCP: feasible assignment)
- neighborhood relation $N(\pi) \subseteq S(\pi) \times S(\pi)$
  (*e.g.*, for GCP: neighboring assignments differ
  in the color at one vertex)

Definition: Local Search Algorithm (2)

- set of memory states $M(\pi)$
  (may consist of a single state, for LS algorithms that do not use memory)
- initialization function *init* $: \emptyset \mapsto \mathcal{D}(S(\pi) \times M(\pi))$
  (specifies probability distribution over initial search positions and memory states)
- step function *step* $: S(\pi) \times M(\pi) \mapsto \mathcal{D}(S(\pi) \times M(\pi))$
  (maps each search position and memory state onto probability distribution over subsequent, neighboring search positions and memory states)
- termination predicate *terminate* $: S(\pi) \times M(\pi) \mapsto \mathcal{D}(\{\top, \bot\})$
  (determines the termination probability for each search position and memory state)

```
procedure LS-Decision(π)
    input: problem instance π ∈ Π
    output: solution s ∈ S′(π) or ∅
    (s, m) := init(π);

    while not terminate(π, s, m) do
        (s, m) := step(π, s, m);
    end



    if s ∈ S′(π) then
        return s
    else
        return ∅
    end
end LS-Decision
```

procedure *LS-Minimization*(π')
  input: *problem instance* $\pi' \in \Pi'$
  output: *solution* $s \in S'(\pi')$ **or** $\emptyset$
  $(s, m) := init(\pi')$;
  $\hat{s} := s$;
  while not $terminate(\pi', s, m)$ **do**
    $(s, m) := step(\pi', s, m)$;
    if $f(\pi', s) < f(\pi', \hat{s})$ **then**
      $\hat{s} := s$;
    end
  end
  if $\hat{s} \in S'(\pi')$ **then**
    return $\hat{s}$
  else
    return $\emptyset$
  end
end *LS-Minimization*