DM812
METAHEURISTICS

Lecture 13

Heuristic Methods for Continuous Optimization

Marco Chiarandini

Department of Mathematics and Computer Science
University of Southern Denmark, Odense, Denmark
<marco@imada.sdu.dk>

Outline

1. An Application Example in Econometrics

2. Heuristics for Continuos Optimization

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1. An Application Example in Econometrics

2. Heuristics for Continuos Optimization

Capital Asset Pricing Model

Tool for pricing an individual asset i

Individual security's reward-to-risk ratio $= \beta_i \cdot$ Market's securities reward-to-risk ratio

$$\big(E(R_i) - R_f\big) = \beta_i \cdot \big(E(R_m) - R_f\big)$$

β_i sensitivity of the asset returns to market returns

Under normality assumption and least squares method:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

Alternatively:

$$R_{it} - R_{ft} = \beta_0 + \beta_1 \cdot (R_{mt} - R_{ft})$$

Use more robust techniques than least squares to determine β_0 and β_1
[Winker, Lyra, Sharpe, 2008]

Least Median of Squares

$$Y_t = \beta_0 + \beta_1 X_t + \epsilon_t$$

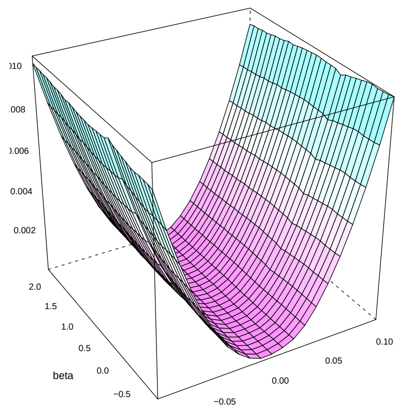
$$\epsilon_t^2 = \left(Y_t - \beta_0 - \beta_1 X_t\right)^2$$

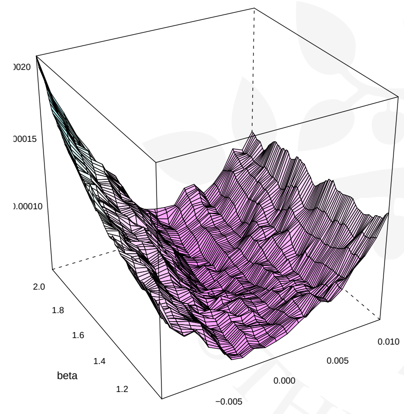
least squares method:

$$\min \sum_{t=1}^{n} \epsilon_t^2$$

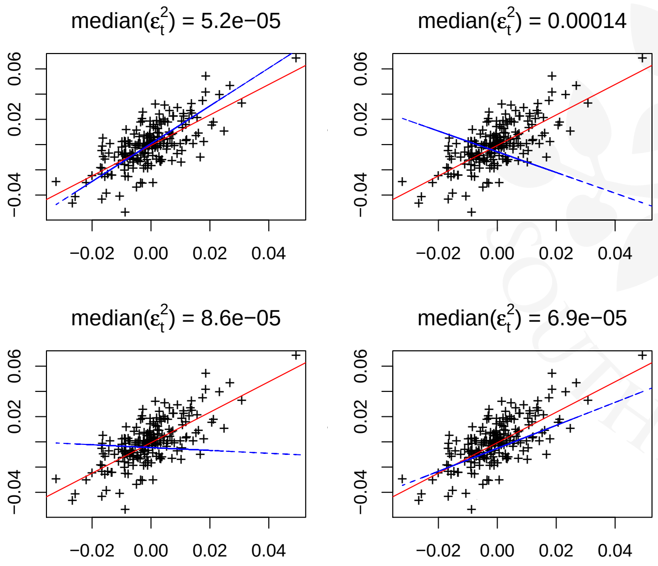
least median of squares method:

$$\min \left\{ \text{median} \left[\epsilon_t^2\right] \right\}$$





Four solutions corresponding to four different local optima
(red line: least squares; blue line: least median of squares)



Outline

1. An Application Example in Econometrics
2. Heuristics for Continuos Optimization

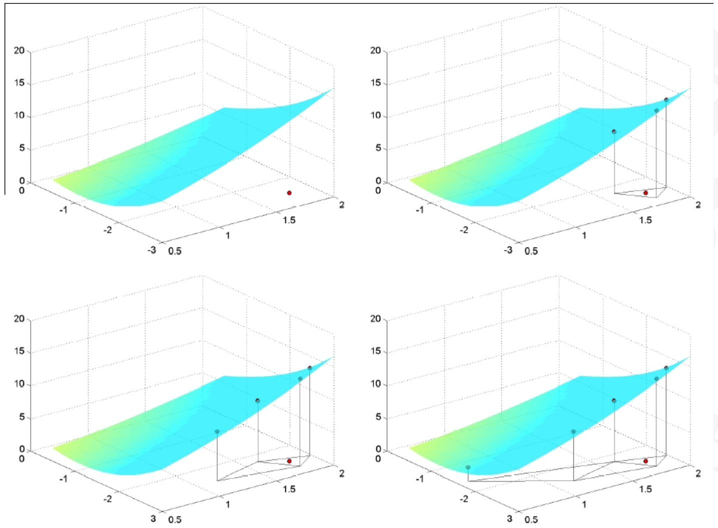
Optimization Heuristics

- Nelder-Mead
- Simulated Annealing
- Differential Evolution
- Particle Swarm Optimization
- Genetic Algorithm
- Ant Colony Optimization

Nelder-Mead

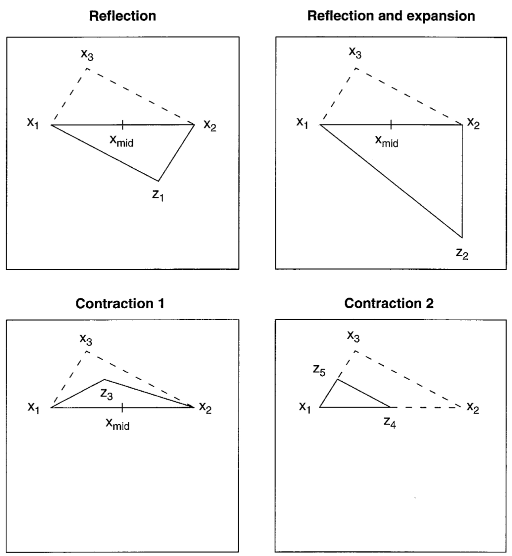
An Example
Heuristics for Continuos Optimization

Simplex based method [Spendley et al. (1962)]



Nelder-Mead (cont.)

An Example
Heuristics for Continuos Optimization



Nelder-Mead (cont.)

An Example
Heuristics for Continuos Optimization

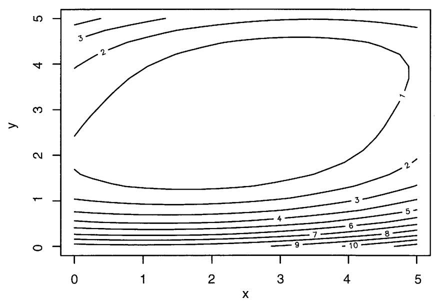
Nelder-Mead simplex method [Nelder and Mead, 1965]:

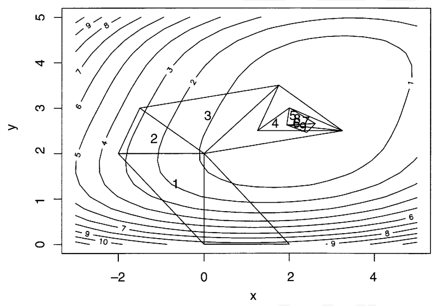
Algorithm 5 Simplex search.

```
Construct vertices x^(1),...,x^(n+1) of starting simplex
repeat
    Rename vertices such that f(x^(1)) ≤ ··· ≤ f(x^(n+1))
    if f(x^R) < f(x^(1)) then
        if f(x^E) < f(x^R) then x* = x^E else x* = x^R
    else
        if f(x^R) < f(x^(n)) then
            x* = x^R
        else
            if f(x^R) < f(x^(n+1)) then
                if f(x^O) < f(x^(n+1)) then x* = x^O else shrink
            else
                if f(x^I) < f(x^(n+1)) then x* = x^I else shrink
            end if
        end if
    end if
    if not shrink then x^(n+1) = x* (Replace worst vertex by x*)
until stopping criteria verified
```

Nelder-Mead (cont.)

Example:





Simulated Annealing

Simulated Annealing (SA):
determine initial candidate solution s
set initial temperature $T = T_0$
while termination condition is not satisfied **do**
  while keep T constant, that is, T_{max} iterations not elapsed **do**
    probabilistically choose a neighbor s' of s
      using proposal mechanism
    accept s' as new search position with probability:

$$p(T, s, s') := \begin{cases} 1 & \text{if } f(s') \leq f(s) \\ \exp \frac{f(s)-f(s')}{T} & \text{otherwise} \end{cases}$$

    update T according to annealing schedule

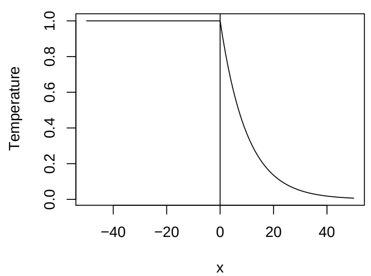
Proposal mechanism
The next candidate point is generated from a Gaussian Markov kernel with scale proportional to the actual temperature.

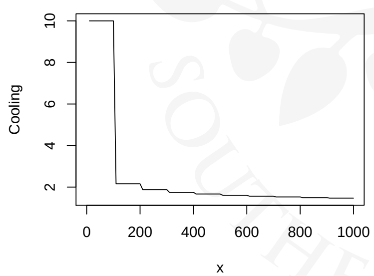
Simulated Annealing

Annealing schedule

- logarithmic cooling schedule [Belisle (1992)]

$$T = \frac{T_0}{\ln(\lfloor \frac{i-1}{I_{max}} \rfloor I_{max} + e)}$$



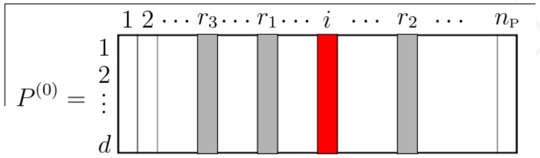


- threshold accepting [Dueck and Scheuer (1990)]
  accept if $\Delta < \tau$

Differential Evolution

Differential Evolution (DE)
determine initial population P
while termination criterion is not satisfied **do**
  for each solution x of P **do**
    generate solution u from three solutions of P by mutation
    generate solution v from u by recombination with solution x
    select between x and v solutions

Differential Evolution (cont.)



- Solution representation: $x = (x_1, x_2, \ldots, x_p)$
- Mutation:

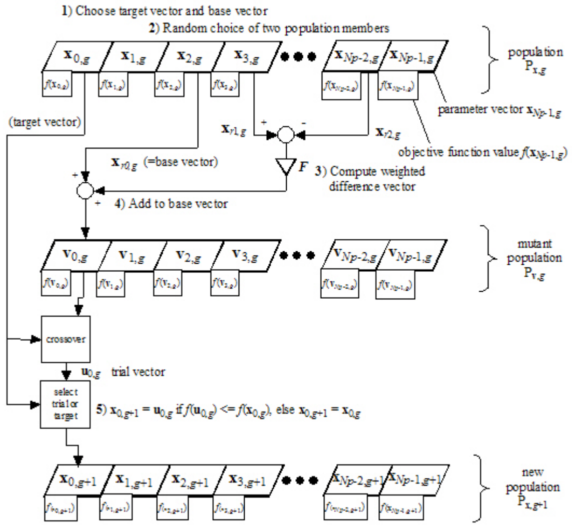
$$\mathbf{u} = \mathbf{r}_1 + F \cdot (\mathbf{r}_2 - \mathbf{r}_3) \qquad F \in [0, 2] \text{ and } (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \in P$$

- Recombination:

$$v_j = \begin{cases} u_j & \text{if } p < CR \text{ or } j = r \\ x_j & \text{otherwise} \end{cases} \qquad j = 1, 2, \ldots, p$$

- Selection: replace x with v if $f(v)$ is better

Differential Evolution (cont.)



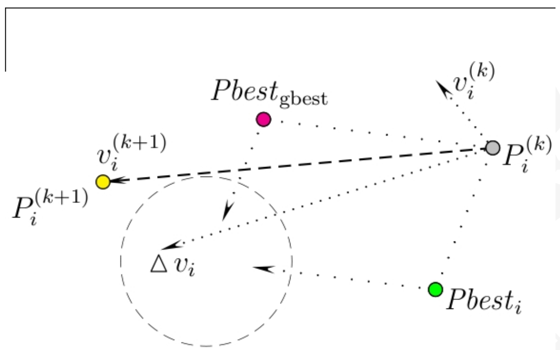
[http://www.icsi.berkeley.edu/~storn/code.html
K. Price and R. Storn, 1995]

Particle Swarm Optimization

Particle Swarm Optimization

Initialize parameters n_P, n_G and c
Initialize particles $P_i^{(0)}$ and velocity $v_i^{(0)}$, $i = 1, \ldots, n_P$
Evaluate objective function $F_i = f(P_i^{(0)})$, $i = 1, \ldots, n_P$
$Pbest = P^{(0)}$, $Fbest = F$, $Gbest = \min_i(F_i)$, $gbest = \mathrm{argmin}_i(F_i)$
for $k = 1$ to n_G **do**
  for $i = 1$ to n_P **do**
    $\triangle v_i = c\,u\,(Pbest_i - P_i^{(k-1)}) + c\,u\,(Pbest_{\mathrm{gbest}} - P_i^{(k-1)})$
    $v_i^{(k)} = v^{(k-1)} + \triangle v_i$
    $P_i^{(k)} = P_i^{(k-1)} + v_i^{(k)}$
  end for
  Evaluate objective function $F_i = f(P_i^{(k)})$, $i = 1, \ldots, n_P$
  for $i = 1$ to n_P **do**
    if $F_i < Fbest_i$ **then** $Pbest_i = P_i^{(k)}$ and $Fbest_i = F_i$
    if $F_i < Gbest$ **then** $Gbest = F_i$ and $gbest = i$
  end for
end for

Particle Swarm Optimization

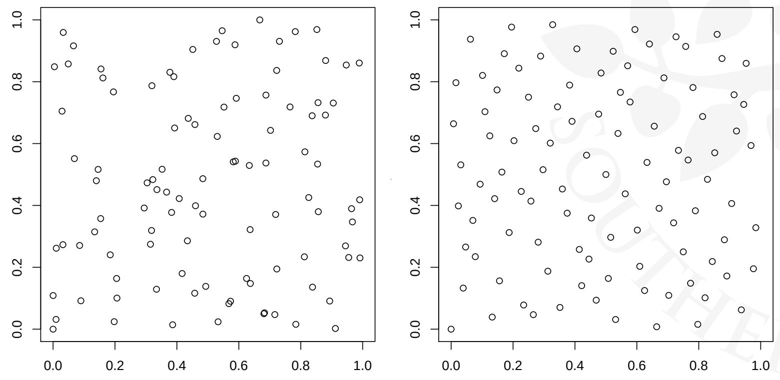


Generation of Initial Solutions

Point generators:

Left: Uniform random distribution (pseudo random number generator)

Right: Quasi-Monte Carlo method: low discrepancy sequence generator
[Bratley, Fox and Niederreiter, 1994]



(for other methods see *spatial point process from spatial statistics*)