Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Advances in Mathematics 187 (2004) 257–319
ADVANCES IN Mathematics
http://www.elsevier.com/locate/aim

$\mathscr{COL}_p$ spaces—the local structure of non-commutative L_p spaces

M. Junge,[a,*,1] N.J. Nielsen,[b,2] Zhong-Jin Ruan,[a,1] and Q. Xu[c]

[a] *Department of Mathematics, University of Illinois, 273 Altgeld Hall MC 382 1409, West Green Street, Urbana-Champaign, IL 61801, USA*

[b] *Department of Mathematics & Computer Science, The University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark*

[c] *Laboratoire de Mathématique, Université de Franche-Comté, Route de Gray, 25030 Besançon, Cedex, France*

Received 9 October 2001; accepted 14 August 2003

Communicated by Virgil Voiculescu

Abstract

Developing the theory of $\mathscr{COL}_p$ spaces (a variation of the non-commutative analogue of $\mathscr{L}_p$ spaces), we provide new tools to investigate the local structure of non-commutative L_p spaces. Under mild assumptions on the underlying von Neumann algebras, non-commutative L_p spaces with Grothendieck's approximation property behave locally like the space of matrices equipped with the p-norm (of the sequences of their singular values). As applications, we obtain a basis for non-commutative L_p spaces associated with hyperfinite von Neumann algebras with separable predual von Neumann algebras generated by free groups, and obtain a basis for separable nuclear C^*-algebras.

MSC: 46L52; 46L07

Keywords: Non-commutative L_p spaces; (cb-)basis; Non-commutative $\mathscr{L}_p$ spaces

*Corresponding author.

E-mail addresses: junge@math.uiuc.edu (M. Junge), njn@imada.sdu.dk (N.J. Nielsen), ruan@math.uiuc.edu (Z.-J. Ruan), qx@math.univ-fcomte.fr (Q. Xu).

[1] Junge and Ruan were partially supported by the National Science Foundation, DMS 00-88928 and DMS 98-77157.

[2] Supported by the Danish Natural Science Research Council, Grant 9801867.

0001-8708/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2003.08.010

0. Introduction

Non-commutative integration theory goes back to the work of Murray and von Neumann and has been investigated in the context of von Neumann algebras by Dixmier [D1], Segal [Se], Kunze [Ku], Nelson [Ne], Connes [C3], Haagerup [Ha2], Kosaki [Ko] and many other researchers. Following the philosophy of quantization, non-commutative L_p spaces could be considered as non-commutative function spaces. In particular, the classical Banach spaces of trace class operators, Hilbert–Schmidt operators and more generally Schatten p-classes share many properties with their commutative counterparts, the classical ℓ_p spaces (see [A1,A2,Fa,GK,TJ]). Since these spaces are not compatible with the usual lattice structure of classical function spaces (except for $p=2$ see [GL,P1]), their local structure has not been investigated as thoroughly as for classical function spaces. The main intention of this paper is to show that non-commutative L_p spaces with the bounded approximation property (BAP) have very nice local properties, for instance, they can be paved out by copies of finite-dimensional non-commutative L_p spaces. This can be achieved under mild assumptions on the underlying von Neumann algebra by combining concepts from the local theory of Banach spaces with more recent tools from the theory of operator spaces. In contrast to the classical theory, these more abstract techniques provide appropriate tools to prove the existence of bases for some important spaces like nuclear (in particular type I) C^*-algebras, preduals of hyperfinite von Neumann algebras, and non-commutative L_p spaces associated with hyperfinite von Neumann algebras or the von Neumann algebra generated by the left regular representation of a countable free group.

Let us first recall the classical notion of $\mathscr{L}_p$ spaces. Following Lindenstrauss and Pełczyński [LP] a Banach space X is called an *$\mathscr{L}_{p,\lambda}$ space* if every finite-dimensional subspace $E\subset X$ is contained in a finite-dimensional subspace $E\subset F\subset X$ such that for $n=\dim(F)$ the Banach–Mazur distance satisfies

$$d(F,\ell_p^n)\leqslant\lambda. \tag{0.1}$$

If this is true for some λ, X is called an *$\mathscr{L}_p$ space*. If this is true for all $\lambda>1$, then X is isometrically isomorphic to $L_p(\Omega,\Sigma,\mu)$ for some measure space (Ω,Σ,μ) and vice versa. For $1<p<\infty$ every separable $\mathscr{L}_p$ space is isomorphic to a complemented subspace of $L_p[0,1]$ and therefore inherits the bounded approximation property. (The absence of the approximation property for general non-commutative L_p spaces is a substantial but interesting drawback in the non-commutative setting.) The 'paving' definition (0.1) is not very practical for showing that the dual of an $\mathscr{L}_p$ space is an $\mathscr{L}_{p'}$ space ($p'=\frac{p}{p-1}$ the conjugate index). However, using the fundamental Kadec–Pełczyński dichotomy and a 'cut and paste' technique, Lindenstrauss and Rosenthal [LR] managed to prove that the dual of an $\mathscr{L}_p$ space is an $\mathscr{L}_{p'}$ space and that the copies of ℓ_p^n in the definition of $\mathscr{L}_p$ spaces may be assumed to be uniformly complemented. In order to underline the different notions in the non-commutative setting, we might call spaces satisfying the complemented condition $\mathscr{C}\mathscr{L}_p$ spaces and note by the remarks above that in the commutative setting $\mathscr{L}_p$ spaces are indeed $\mathscr{C}\mathscr{L}_p$ spaces.

In the non-commutative setting a Kadec–Pełczyński dichotomy (see [KP]) is not available at the time of this writing. This forces us to introduce the class of $\mathscr{C}\mathscr{O}\mathscr{L}_p$ spaces, the non-commutative analogue of $\mathscr{C}\mathscr{L}_p$-space. In contrast to the $\mathscr{O}\mathscr{L}_p$-spaces defined by Effros and Ruan [ER1], we assume in addition that the finite-dimensional copies of non-commutative L_p spaces are uniformly (completely) complemented. This class of $\mathscr{C}\mathscr{O}\mathscr{L}_p$ spaces (for a precise definition see Section 2) seems to be the right substitute for the class of $\mathscr{L}_p$ spaces in Banach space theory. We refer to the end of Section 2 for a discussion of these two notions. In the commutative theory Johnson et al. [JRZ] showed that a separable $\mathscr{L}_p$ space admits a basis. Refining their techniques Nielsen and Wojtaszczyk showed that this basis locally looks like the basis of ℓ_p^n. We use this approach as a guideline to discover the local structure of a separable $\mathscr{C}\mathscr{O}\mathscr{L}_p$ space and construct (very nice operator space) bases therein. Let us note that due to the work of Bourgain [Bo], Bourgain et al. [BRS] many non-isomorphic $\mathscr{L}_p$ spaces are known, and thus many of them are not isomorphic to standard examples $L_p[0,1]$, ℓ_p or $\ell_p \oplus \ell_2$. Therefore $\mathscr{L}_p$ spaces have a very rich global structure.

The right framework for the investigation of the local structure of non-commutative L_p spaces is the category of operator spaces. We will now indicate some elementary operator space notations and in particular the notion of $\mathscr{O}\mathscr{L}_p$-spaces, introduced by Effros and Ruan [ER1]. An *operator space* X is a norm closed subspace of some $B(H)$ equipped with the distinguished operator space matrix norm inherited from $M_n(X) \subset B(\ell_2^n(H))$. An abstract matrix norm characterization of operator spaces was given by Ruan (see e.g. [ER2]). The morphisms in the category of operator spaces are completely bounded maps. Given operator spaces X and Y, a linear map $T : X \to Y$ is *completely bounded* if the corresponding linear maps $T_n : M_n(X) \to M_n(Y)$ defined by $T_n([x_{ij}]) = [T(x_{ij})]$ are uniformly bounded, i.e.,

$$||T||_{cb} = \sup_{n \in \mathbb{N}} ||T_n|| < \infty.$$

A map T is a *complete contraction* (respectively, a *complete isometry*, or a *complete quotient*) if $||T||_{cb} \leqslant 1$ (respectively, if each T_n is an isometry, or a quotient map). A map T is said to be a *complete isomorphism* if it is a completely bounded linear isomorphism with a completely bounded inverse. In this case, we let

$$d_{cb}(X,Y) = \inf\{||T||_{cb}||T^{-1}||_{cb} \colon T \text{ a complete isomorphism from } X \text{ onto } Y\}$$

denote the *completely bounded Banach–Mazur distance* (in short *cb*-distance) of X and Y (see [P4]).

Variations of Grothendieck's approximation property inspired crucial developments in operator algebras and operator spaces. An operator space $X \subset B(H)$ has the *operator space approximation property*, in short OAP, if there exists a net of finite rank maps (T_i) such that $id_{\mathscr{K}} \otimes T_i$ converges in the point-norm topology to the identity on $\mathscr{K}\otimes_{\min}X\subset B(\ell_2\otimes H)$, where $\mathscr{K}$ denotes the space of compact operators on ℓ_2. An operator space has the *completely bounded approximation property* (in short CBAP) if there exists a net (T_i) of finite rank maps converging in the point-norm topology to the identity on X and $\sup_i \|T_i\|_{cb}<\infty$. We say that an operator space X has a *cb-basis* if, X has a basis (x_n) and the natural projection maps

$$P_n\left(\sum_{k=1}^{\infty}\alpha_k x_k\right)=\sum_{k=1}^{n}\alpha_k x_k$$

satisfy $K=\sup_n \|P_n\|_{cb}<\infty$. In this case we call (x_n) a *K-cb-basis*.

For non-commutative L_p spaces Pisier [P5] introduced a very natural operator space structure by interpolation (see [BL] for interpolation theory). Indeed, it is well-known that the Schatten p-classes S_p can be obtained by complex interpolation

$$S_p=[\mathscr{K},\mathscr{T}]_{\frac{1}{p}}.$$

Here $\mathscr{T}=S_1$ denotes the space of trace class operators and $\mathscr{K}=S_\infty$ the space of compact operators. Moreover, the natural (operator space structure preserving) duality between $x=[x_{ij}]\in\mathscr{K}$ and $y=[y_{ij}]\in\mathscr{T}$ is given by

$$\langle x,y\rangle=\sum_{ij}x_{ij}y_{ij}=tr(xy^t).$$

Pisier [P5] proved that

$$M_n(S_p)=[M_n(\mathscr{K}),M_n(\mathscr{T})]_{\frac{1}{p}}$$

define matrix norms on S_p which satisfy Ruan's abstract characterization for operator spaces. Therefore, there is an isometric embedding $j_p:S_p\to B(\ell_2)$ inducing these matrix norms and this is nowadays called *the natural operator space structure of* S_p. We refer to [P5] for many nice features. Similarly, we may obtain a natural operator space structure on $L_p(A)$ for every finite-dimensional C^*-algebra A.

Let us recall the operator space analogue of $\mathscr{L}_p$ spaces. An operator space X is called an operator $\mathscr{L}_p$ space (in short $\mathcal{OL}_{p,\lambda}$ space) if X can be paved out by copies of finite-dimensional L_p spaces, where the cb-distance is uniformly controlled by λ. An operator space X is called an $\mathcal{OL}_p$ if it is an $\mathcal{OL}_{p,\lambda}$ for some $\lambda>1$. In this case, we use the parameter $\mathcal{OL}_p(X)=\inf\lambda$, where the infimum is taken over all λ's above. For a precise definition see Section 2.

During the last few years, $\mathcal{OL}_1$ spaces have been intensively studied in [ER1,JOR,NO]. In particular, it was proved in [ER1] that the predual N_* of a von Neumann algebra N is an $\mathcal{OL}_1$ space if and only if N is hyperfinite. Moreover, a separable operator space X is an $\mathcal{OL}_1$ space with $\mathcal{OL}_1(X)=1$ if and only if it is the operator predual of a hyperfinite von Neumann algebra (see [NO]).

Concerning $\mathcal{OL}_\infty$ space, we recall that by Szankowski's result (see [Sz1]) the space $B(H)$ does not have Grothendieck's approximation property and hence is not an

$\mathcal{OL}_\infty$ space. In fact contrary to the commutative case, the $\mathcal{OL}_\infty$ property for C^*-algebras is very restrictive. More precisely, according to results by Pisier [P4], Effros and Ruan [ER1], Kirchberg [Ki2], and Junge et al. [JOR] we know that a C^*-algebra A is an $\mathcal{OL}_{\infty,\lambda}$ space for some λ if and only if A is nuclear. In Theorem 3.11, we will improve a recent result in [JOR] by showing $\mathcal{OL}_\infty(A)\leqslant 3$. The microscopic index λ even provides some additional information on the structure of the underlying C^*-algebra. For example, a C^*-algebra is stably finite if $\lambda\leqslant(\frac{1+\sqrt{5}}{2})^{\frac{1}{2}}$ (see [JOR]). From these results, we can see that the local operator space structure provides a very important tool for the investigation of operator algebras.

However, not much work has been done for $\mathcal{OL}_p$ spaces in the range $1<p<\infty$. It is known that every $\mathcal{OL}_p$ space is completely complemented in some non-commutative L_p space. However, it can be derived from Szankowski's work [Sz2] that there are finite von Neumann algebras with separable predual such that $L_p(N)$ does not have the approximation property (see Theorem 2.19). Moreover, it is not known whether every $\mathcal{OL}_p$ space has the CBAP. In order to use the concepts from Banach space theory, we will work with the analogue of $\mathcal{CL}_p$ spaces. An operator space X is called a $\mathcal{COL}_{p,\lambda}$ space if it is paved by complemented copies of $L_p(A)$'s where *cb*-distance and the *cb*-norm of the projections are uniformly controlled by λ. If this is true for some λ, X is called a $\mathcal{COL}_p$ space. If we can replace the $L_p(A)$'s by S_p^n's, we call this a $\mathcal{COS}_{p,\lambda}$, $\mathcal{COS}_p$ space, respectively. Again we refer to Section 2 for a precise definition. Combining Banach space techniques from [JRZ] with applications of the Fubini Theorem from [Ju2], we obtain the following results on $\mathcal{COL}_p$ spaces.

Theorem 0.1. *Let $1<p<\infty$ and X an operator space. X is a $\mathcal{COS}_p$ space if and only if X has the CBAP, id_X admits a cb-factorization through an ultrapower of S_p, and X contains completely complemented S_p^n's uniformly.*

Theorem 0.2. *Let $1<p,\ p'<\infty$ with $\frac{1}{p}+\frac{1}{p'}=1$ and X an operator space. Then X is a $\mathcal{COL}_p$ space if and only if X^* is a $\mathcal{COL}_{p'}$ space.*

The cases $p=1,\ p=\infty$ remain true if we assume in addition that X^* has the CBAP and X is locally reflexive (in the operator space sense). Using an idea of Kirchberg, we can construct an operator space X such that X^* is $\mathcal{COL}_1$ but X does not have the CBAP. In Section 4, we extend the results of Johnson et al. [JRZ], Nielsen and Wojtaszczyk [NW] to $\mathcal{COL}_p$ spaces.

Theorem 0.3. *Let $1\leqslant p\leqslant\infty$ and X a separable $\mathcal{COL}_p$ space (such that in addition X^* has the CBAP and X is locally reflexive for $p\in\{1,\infty\}$). Then X has a cb-basis.*

Before we state our main application to non-commutative L_p spaces, we have to clarify the 'mild assumptions' on the underlying von Neumann algebra N. A C^*-algebra A has the *weak expectation property of Lance* (in short WEP) if for the universal representation $A \subset A^{**} \subset B(H)$ there is a contraction $P : B(H) \to A^{**}$ such that $P|_A = id_A$. A C^*-algebra B is said to be *QWEP* if there exists a C^*-algebra A with the WEP and a closed two-sided ideal I such that $B = A/I$. It is a long standing open problem whether every C^*-algebra is QWEP (see [Ki1] for many equivalent formulations). Note that a hyperfinite von Neumann algebra is injective, hence has WEP and thus is QWEP.

Theorem 0.4. *Let N be a QWEP von Neumann algebra with separable predual. Then for $1<p<\infty$ the following are equivalent*

(i) $L_p(N)$ *has the OAP*;
(ii) $L_p(N)$ *has the CBAP*;
(iii) $L_p(N)$ *is a* $\mathscr{COL}_p$ *space*;
(iv) $L_p(N)$ *has a cb-basis.*

In particular, if one of the conditions above is satisfied, then $L_p(N)$ *is an* $\mathscr{OL}_p$ *space.*

We apply Haagerup's pioneering work [CH,Ha3] on approximation properties and an interpolation argument (see e.g. [JR]) in order to obtain a result for L_p spaces associated to the von Neummann algebra $VN(\mathbb{F}_n)$ generated by the left regular representation of the free group $\mathbb{F}_n$. As so often in harmonic analysis, the spaces $L_p(VN(\mathbb{F}_n))$ behave much nicer for $1<p<\infty$ than for the border cases $p \in \{1, \infty\}$. Indeed, here $L_1(VN(\mathbb{F}_n))$ is not an $\mathscr{OL}_1$ space and $C_{\mathrm{red}}(\mathbb{F}_n)$ is not an $\mathscr{OL}_\infty$ space because $\mathbb{F}_n$ is not amenable.

Theorem 0.5. *Let $1<p<\infty$ and $\mathbb{F}_n$ the free group with n generators. Then $L_p(VN(\mathbb{F}_n))$ is a $\mathscr{COL}_p$ space (hence an $\mathscr{OL}_p$ space) and has a cb-basis.*

We note that the existence of a basis for $L_1(VN(\mathbb{F}_n))$ or $C_{\mathrm{red}}(\mathbb{F}_n)$ is an open problem. In contrast to the commutative theory a non-commutative C^*-algebra A might not have enough orthogonal finite-dimensional representations. Using the operator space structure of A^* instead, we can obtain sufficiently many information about the local structure of A in the cases of nuclear C^*-algebras.

Theorem 0.6. *Every separable nuclear C^*-algebra has a cb-basis.*

For researchers interested only in Banach space theory, we should mention that all the results hold in the Banach space sense. For example in Theorem 0.4, $L_p(N)$ has Grothendieck's approximation property iff it has a basis. A positive solution to the basis problem for non-commutative L_p spaces has previously only been known for the class of type I von Neumann algebras and the hyperfinite II_1 and II_∞ factors (see [Su]). However, we note that passing to tensor products of $\mathscr{COL}_p$ spaces already requires *cb*-norm estimates of the basis projections and thus operator space techniques are very natural (and useful) in this setting. However, our project seems to be the first attempt to provide more specific information on the local structure of

$L_p(N)$ spaces even on a purely Banach space level. We are indebted to W. B. Johnson for stressing the fact that the existence of a basis in $\mathscr{L}_p$ spaces can be proved by entirely local arguments. Indeed, this entirely local approach unifies the construction of bases for $\mathscr{L}_p$ spaces for all values $1 \leqslant p \leqslant \infty$ even in the commutative case (using the appropriate new notion of containing ℓ_p^n's 'far out').

In order to make this paper more accessible (for researchers with a Banach space background), we postpone arguments using modular theory of von Neumann algebras to the end of Section 5. In the subsequent paper [JRX], we will investigate the isometric theory in the hyperfinite (non-semifinite) case. Further applications of L_p spaces associated with discrete groups will be given in [JR].

1. Notation and preliminary results

We will use standard notation in operator algebras [D2,KR,Pe,Ta], and Banach space theory [LT]. In particular, given a Hilbert space H, we let $B(H)$ denote the space of all bounded linear operators on H. Our general references for operator spaces are [ER2,P6]. Let us recall some basic notations. A completely bounded map P on an operator space X is a *completely bounded projection* if $P^2 = P$. A subspace X of an operator space Y is called a *completely complemented* (respectively, a *completely contractively complemented*) subspace in Y if there is a completely bounded (respectively, completely contractive) projection from Y onto X. If X is an operator space, then its dual space X^* is an operator space with matrix norms given by the isometric identifications

$$M_n(X^*) = CB(X, M_n)$$

(see [BP,ER2]). This operator space structure on X^* is called the *operator* (*space*) *dual* of X. If X is an operator space, then the canonical embedding $\iota : X \to X^{**}$ is *a completely isometric injection*, i.e. $id_{M_n} \otimes \iota$ is isometric for all $n \in \mathbb{N}$. If $T : X \to Y$ is a completely bounded map, then its adjoint map $T^* : Y^* \to X^*$ is also completely bounded with $||T^*||_{cb} = ||T||_{cb}$. Using the Arveson–Wittstock–Hahn–Banach theorem [ER2,Pa], it is easy to show that if T is a complete isometry, then T^* is *a complete quotient map*, i.e. $id_{M_n} \otimes T^*$ maps the open unit ball onto the open unit ball for all $n \in \mathbb{N}$. Similarly, if T is a complete quotient map, then T^* is a complete isometry. Given a von Neumann algebra N, the canonical embedding $\iota : N_* \hookrightarrow N^*$ induces an operator space structure on N_*. With these matrix norms, we have the complete isometry

$$N = (N_*)^*.$$

In the following, we will use the notation S_p (resp. S_p^n) for the spaces of all compact operators on the Hilbert spaces $\ell_2 = \ell_2(\mathbb{N})$ (resp. ℓ_2^n) such that

$$||x||_p = [tr((x^*x)^{\frac{p}{2}})]^{\frac{1}{p}} < \infty .$$

We will always work with the canonical duality between S_p and $S_{p'}$ $(\frac{1}{p}+\frac{1}{p'}=1)$ given by

$$\langle [a_{ij}],[b_{ij}]\rangle = \sum_{ij} a_{ij}b_{ij} \tag{1.1}$$

and obtain a complete isometry $S_p^* = S_{p'}$. Similarly, if A is a finite-dimensional C^*-algebra given by

$$A = M_{n_1} \oplus_\infty \cdots \oplus_\infty M_{n_l},$$

then we have

$$A^* = S_1^{n_1} \oplus_1 \cdots \oplus_1 S_1^{n_l},$$

where $\oplus_1$ is the operator space l_1-direct sum. Let $n = n_1 + \cdots + n_l$. The canonical projection of S_∞^n onto A is completely contractive on S_∞^n and the same map is also completely contractive on S_1^n. Therefore, we may apply complex interpolation for the compatible pair $(A, A^*) \subset (S_\infty^n, S_1^n)$ and obtain the natural operator space structure on

$$L_p(A) = [A, A^*]_{\frac{1}{p}} = S_p^{n_1} \oplus_p \cdots \oplus_p S_p^{n_l} \subset S_p^n.$$

We refer to [BL] for the complex interpolation method. Note that by complementation, we still have a complete isometry

$$L_p(A)^* = L_{p'}(A)$$

and $L_p(A)$ is a completely contractively complemented subspace of S_p^n for $1 \leqslant p \leqslant \infty$. In the sequel, we will also use an infinite-dimensional analogue of these spaces. Let $\mathbf{m} = (m(n))_{n\in\mathbb{N}}$ be a sequence of natural numbers and

$$\mathfrak{b}(\mathbf{m}) = \prod_n M_{m(n)},$$

the von Neumann algebra obtained as block diagonals in $B(\ell_2)$. In the Banach space literature one may also write $\mathfrak{b}(\mathbf{m}) = (\sum_n \oplus M_{m(n)})_\infty$. Then the predual of $\mathfrak{b}(\mathbf{m})$ is $\mathfrak{s}_1(\mathbf{m}) = (\sum_n \oplus S_1^{m(n)})_1$, i.e. the block diagonals in S_1. Since the projection onto these block diagonals is completely contractive in both cases, we see that

$$\mathfrak{s}_p(\mathbf{m}) = \left(\sum_n \oplus S_p^{m(n)}\right)_p = [\mathfrak{b}(\mathbf{m}), \mathfrak{s}_1(\mathbf{m})]_{\frac{1}{p}} = L_p(\mathfrak{b}(\mathbf{m}))$$

is completely contractively complemented in S_p. For $p=\infty$, we use the notation $\mathfrak{s}_\infty(\mathbf{m})$ for the c_0 sum. In the special case where $\mathbf{m}$ is given by $m(n)=n$ for all $n\in\mathbb{N}$, we will simply use the notation $\mathfrak{s}_p$.

As in Banach space theory, ultraproducts turn out to be a useful tool in the study of operator spaces (see for example [P3]). Let us recall that if $\mathscr{U}$ is a free ultrafilter on an infinite index set I and $(X_i)_{i\in I}$ is a family of operator spaces, then we consider the ultraproduct

$$\prod_{\mathcal{U}} X_i = \prod_{i\in I} X_i / J_{\mathcal{U}},$$

where $\prod_{i\in I} X_i = \{(x_i) \mid \sup_i \|x_i\| < \infty\}$ is the space of all bounded families, and

$$J_{\mathcal{U}} = \left\{ (x_i) \in \prod_{i\in I} X_i \,\middle|\, \lim_{\mathcal{U}} \|x_i\|_{X_i} = 0 \right\}$$

is the norm closed subspace in $\prod_{i\in I} X_i$ of families tending to 0 along $\mathcal{U}$. An ultraproduct $\prod_{\mathcal{U}} X_i$ of operator spaces carries canonical matrix norms given by

$$M_n\left(\prod_{\mathcal{U}} X_i\right) = \prod_{\mathcal{U}} M_n(X_i).$$

For details see [ER2,P3,P5]. If (A_i) is a family of C^*-algebras, it is well-known that $\prod_{\mathcal{U}} A_i$ is again a C^*-algebra. It is also known (see [Gr,Ra1,Ra2]) that for $1 \leqslant p < \infty$, we have $\prod_{\mathcal{U}} S_p = L_p(N)$ for some von Neumann algebra N. The following result is due to Junge [Ju2] and holds only for $p \in (1, \infty)$.

Theorem 1.1. *Let E and F be finite-dimensional operator spaces and $1 < p < \infty$. If we have a commuting diagram of completely bounded maps*

$$\begin{tikzcd} & \prod_{\mathcal{U}} S_p \arrow[dr, "r"] & \\ E \arrow[ur, "s"] \arrow[rr, "T"] & & F \end{tikzcd}$$

then for any $\varepsilon > 0$, there exist an integer n and a commuting diagram of completely bounded maps

$$\begin{tikzcd} & S_p^n \arrow[dr, "\tilde{r}"] & \\ E \arrow[ur, "\tilde{s}"] \arrow[rr, "T"] & & F \end{tikzcd}$$

such that

$$\|\tilde{r}\|_{cb}\|\tilde{s}\|_{cb} < \|r\|_{cb}\|s\|_{cb} + \varepsilon.$$

Approximation properties play an important role in operator algebras and operator spaces. Let X and Y be operator spaces. A linear map $T : X \to Y$ is said to have the *completely bounded approximation property* (in short *CBAP*) if there exists a constant λ and a net of finite rank maps $T_i : X \to Y$ such that $T_i \to T$ in the point-norm topology and $\sup_i \|T_i\|_{cb} \leqslant \lambda$. In this case, we let

$$\Lambda(T) = \inf \lambda$$

denote the infimum of all λ as above. If $T = id_X$ we say that X has the CBAP and let

$$\Lambda(X) = \Lambda(id_X).$$

Let $1 \leqslant p \leqslant \infty$. A linear map $T : X \to Y$ is said to have the γ_p-*approximation property* (in short γ_p-*AP*) (i.e. can be approximately factored through S_p^n spaces) if there exist diagrams of completely bounded maps

$$\begin{tikzcd} & S_p^{n_i} \arrow[dr, "r_i"] & \\ X \arrow[ur, "s_i"] \arrow[rr, "T"] & & Y \end{tikzcd}$$

which converges in the point-norm topology to T and satisfies $\sup_i \|r_i\|_{cb}\|s_i\|_{cb} \leqslant \lambda$ for some constant $\lambda < \infty$. We let as above

$$\gamma_p^{ap}(T) = \inf \lambda.$$

If $T = id_X$, we say that X has the γ_p-*approximation property* and let

$$\gamma_p^{ap}(X) = \gamma_p^{ap}(id_X).$$

It is clear that if T has the γ_p-AP, then T has the CBAP with

$$\Lambda(T) \leqslant \gamma_p^{ap}(T).$$

In the analysis of approximation properties, small perturbation arguments provide an essential technical tool. Let us recall the following operator space analogue of a classical Banach space argument due to Pisier [P5].

Lemma 1.2. *Let X be an operator space and $E \subset X$ an n-dimensional subspace with a biorthogonal system $x_1, \dots, x_n, x_1^*, \dots, x_n^*$ (i.e. $\|x_i\| \leqslant 1$, $\|x_j^*\| \leqslant 1$ and $x_i^*(x_j) = \delta_{ij}$ for all $i,j = 1, \dots, n$). Let $0 < \varepsilon < 1$, and $T : E \to X$ a linear map such that*

$$\|T(x_i) - x_i\| \leqslant \frac{\varepsilon}{n}$$

for all $i = 1, \dots, n$. Then there exists a complete isomorphism $W : X \to X$ such that

$$WT(x) = x$$

for all $x \in E$ *and*

$$\|W\|_{cb} \leqslant (1-\varepsilon)^{-1}, \quad \|W - id\|_{cb} \leqslant \frac{\varepsilon}{1-\varepsilon} \text{ and } \|W^{-1}\|_{cb} \leqslant (1+\varepsilon). \tag{1.2}$$

As in the category of Banach spaces, we may obtain the following result.

Corollary 1.3. *Let X and F be operator spaces with F finite-dimensional and let $r : F \to X$ and $s : X \to F$ be maps such that $sr = id_F$. If $E \subset X$ is an n-dimensional subspace and $0 < \varepsilon < \frac{1}{2}$ such that*

$$\|rs(x) - x\| \leqslant \frac{\varepsilon \|x\|}{n\|r\|_{cb}\|s\|_{cb}}$$

for all $x \in E$. Then there exist maps $\tilde{r} : F \to X$ and $\tilde{s} : X \to F$ such that $\tilde{s}\tilde{r} = id_F$, $\tilde{r}\tilde{s}|_E = id_E$ and

$$\|\tilde{r}\|_{cb}\|\tilde{s}\|_{cb} \leqslant (1-\varepsilon)^{-1}(1-2\varepsilon)^{-1}\|r\|_{cb}\|s\|_{cb}.$$

Proof. Applying Lemma 1.2 to $\varepsilon' = \frac{\varepsilon}{\|r\|_{cb}\|s\|_{cb}} \leqslant \frac{1}{2}$ and $T = rs$, we may obtain a complete isomorphism $W : X \to X$ such that $\|id - W\|_{cb} \leqslant \frac{\varepsilon'}{1-\varepsilon'}$ and $WT(x) = x$ for all $x \in E$. Then we deduce

$$\|id_F - sWr\|_{cb} = \|s(id - W)r\|_{cb} \leqslant \|r\|_{cb}\|s\|_{cb} \frac{\varepsilon'}{1-\varepsilon'} \leqslant 2\varepsilon.$$

Hence, for $b = (sWr)^{-1}$ we obtain the estimate

$$\|b\|_{cb} \leqslant (1-2\varepsilon)^{-1}.$$

We define $\tilde{s} = bs$ and $\tilde{r} = Wr$. Clearly, $\tilde{s}\tilde{r} = bsWr = id_F$. For $x \in E$, we observe that

$$sWrs(x) = sWT(x) = s(x).$$

Hence, $sWr|_{s(E)} = id_{s(E)}$ and therefore $b|_{s(E)} = id_{s(E)}$. Thus, we get

$$\tilde{r}\tilde{s}(x) = Wrbs(x) = Wrs(x) = WT(x) = x$$

for all $x \in E$. Using the *cb*-norm estimates for b and W, we obtain the assertions. □

For Banach spaces (and nowadays also for operator spaces) it is well-known that Lemma 1.2 implies that the 'point-norm approximation' can be improved to obtain finite rank maps which are the identity on a given finite-dimensional subspace.

Lemma 1.4. *Let $T : X \to Y$ be a completely bounded map.*

(i) *$T : X \to Y$ has the CBAP with $\Lambda(T) < \lambda$ if and only if for every finite-dimensional subspace $E \subseteq X$, there exists a finite rank map $u : X \to Y$ such that $\|u\|_{cb} < \lambda$ and $u(x) = T(x)$ for all $x \in E$.*

(ii) *$T : X \to Y$ has the γ_p-AP with $\gamma_p^{ap}(T) < \lambda$ if and only if for every finite-dimensional subspace $E \subseteq X$, there exist $n \in \mathbb{N}$ and maps $u : X \to S_p^n$, $v : S_p^n \to Y$ such that $\|u\|_{cb}\|v\|_{cb} < \lambda$ and $vu(x) = T(x)$ for all $x \in E$.*

Proof. Obviously the second assertion in (i), (ii) implies the CBAP, γ_p-AP, respectively. Since the arguments are very similar, we will only show the missing implication in (i). If E is a finite-dimensional subspace of X, then $T(E)$ is a finite-dimensional subspace of Y. We can find vectors $x_1, \dots, x_k$ in E such that $(T(x_i))_{i=1}^k$ is part of a biorthogonal system in $T(E)$. Choose $0 < \delta < 1$ such that $(1+\delta)^2 \Lambda(T) < \lambda$. Since T has the CBAP, there exists a finite rank map $\tilde{T} : X \to Y$ such that $\|\tilde{T}\|_{cb} < (1+\delta)\Lambda(T)$ and $\|\tilde{T}(x_i) - T(x_i)\| < \frac{\delta}{k}$ for all $i = 1, \dots, k$. It follows from Lemma 1.2 that there exists a complete isomorphism $W : Y \to Y$ such that $WT(x_i) = \tilde{T}(x_i)$ for $(i = 1, \dots, k)$ and $\|W^{-1}\|_{cb} < (1+\delta)$. Hence, $u = W^{-1}\tilde{T} : X \to Y$ is a finite rank map which satisfies the requirement of the assertion. $\square$

Using the uniform convexity of S_p (see [TJ]) it is easy to prove the following well-known fact. We refer to [ER1] for the details.

Lemma 1.5. *Let $1 < p < \infty$. Then $\prod_{\mathscr{U}} S_p$ is reflexive for every ultrafilter $\mathscr{U}$. Moreover, every $\mathcal{OL}_p$ space is completely contractively complemented in some $\prod_{\mathscr{U}} S_p$ and thus reflexive.*

Proposition 1.6. *Let $1 < p < \infty$ and X an operator space. Then X has the γ_p-AP if and only if X has the CBAP and there exists a free ultrafilter $\mathscr{U}$ on some index set I such that X is completely complemented in $\prod_{\mathscr{U}} S_p$, i.e. there exists a commuting diagram of completely bounded maps*

$$\begin{tikzcd} & \prod_{\mathscr{U}} S_p \arrow[dr, "r"] & \\ X \arrow[ur, "s"] \arrow[rr, "id_X"] & & X \end{tikzcd} \tag{1.3}$$

Proof. If X has the γ_p-AP, then X has the CBAP, and there exist diagrams of completely bounded maps

$$\begin{tikzcd} & S_p^{n_i} \arrow[dr, "r_i"] & \\ X \arrow[ur, "s_i"] \arrow[rr, "id_X"] & & X \end{tikzcd}$$

which approximately commute in the point-norm topology and in addition satisfy

$$\sup \|r_i\|_{cb}\|s_i\|_{cb} = \lambda < \infty.$$

If we let $\mathscr{U}$ be a free ultrafilter on the index set I, then we obtain a commuting diagram of completely bounded maps

$$\begin{tikzcd} & \prod_{\mathscr{U}} S_p^{n_i} \arrow[dr, "r"] & & \\ X \arrow[ur, "s"] \arrow[rr, "id_X"] & & X \arrow[r, hook, "\iota"] & X^{**} \end{tikzcd} \tag{1.4}$$

where we let $s : X \to \prod_{\mathscr{U}} S_p^{n_i}$ be the map given by $s(x) = (s_i(x))_{\mathscr{U}}$ and $r : \prod_{\mathscr{U}} S_p^{n_i} \to X^{**}$ the map given by

$$\langle r((z_i)_{\mathscr{U}}), x^* \rangle = \lim_{\mathscr{U}} x^*(r_i(z_i))$$

for all $x^* \in X^*$. Since each $S_p^{n_i}$ is completely contractively complemented in S_p, $\prod_{\mathscr{U}} S_p^{n_i}$ is completely contractively complemented in $\prod_{\mathscr{U}} S_p$, and thus we can actually replace $\prod_{\mathscr{U}} S_p^{n_i}$ in (1.4) by $\prod_{\mathscr{U}} S_p$. Hence X is isomorphic to $s(X) \subset \prod_{\mathscr{U}} S_p$ and thus reflexive according to Lemma 1.5. Thus we obtain the commuting diagram

$$\begin{tikzcd} & \prod_{\mathscr{U}} S_p \arrow[dr, "r"] & \\ X \arrow[ur, "s"] \arrow[rr, "id_X"] & & X \end{tikzcd}$$

On the other hand, let us assume that X has the CBAP and satisfies diagram (1.3) with $\|r\|_{cb}\|s\|_{cb} \leqslant C$. It follows from Lemma 1.4 that for any finite-dimensional subspace $E \subseteq X$ and $\varepsilon > 0$, there exists a finite rank map $u : X \to X$ such that $\|u\|_{cb} < (1+\varepsilon)\Lambda(X)$ and $u|_E = id_E$ for all $x \in E$. In particular $u^2|_E = id_E$ and it suffices to show that u^2 factors through S_p^m. Let us consider the finite-dimensional operator space $G = X/ker(u)$ with quotient map $q_G : X \to G$ and the induced map $\hat{u} : G \to X$ such that $u = \hat{u} q_G$. Note that $\hat{u}$ has the same *cb*-norm as u. Let $F = u(X) \subset X$ with inclusion map $i_F : F \to X$. Then $u\hat{u} = urs\hat{u} : G \to F$ satisfies the assumption of Theorem 1.1 and hence admits a factorization $u\hat{u} = vw$, $w : G \to S_p^m$, $v : S_p^m \to F$ such that

$$\|v\|_{cb}\|w\|_{cb} \leqslant (1+\varepsilon)\|u\|_{cb}^2\|r\|_{cb}\|s\|_{cb} \leqslant (1+\varepsilon)^3 \Lambda(X)^2 C.$$

Thus $u^2 = i_F vwq_G$ factors through S_p^m and satisfies the corresponding *cb*-norm estimate. Therefore, X has the γ_p-AP with $\gamma_p^{ap}(X) \leqslant C\Lambda(X)^2$. $\square$

Remark 1.7. The same result holds if we replace the *cb*-norm by the operator norm in all instances above. Indeed, in Theorem 1.1, this can be easily proved by using finite δ-nets in the unit ball of E and F^*.

In the category of operator spaces Proposition 1.6 is no longer true for $p = 1$. Indeed, if N is a von Neumann algebra, then $\gamma_1^{ap}(N_*) < \lambda$ if and only if N is λ-semidiscrete, and thus injective by Pisier [P4] or Christensen and Sinclair [CS]. Let $\mathbb{F}_n$ denote the free group of n generators. It is known that the von Neumann algebra $VN(\mathbb{F}_n)$ is not injective (for any $\lambda > 1$), but satisfies $\Lambda(VN(\mathbb{F}_n)_*) = 1$ (see [Ha3]). Using an argument of Wassermann (or the fact that $VN(\mathbb{F}_n)$ is QWEP with the results in [EJR, Section 7] and [NO]), we see that there are complete contractions $r : VN(\mathbb{F}_n)_* \to \prod_{\mathscr{U}} S_1$, $s : \prod_{\mathscr{U}} S_1 \to VN(\mathbb{F}_n)_*$ such that

$$\begin{tikzcd} & \prod_{\mathscr{U}} S_1 \arrow[dr, "r"] & \\ VN(\mathbb{F}_n)_* \arrow[ur, "s"] \arrow[rr, "id_X"] & & VN(\mathbb{F}_n)_* \end{tikzcd}.$$

Hence $VN(\mathbb{F}_n)_*$ satisfies the assumptions of Proposition 1.6 without having the γ_1-AP. In Theorem 5.7, we will show that for $1 < p < \infty$, $L_p(VN(\mathbb{F}_n))$ has the γ_p-AP. This indicates that, as so often in harmonic analysis, the L_p spaces in the range $1 < p < \infty$ behave much nicer than the extreme cases $p = 1$ and $p = \infty$.

2. $\mathscr{COL}_p$ and $\mathscr{OL}_p$ spaces

In this and the following sections (unless stated explicitly otherwise) we will work in the category of operator spaces. This means that all linear maps, inclusions, quotient maps and projections are to be understood as completely bounded maps, complete isomorphisms with values in the images, complete quotient maps and completely bounded projections, respectively. This convention will simplify our presentation but is by no means necessary. Let us point out that all the results (stated here in terms of operator spaces) hold true in the category of Banach spaces. Some of the proofs are slightly easier for Banach spaces or can be found in the literature, namely in [JRZ,NW]. Therefore, we decided to emphasize the modifications required for operator spaces.

An operator space X is called an *operator $\mathscr{L}_p$ space* (in short *$\mathscr{OL}_p$ space*) if there exists a constant $\lambda > 1$ and a family $(F_i)_{i \in I}$ of finite-dimensional subspace such that $\bigcup_i F_i$ is dense in X and for every index i there exists a finite-dimensional C^*-algebra A_i such that

$$d_{cb}(L_p(A_i), F_i) \leqslant \lambda. \tag{2.1}$$

In this case, we denote by $\mathscr{OL}_p(X) = \inf \lambda$, where the infimum is taken over all λ as above. Moreover, we say that X is *an $\mathscr{OL}_{p,\lambda}$-space*, if $\mathscr{OL}_p(X) \leqslant \lambda$. We call X an *$\mathscr{OS}_{p,\lambda}$ space* if we can replace the $L_p(A_i)$'s in (2.1) by $S_p^{n_i}$'s.

An operator space X is called a *completely complemented* $\mathcal{OL}_{p,\lambda}$ *space* (in short $\mathcal{COL}_{p,\lambda}$ *space*) for some constant $\lambda > 1$ if there exist a family of finite-dimensional C^*-algebras (A_i) and commuting diagrams of completely bounded maps

$$\begin{tikzcd} & X \arrow[dr, "s_i"] & \\ L_p(A_i) \arrow[ur, "r_i"] \arrow[rr, "id_{L_p(A_i)}"] & & L_p(A_i) \end{tikzcd} \tag{2.2}$$

such that $r_i s_i \to id_X$ in the point-norm topology on X and $\|r_i\|_{cb}\|s_i\|_{cb} \leqslant \lambda$. We call X a *completely complemented* $\mathcal{OS}_{p,\lambda}$ *space* (in short $\mathcal{COS}_{p,\lambda}$ *space*) if we can replace $L_p(A_i)$ in (2.2) by $S_p^{n_i}$. We say that X is a $\mathcal{COL}_p$ space (respectively, a $\mathcal{COS}_p$ space) if it is a $\mathcal{COL}_{p,\lambda}$ space (respectively, a $\mathcal{COS}_{p,\lambda}$ space) for some $\lambda \geqslant 1$. In this case, we denote by $\mathcal{COL}_p(X) = \inf \lambda$ (respectively, $\mathcal{COS}_p(X) = \inf \lambda$), where the infimum is taken over all λ such that X is a $\mathcal{COL}_{p,\lambda}$ space (respectively, a $\mathcal{COS}_{p,\lambda}$ space). The following perturbation result (Lemma 2.1) shows that these definitions of $\mathcal{OL}_p$ (respectively, $\mathcal{COL}_p$ spaces) are consistent with the idea of paving out the operator space X by copies (respectively, complemented copies) of finite-dimensional non-commutative L_p spaces. Since the proof is very similar to the proof of Lemma 1.4 we will leave the details of the proof of Lemma 2.1 to the reader.

Lemma 2.1. *Let X be an operator space and $\lambda > 1$.*

(i) *X is an $\mathcal{OL}_p$ space with $\mathcal{OL}_p(X) < \lambda$ if and only if there exists a $\lambda' < \lambda$ such that for every finite-dimensional subspace E of X there exists a finite-dimensional space $E \subset F \subset X$ and a finite-dimensional C^*-algebra A such that*

$$d_{cb}(L_p(A), F) < \lambda'.$$

(ii) *X is a $\mathcal{COL}_p$ space satisfying $\mathcal{COL}_p(X) < \lambda$ if and only if there exists a $\lambda' < \lambda$ such that for every finite-dimensional subspace $E \subseteq X$, there exist a finite-dimensional C^*-algebra A and a commuting diagram of completely bounded maps*

$$\begin{tikzcd} & X \arrow[dr, "s"] & \\ L_p(A) \arrow[ur, "r"] \arrow[rr, "id_{L_p(A)}"] & & L_p(A) \end{tikzcd}$$

with $\|r\|_{cb}\|s\|_{cb} \leqslant \lambda'$ and $rs(x) = x$ for all $x \in E$.

A similar result holds for $\mathcal{OS}_p$ spaces and $\mathcal{COS}_p$ spaces.

It follows from Lemma 2.1 that every $\mathcal{COL}_p$ space is an $\mathcal{OL}_p$ space. For $p = \infty$, the two notions are equivalent by the injectivity of finite-dimensional C^*-algebras. (However, if we consider the Banach space versions of spaces paved out by Banach space copies of finite-dimensional C^*-algebras, it is not clear whether they might be assumed to be norm complemented.) In the context of operator space the two notions are equivalent for $p=1$ if λ is sufficiently close to 1 (see [Oz]). It is not known whether these notions are still equivalent for large λ. We refer to the end of this section for more open problems. Let us state the main result in this section.

Theorem 2.2. *Let $1 \leqslant p \leqslant \infty$ and X an operator space with the γ_p-AP. If X contains S_p^n's (respectively, complemented S_p^n's) then X is an $\mathcal{OS}_p$ space (respectively, a $\mathcal{COS}_p$ space).*

Here X *is said to contain S_p^n's* if there exists a constant C such that for every $n \in \mathbb{N}$, we can find $G_n \subset X$ such that

$$d_{cb}(G_n, S_p^n) \leqslant C.$$

We note that in the Banach space literature the term 'X contains ℓ_p^n's uniformly' (respectively 'X contains ℓ_p^n's uniformly complemented') is in use. If we want to specify the constant C we say X *contains S_p^n's with constant* C. Accordingly, we say that X *contains complemented S_p^n's* (with constant C) if for every $n \in \mathbb{N}$ there are $r_n : S_p^n \to X$ and $s_n : X \to S_p^n$ such that

$$s_n r_n = id_{S_p^n} \quad \text{and} \quad \|r_n\|_{cb} \|s_n\|_{cb} \leqslant C.$$

As a technical (but important) modification we say that X *contains complemented S_p^n's with respect to* Y if $Y \subset X^*$ and $s_n^*(S_{p'}^n) \subset Y$ for all $n \in \mathbb{N}$.

Although this clarifies the assumptions of Theorem 2.2, the proof requires 'sufficiently many orthogonal' copies of S_p^n with respect to any finite-dimensional subspace of X. Note that in the commutative setting this is an immediate consequence of the Kadec–Pełczyński dichotomy. In our setting, we have to use a formal definition of 'sufficiently orthogonal'. We say that an operator space X *contains complemented S_p^n's far out* if there exists a constant $C>0$ such that for every finite-dimensional subspace $E \subset X$ and for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $r_n : S_p^n \to X$, $s_n : X \to S_p^n$ such that

$$s_n r_n = id_{S_p^n}, \quad \|r_n s_n|_E\|_{cb} \leqslant \varepsilon \quad \text{and} \quad \|r_n\|_{cb} \; \|s_n\|_{cb} \leqslant C.$$

Again, we use 'with constant C' and 'with respect to Y' as above. Similarly, we say that X *contains S_p^n's far out* (*with constant* C) if for every finite rank map $T : X \to X$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $G_n \subset X$ such that

$$d_{cb}(G_n, S_p^n) \leqslant C \quad \text{and} \quad \|T|_{G_n}\|_{cb} \leqslant \varepsilon.$$

Note that it suffices to have $\|T|_{G_n}\| \leqslant \varepsilon$ because G_n is finite dimensional. Indeed, using a biorthogonal system one can easily prove that for a linear map $T : E \to X$ on a d-dimensional operator space E (see e.g. [EH])

$$\|T\|_{cb} \leqslant d\|T\|. \tag{2.3}$$

Similarly, the condition $\|r_n s_n|_E\|_{cb} \leqslant \varepsilon$ can be weakened to $\|r_n s_n|_E\| \leqslant \varepsilon$. Let us start with the most natural class of examples (see the preliminaries for the definition of $\mathfrak{s}_p$.)

Example 2.3. Let $1 \leqslant p \leqslant \infty$. The space $\mathfrak{s}_p$ is a $\mathscr{COS}_p$-space with constant $\mathscr{COS}_p(\mathfrak{s}_p) = 1$. $\mathfrak{s}_p$ contains complemented S_p^n's far out. Moreover, every operator space containing $\mathfrak{s}_p$ completely complemented contains complemented S_p^n's far out.

Proof. For $n \in \mathbb{N}$, we denote by $r_n : (\sum_{k=1}^n \oplus S_p^k)_p \to \mathfrak{s}_p$ the natural completely isometric inclusion map and by $s_n : \mathfrak{s}_p \to (\sum_{k=1}^n \oplus S_p^k)_p$ the completely contractive projection. Then $r_n s_n$ tends to the identity map in the point-norm topology and the assertion follows from the fact that $(\sum_{k=1}^n \oplus S_p^k) = L_p(A_n)$ for the finite-dimensional C^*-algebra $A_n = M_1 \oplus_\infty \cdots \oplus_\infty M_n$. In order to prove the second assertion, we consider the map $v_k : S_p^k \to \mathfrak{s}_p$ which maps S_p^k in the kth block and the natural projection $u_k : \mathfrak{s}_p \to S_p^k$ on the kth block. Clearly $u_k v_k = id_{S_p^k}$ for all k and the sequence of projections (P_k) defined by $P_k = v_k u_k$ satisfies $\lim_k P_k(x) = 0$ for all $x \in \mathfrak{s}_p$ by the density of elements with finitely many entries in $\mathfrak{s}_p$. The last assertion is an obvious consequence. $\square$

In our context the techniques developed by Lindenstrauss and Rosenthal [LR] in the commutative setting yield the following key result.

Proposition 2.4. *Let $1 \leqslant p \leqslant \infty$ and X an operator space with the γ_p-AP and containing complemented S_p^n's far out with constant C. Then X is a $\mathscr{COS}_p$ space satisfying $\mathscr{COS}_p(X) \leqslant (1+2C)(1+2\gamma_p^{ap}(X))$.*

Proof. Let E be a finite-dimensional subspace of X and $0 < \varepsilon < \frac{1}{2}$. Since X has the γ_p-AP, we can apply Lemma 1.4 to obtain maps $u : X \to S_p^n$ and $v : S_p^n \to X$ such that

$$vu|_E = id_E, \quad \|u\|_{cb} = 1 \quad \text{and} \quad \|v\|_{cb} \leqslant (1+\varepsilon)\gamma_p^{ap}(X).$$

Let $0 < \delta < \varepsilon(4(C+1)\gamma_p^{ap}(X)n^2)^{-1}$, where C is the constant from the 'far out' definition. Let $F = v(S_p^n)$. According to the assumption, we may find $r : S_p^n \to X$ and $s : X \to S_p^n$ such that

$$sr = id_{S_p^n}, \quad \|r\|_{cb} = 1, \quad \|s\|_{cb} \leqslant C \quad \text{and} \quad \|rs|_F\|_{cb} \leqslant \delta.$$

We let $P = rs : X \to X$ denote the completely bounded projection from X onto the range of r, and let $\tilde{r} : S_p^n \to X$ and $\tilde{s} : X \to S_p^n$ be completely bounded maps given by

$$\tilde{r} = v + r(id_{S_p^n} - uv) \quad \text{and} \quad \tilde{s} = u(id_X - P) + s.$$

However, in general $\tilde{s}\tilde{r}$ need not be the identity map on S_p^n. Since $sr = id_{S_p^n}$ and $(id_X - P)r = 0$, we have

$$\begin{aligned}\tilde{s}\tilde{r} &= [u(id_X - P) + s][v + r(id_{S_p^n} - uv)]\\ &= uv - uPv + sv + (id_{S_p^n} - uv)\\ &= id_{S_p^n} - uPv + srsv = id_{S_p^n} + (s-u)Pv.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\tilde{s}\tilde{r} - id_{S_p^n}\|_{cb} &\leqslant \|(s-u)Pv\|_{cb} \leqslant \|s-u\|_{cb}\|P|_F\|_{cb}\|v\|_{cb}\\ &\leqslant (C+1)\delta 2\gamma_p^{ap}(X) \leqslant \frac{\varepsilon}{2n^2}.\end{aligned}$$

According to Lemma 1.2 we can find an isomorphism $w : S_p^n \to S_p^n$ such that $w\tilde{s}\tilde{r} = id_{S_p^n}$ and

$$\|w\|_{cb} \leqslant \frac{1}{1-\frac{\varepsilon}{2}} \leqslant (1+\varepsilon).$$

If we define $r_{E,\varepsilon} = \tilde{r}$ and $s_{E,\varepsilon} = w\tilde{s}$, then we deduce

$$\begin{aligned}\|r_{E,\varepsilon}\|_{cb} &\leqslant \|v\|_{cb} + \|r\|_{cb}(1 + \|u\|_{cb}\|v\|_{cb})\\ &\leqslant (1+\varepsilon)\gamma_p^{ap}(X) + (1 + (1+\varepsilon)\gamma_p^{ap}(X))\\ &\leqslant (1+\varepsilon)(1 + 2\gamma_p^{ap}(X))\end{aligned}$$

and

$$\|s_{E,\varepsilon}\|_{cb} \leqslant (1+\varepsilon)\|u - uP + s\|_{cb} \leqslant (1+\varepsilon)(1 + C + C).$$

Finally, we have to check that $r_{E,\varepsilon}s_{E,\varepsilon}(x) = \tilde{r}w\tilde{s}(x) = x$ for all $x \in E$. If we let $G = u(E)$, then for $y = u(x) \in G$, we have

$$uv(y) = u(vu(x)) = u(x) = y,$$

hence

$$\tilde{r}(y) = v(y) = vu(x) = x.$$

This implies that for all $x \in E$

$$u(x) = y = (w\tilde{s}\tilde{r})(y) = w\tilde{s}(x).$$

In particular,

$$\tilde{r}w\tilde{s}(x) = \tilde{r}(u(x)) = \tilde{r}(y) = x.$$

We have checked the conditions for $\mathscr{COS}_p$ formulated in the introduction. Thus the assertion is proved. $\square$

Remark 2.5. For a fixed subspace $Y \subset X^*$, we may also define the γ_p-*AP with respect to* Y by requiring that the factorizations $r_i : S_p^n \to X$ and $s_i : X \to S_p^n$ with $r_i s_i$ tending to id_X satisfy the additional property $s_i^*(S_{p'}^n) \subset Y$. The argument above shows that if X has the γ_p-AP with respect to Y and contains complemented S_p^n's with respect to Y then X is a $\mathscr{COS}_p$ space with respect to Y (defined as above). Let us point out that these technical modifications are essential for the interesting applications in the cases $p = 1$ or $p = \infty$.

Remark 2.6. For a fixed $n \in \mathbb{N}$, let us consider the following stronger version of the γ_p-AP. We say that X has the $\gamma_{p,n}$-AP if there exist diagrams of completely bounded maps

$$\begin{tikzcd} & \ell_p^{k_i}(S_p^n) \arrow[dr, "r_i"] & \\ X \arrow[ur, "s_i"] \arrow[rr, "id_X"] & & X \end{tikzcd}$$

which converges in the point-norm topology to id_X and satisfies the inequalities $\sup_i \|r_i\|_{cb}\|s_i\|_{cb} \leqslant \gamma_{p,n}^{ap}(X) < \infty$. Similarly, we say that X contains complemented $\ell_p^k(S_p^n)$'s far out with constant C if for every finite-dimensional subspace $E \subset X$, for every $k \in \mathbb{N}$ and $\varepsilon > 0$, there exist $r : \ell_p^k(S_p^n) \to X$ and $s : X \to \ell_p^k(S_p^n)$ such that

$$sr = id_{\ell_p^k(S_p^n)}, \quad \|rs|_E\|_{cb} \leqslant \varepsilon \quad \text{and} \quad \|r\|_{cb}\|s\|_{cb} \leqslant C.$$

The same proof as above shows that an operator space X with the $\gamma_{p,n}$-AP and containing complemented $\ell_p^k(S_p^n)$'s far out with constant C is a $\mathscr{COL}_p$ space with constant

$$\mathscr{COL}_p(X) \leqslant (1 + 2C)(1 + 2\gamma_{p,n}^{ap}(X)).$$

As an application of Proposition 2.4, we deduce that every operator space X with the γ_p-AP can be enlarged to provide an example of a $\mathscr{COS}_p$ space. This method provides many interesting examples of $\mathscr{COS}_p$ spaces. We refer to Example 2.3 for the obvious fact that $\mathfrak{s}_p$ contains complemented S_p^n's far out.

Corollary 2.7. *Let $1<p<\infty$ and X an operator space. Then the following are equivalent.*

(i) *X has the CBAP and X is completely isomorphic to a completely complemented subspace of $\prod_{\mathscr{U}} S_p$ for some ultrapower of S_p;*
(ii) *X has the γ_p-AP.*
(iii) *$X \oplus_p \mathfrak{s}_p$ is a $\mathscr{COS}_p$ space;*
(iv) *X is completely complemented in a $\mathscr{COS}_p$ space.*

Proof. For the implication (i) $\Rightarrow$ (ii), we note that X has the γ_p-AP according to Proposition 1.6. The implication (ii) $\Rightarrow$ (iii) follows from Proposition 2.4 because X and $\mathfrak{s}_p$ have the γ_p-AP and the space $\mathfrak{s}_p$ contains complemented S_p^n's far out, see Example 2.3. The implication (iii) $\Rightarrow$ (iv) is obvious because X is completely contractively complemented in $X \oplus_p \mathfrak{s}_p$. For the implication (iv) $\Rightarrow$ (i) it suffices to note that every $\mathscr{COL}_p$ space Y has the CBAP and according to Proposition 1.6 is completely complemented in some $\prod_{\mathscr{U}} S_p$. Both properties pass to completely complemented subspaces. $\square$

Similarly as for $\mathscr{COL}_p$ spaces in Proposition 2.4, we can obtain a result in the context of $\mathscr{OL}_p$ spaces.

Proposition 2.8. *Let $1\leqslant p\leqslant\infty$ and X an operator space with the γ_p-AP. If X contains S_p^n's far out, then X is an $\mathscr{OS}_p$ space.*

Proof. Let assume that X contains S_p^n's far out with constant C. Let $0<\varepsilon<(3C)^{-1}(1+2\gamma_p^{ap}(X))^{-1}$ and a finite-dimensional subspace $E\subset X$ be given. Choose $u: X\to S_p^n$ and $v: S_p^n \to X$ such that

$$\|u\|_{cb}\leqslant 1, \quad \|v\|_{cb}\leqslant (1+\varepsilon)\gamma_p^{ap}(X) \quad \text{and} \quad vu|_E = id_E.$$

Put $F=v(S_p^n)$ and apply Lemma 1.4 (ii) to find a finite rank map $T: X\to X$ such that $T|_F = id_F$ and $\|T\|_{cb}\leqslant(1+\varepsilon)\Lambda(X)\leqslant(1+\varepsilon)\gamma_p^{ap}(X)$. By the assumptions there is finite-dimensional $G\subset X$ such that

$$d_{cb}(G, S_p^n)\leqslant C \quad \text{and} \quad \|T\big|_G\|_{cb}\leqslant\varepsilon.$$

Let $w: S_p^n\to G$ be an isomorphism such that $\|w\|_{cb}\leqslant C$ and $\|w^{-1}\|_{cb}\leqslant 1$. We define $R: S_p^n\to X$ by $R = v + w(id_{S_p^n} - uv)$. Then we have $E\subset R(S_p^n)$ as in the proof of Proposition 2.4. Thus it remains to show that R is an isomorphism from S_p^n onto its range. To this end, fix an $m\in\mathbb{N}$ and a unit vector $x\in M_m(S_p^n)$. Let $\delta=\frac{\|T\|_{cb}}{1+2\|T\|_{cb}}$. Note that $\delta\geqslant\frac{1}{3}$. We consider two cases: $\|(id_{M_m}\otimes v)x\|>\delta$ or $\|(id_{M_m}\otimes v)(x)\|\leqslant\delta$. If the former occurs, then by the choice of T (with $id = id_{M_m}$)

$$\begin{aligned}||T||_{cb}||id\otimes R(x)|| &\geqslant ||(id\otimes Tv)x + (id\otimes T)(id\otimes w(id_{S_p^n} - vu))x|| \\ &\geqslant ||(id\otimes v)x|| - ||w||_{cb}(1+||v||_{cb}||u||_{cb})\varepsilon \\ &\geqslant \delta - C(1+2\gamma_p^{ap}(X))\varepsilon.\end{aligned}$$

If we are in the latter case, then

$$\begin{aligned}||(id\otimes w)(id\otimes(id_{S_p^n} - uv))x|| &\geqslant ||(id\otimes(id_{S_p^n} - uv))x|| \\ &\geqslant ||x|| - ||u||_{cb}||(id\otimes v)(x)|| \\ &\geqslant 1-\delta.\end{aligned}$$

Therefore, we get

$$\begin{aligned}||(id\otimes R)(x)|| &\geqslant ||(id\otimes w)(id\otimes(I_{S_p^n} - uv))x|| - ||(id\otimes v)x|| \\ &\geqslant 1-\delta-\delta = 1-2\delta = \frac{1}{1+2||T||_{cb}}.\end{aligned}$$

This shows that

$$\begin{aligned}||R^{-1}||_{cb} &\leqslant \max\left\{1+2||T||_{cb}, \frac{\delta}{\delta-\varepsilon C(1+2\gamma_p^{ap}(X))}\,\delta^{-1}||T||_{cb}\right\} \\ &\leqslant \frac{\delta}{\delta-\varepsilon C(1+2\gamma_p^{ap}(X))}(1+2||T||_{cb}).\end{aligned}$$

The assertion is proved and since $\varepsilon > 0$ is arbitrary, we obtain

$$\mathcal{O}\mathcal{L}_p(X) \leqslant (1+2\Lambda(X))(\gamma_p^{ap}(X) + C(1+\gamma_p^{ap}(X))).$$

The assertion is proved. $\square$

Apart from introducing the notion of containing S_p^n's far out, the main new ingredient in the proof of Theorem 2.2 is the fact that the 'far out' properties can be derived from more natural, weaker assumptions. After a first version of this paper circulated, E. Ricard considerably improved a technical lemma crucial for this kind of results. We want to thank him for the permission to publish his refinement of our result which turned out to be crucial for the final version of Theorem 4.10.

Lemma 2.9. *Let* $1 \leqslant p \leqslant \infty$ *and* $n, k, l, m \in \mathbb{N}$ *such that the integer part* $[\frac{m}{k}]$ *satisfies* $[\frac{m}{k}] > lkn^2$. *Let F be a vector space and* $T : \ell_p^m(S_p^n) \to F$ *a linear map with* $rk(T) \leqslant l$. *Then there exists a subspace* $E \subset \ell_p^m(S_p^n)$ *completely isometric to* $\ell_p^k(S_p^n)$ *and completely* *contractively complemented such that*

$$T|_E = 0.$$

Proof. We may assume that $\dim(F) = l$ and that $f_1^*, \dots, f_l^*$ are linear independent vectors in F^*. We may assume $m = vk + r$, $r < k$ and $v > lkn^2$. Let $(h_i)_{i=1}^m$ be the unit vector basis in ℓ_p^m and $(e_{st})_{1 \leqslant s,t \leqslant n}$ denotes the matrix units in S_p^n. Consider the matrix $\alpha_{u,(w,s,t,j)} = f_j^*(T(h_{uk+w} \otimes e_{st}))$ where $1 \leqslant u \leqslant v$, $1 \leqslant w \leqslant k, 1 \leqslant s, t \leqslant n$ and $1 \leqslant j \leqslant l$. Since $v > lkn^2$, there exists a non-trivial solution $(a_1, \dots, a_v)$ of scalars such that

$$\sum_{u=1}^{v} a_u \alpha_{u,(w,s,t,j)} = 0$$

for all $1 \leqslant s, t \leqslant n$, $1 \leqslant w \leqslant v$, $1 \leqslant j \leqslant l$. We may assume $||(a_u)||_p = 1$ and then

$$E = \left\{ \sum_{1 \leqslant u \leqslant v, 1 \leqslant w \leqslant k, 1 \leqslant st \leqslant n} a_u b_{wst} h_{uk+w} \otimes e_{st} | b_{wst} \in \mathbb{C} \right\}$$

is completely isometrically isomorphic to $\ell_p^k(S_p^n)$ and T vanishes on E. Using a sequence $(b_u)_{u=1}^v$ such that $\sum_u a_u b_u = 1$ and $||(b_u)||_{p'} = 1$, we see that

$$P\left(\sum_{i=1}^{m} h_i \otimes x_i\right) = \sum_{1 \leqslant u \leqslant v, 1 \leqslant w \leqslant k} a_u h_{uk+w} \otimes \left(\sum_{u'=1}^{l} b_{u'} x_{u'k+w}\right)$$

is a completely contractive projection. $\square$

The following lemma can also be proved by using Ramsey-type arguments and ultraproduct techniques (see [RX]), but our proofs based on Lemma 2.9 are significantly more elementary.

Lemma 2.10. *Let $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$ fixed and X an operator space.*

(i) *If X contains $\ell_p^k(S_p^n)$'s for all k, then X contains $\ell_p^k(S_p^n)$'s far out with $\varepsilon = 0$.*
(ii) *If X contains complemented $\ell_p^k(S_p^n)$'s, then X contains complemented $\ell_p^k(S_p^n)$'s far out with $\varepsilon = 0$.*

In particular, if X contains S_p^k's (complemented S_p^k's), then it contains S_p^k's far out, complemented S_p^k's far out, respectively.

Proof. (i) We assume that X contains $\ell_p^k(S_p^n)$'s with constant C. Let $T : X \to X$ be a finite rank map, $k \in \mathbb{N}$. Choose m such that $[\frac{m}{k}] \geqslant rk(T)kn^2$. Let $G_m \subset X$ such that $d_{cb}(G_m, \ell_p^m(S_p^n)) \leqslant C$. Let $r : \ell_p^m(S_p^n) \to G_m$ and $s : G_m \to \ell_p^m(S_p^n)$ such that $sr = id$ and $||r||_{cb} ||s||_{cb} \leqslant C$. According to Lemma 2.9, there exists a subspace $E \subset \ell_p^m(S_p^n)$

completely isometric to $\ell_p^k(S_p^n)$ such that $Tr|_E = 0$. Hence, $F = r(E)$ is C-cb-isomorphic to $\ell_p^k(S_p^n)$ and $T|_F = 0$. In order to prove (ii) we assume that X contains complemented $\ell_p^m(S_p^n)$'s with constant C. Let $F \subset X$ be a l-dimensional subspace. Let $[\frac{m}{k}] > lkn^2$. Let $u : \ell_p^m(S_p^n) \to X$ and $v : \ell_p^m(S_p^n)$ such that

$$vu = id_{\ell_p^m(S_p^n)}, \quad ||u||_{cb} \leqslant 1 \quad \text{and} \quad ||v||_{cb} \leqslant C.$$

Let $\iota_F : F \to X$. We apply Lemma 2.9 to $(v\iota_F)^* : \ell_{p'}^m(S_{p'}^n) \to F^*$ and find a completely contractively complemented copy G of $\ell_{p'}^k(S_{p'}^n)$ such that $(v\iota_F)^*|_G = 0$. Using either the proof of Lemma 2.9 or a simple duality argument, we find a completely contractive projection $Q : \ell_p^m(S_p^n) \to \ell_p^m(S_p^n)$ such that $Q(\ell_p^m(S_p^n))$ is completely isometric to $\ell_p^k(S_p^n)$ and $Qv|_F = 0$. Then, we deduce that $P = uQv$ is a projection satisfying $P|_F = 0$ and $id_{Q(\ell_p^m(S_p^n))} = Qvu$. This concludes the proof of (b). For the particular part, we only have to observe that $\ell_p^m(S_p^n)$ is completely contractively complemented in S_p^{nm}. Hence for all n the assumptions are satisfied. $\square$

Remark 2.11. In (a) and in (b), we may add 'with respect to Y' in every place.

Proof of Theorem 2.2. Combine Proposition 2.4 and Lemma 2.10 in the complemented case and Proposition 2.8 and Lemma 2.10 in the non-complemented case. $\square$

Remark 2.12. In the complemented case, we may again add 'with respect to Y' everywhere.

Corollary 2.13. *Let* $1 \leqslant p \leqslant \infty$.

(i) *Let X be a complemented subspace of a $\mathscr{COL}_p$ space containing complemented S_p^n's. Then X is a $\mathscr{COS}_p$ space.*
(ii) *Let X be an operator space with the CBAP and containing S_p^n's. If X is a complemented subspace of an $\mathscr{OL}_p$ space, then X is an $\mathscr{OS}_p$ space.*

Proof. In case (i), it suffices to note that a complemented subspace of a $\mathscr{COL}_p$ space has the γ_p-AP and thus Theorem 2.2 yields the assertion. In case (ii) again by Theorem 2.2, it remains to prove that X has the γ_p-AP. Let $X \subset Y$ such that Y is an $\mathscr{OL}_p$ space. Let $E \subset X$ be a finite-dimensional subspace and a finite rank map $T : X \to X$ such that $T|_E = id_E$ according to Lemma 1.2. Then $T(X) \subset X \subset Y$ is a finite-dimensional subspace and we can find a finite-dimensional C^*-algebra A and $T(X) \subset F \subset Y$ such that $d_{cb}(F, L_p(A)) \leqslant C$. Let $v : L_p(A) \to F$ and $u : F \to L_p(A)$ such that $u = v^{-1}$, then we deduce for the inclusion map $\iota_X : X \to Y$ that

$$\iota_X T = \iota_F vuT$$

factors through $L_p(A)$, and thus factors through S_p^m for m large enough. Let $P: Y \to X$ be a completely bounded projection. Then

$$T = P\iota_X = P\iota_F vuT$$

factors through S_p^m and X has the γ_p-AP. $\square$

Resuming Lemma 1.5, Propositions 1.6, 2.4 and 2.8, we can formulate the following result.

Theorem 2.14. *Let $1<p<\infty$ and X an operator space with the CBAP. Then,*

(i) *X is a $\mathscr{COS}_p$ space if and only if X is completely complemented in $\prod_{\mathscr{U}} S_p$ and contains complemented S_p^n's.*
(ii) *X is an $\mathscr{OS}_p$ space if and only if X is completely complemented in $\prod_{\mathscr{U}} S_p$ and contains S_p^n's.*

As mentioned above our main motivation is the investigation of non-commutative L_p spaces. Let us recall some definitions. A von Neumann algebra N is called *semifinite* if there exists a normal semifinite faithful (in short n.s.f.) trace, i.e. a positive homogeneous and additive function on $N_+ = \{x^*x \mid x \in N\}$, the cone of positive elements of N, such that for all increasing nets $(x_i)_i$ with supremum in N and for all $x \in N_+$

n. $\tau(\sup_i x_i) = \sup_i \tau(x_i)$;
s. For every $0<x$ there exists $0<y<x$ such that $\tau(y)<\infty$;
f. $\tau(x) = 0$ implies $x = 0$;
t. For all unitaries $u \in N$: $\tau(uxu^*) = \tau(x)$.

A positive homogeneous and additive function w: $N_+ \to [0, \infty]$ satisfying n.s.f. but not the last property t. is called an n.s.f. (*normal semifinite faithful*) weight. If τ is an n.s.f. trace then

$$m(\tau) = \left\{ \sum_{i=1}^{n} y_i x_i \mid n \in \mathbb{N}, \sum_{i=1}^{n} [\tau(y_i^* y_i) + \tau(x_i^* x_i)] < \infty \right\}$$

is the definition ideal on which there exists a unique linear extension $\tau : m(\tau) \to \mathbb{C}$ which satisfies $\tau(xy) = \tau(yx)$. The L_p-norm is defined for $x \in m(\tau)$ and $1 \leqslant p < \infty$ by

$$\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}.$$

Then $L_p(N, \tau)$ is the completion of $m(\tau)$ with respect to the L_p-norm. For two faithful traces τ_1 and τ_2 on N, we can find an element d affiliated with the center of N such that $\tau_1(x) = \tau_2(dx)$. Thus the space $L_p(N, \tau_1)$ and $L_p(N, \tau_2)$ are (completely) isometrically isomorphic. Therefore, we will often use the notation $L_p(N)$ for this (class of) operator space(s). We use the convention $L_\infty(N, \tau) = N$. We refer to

[D1,FK,Ku,Ne,Se,Te] for more on this and for information on the topological algebra of τ-measurable operators affiliated with N in which all the spaces $L_p(N,\tau)$ embed topologically. It is well-known that the complex interpolation method yields

$$L_p(N,\tau) = [N, L_1(N,\tau)]_{\frac{1}{p}}.$$

Here $L_1(N,\tau)$ inherits the natural operator space structure from N_*^{op} via the map

$$\beta(x)(y) = \tau(yx).$$

(Note that N and N^{op} coincide as Banach spaces.) Then, we have

$$\begin{aligned} \|[\beta(x_{ij})]\|_{S_1^n \widehat{\otimes} N_*^{\mathrm{op}}} &= \sup_{\|[y_{ij}]\|_{M_n(N^{\mathrm{op}})} \leqslant 1} \left| \sum_{ij} \tau(x_{ij} y_{ij}) \right| \\ &= \sup_{\|[y_{ij}]\|_{M_n(N)} \leqslant 1} \left| \sum_{ij} \tau(x_{ij} y_{ji}) \right| = \|[x_{ij}]\|_{L_1(M_n \otimes N, tr_n \otimes \tau)}. \end{aligned}$$

Here tr_n denotes the non-normalized trace on M_n. The complex interpolation (as explained in the first section for the finite-dimensional case) defines the natural operator space structure

$$M_n(L_p(N,\tau)) = [M_n(L_\infty(N,\tau)), M_n(L_1(N,\tau))]_{\frac{1}{p}}$$

on $L_p(M,\tau)$. Following [P5, Corollary 1.4 and Lemma 1.7], we obtain the following formula:

$$\|x\|_{M_n(L_p(N,\tau))} = \sup_{\|a\|_{S_{2p}^n} \leqslant 1, \|b\|_{S_{2p}^n} \leqslant 1} \|(a \otimes 1_N) x (b \otimes 1_N)\|_{L_p(M_n \otimes N, tr_n \otimes \tau)}. \tag{2.4}$$

Note that these formulas slightly differ from [Fi] but are more consistent with [P5]. In particular, for every linear map $T : L_p(N_1,\tau_1) \to L_p(N_2,\tau_2)$, we deduce

$$\|T\|_{cb} = \|id \otimes T : L_p(B(\ell_2) \otimes N_1, tr \otimes \tau_1) \to L_p(B(\ell_2) \otimes N_2, tr \otimes \tau_2)\|. \tag{2.5}$$

This shows, as it should be, that the cb-norm can be obtained by replacing scalars with matrix-valued coefficients. A corresponding formula also holds for maps defined on a subspace of $L_p(N_1,\tau_1)$. Let us note that if N admits a central decomposition $N = N_1 \oplus N_2$, then we have a direct sum

$$L_p(N,\tau) = L_p(N_1,\tau) \oplus_p L_p(N_2,\tau). \tag{2.6}$$

Indeed, every matrix $x \in L_p(B(\ell_2) \otimes N)$ has two components $x_1 \in L_p(B(\ell_2) \otimes N_1)$ and $x_2 \in L_p(B(\ell_2) \otimes N_2)$ satisfying

$$\|x\|_p^p = \|x_1\|_p^p + \|x_2\|_p^p.$$

According to [P5] this can be retranslated in terms of $M_n(L_p(N,\tau))$ and is then equivalent to the usual definition of the operator space $X_1 \oplus_p X_2 = [X_1 \oplus_\infty X_2, X_1 \oplus_1 X_2]_{\frac{1}{p}}$.

The definition of non-commutative L_p spaces in the non-semifinite case is more involved. If $\mathcal{N}$ is σ-finite, one can use Kosaki's approach via interpolation [Ko]. However, the Haagerup (or Connes) L_p spaces (see [Ha4] and [C3]) provide the most general and algebraically striking presentation. For the operator space structure of Haagerup's abstract L_p spaces including the non-semifinite case, we use as above the natural operator space structure on $L_1(N)$ defined by the map $\beta : L_1(N) \to N_*^{\mathrm{op}}$ given by $\beta(D)(x) = tr(Dx)$ and then interpolation. We refer to [Ju2,JRX] for more details. Since the type decomposition $N = N_{\mathrm{I}} \oplus N_{\mathrm{II}} \oplus N_{\mathrm{III}}$ comes with central projections, we obtain a direct ℓ_p-sum on the level of L_p space:

$$L_p(N) = L_p(N_{\mathrm{I}}) \oplus_p L_p(N_{\mathrm{II}}) \oplus_p L_p(N_{\mathrm{III}}).$$

However, in this paper we focus on the semifinite case and only mention the more general situation in passing. The letter $\mathcal{R}$ will be reserved for the hyperfinite II_1 factor defined as the σ-weak closure of the infinite tensor product $\tau_{\mathcal{R}} = \otimes_{n\in\mathbb{N}} M_2$ in the GNS-construction with respect to the tracial state $\tau = \otimes_{n\in\mathbb{N}} \frac{tr}{2}$ (see [KR]).

Example 2.15. Let $\mathcal{R}$ be the hyperfinite II_1 factor and $1 \leqslant p < \infty$. Then $\mathfrak{s}_p$ is completely contractively complemented in $L_p(R, \tau_{\mathcal{R}})$. Consequently, $L_p(\mathcal{R}, \tau_{\mathcal{R}})$ contains complemented S_p^n's far out.

Proof. Since the hyperfinite II_1 factor is unique (see [KR]) $\mathcal{R} \otimes \mathcal{R}$ is isomorphic to $\mathcal{R}$. On the other hand, $\mathcal{R}$ clearly contains $L_\infty(\{-1,1\}^{\mathbb{N}})$ on the diagonal and the trace induces the Haar measure μ on $\{-1,1\}^{\mathbb{N}}$. Let $(A_n)_{n\in\mathbb{N}}$ be a family of disjoint measurable sets of positive measure in $\{-1,1\}^{\mathbb{N}}$. Then $(\chi_{A_n} \otimes 1_{\mathcal{R}})_{n\in\mathbb{N}}$ is a family of mutually orthogonal non-zero projections of $\mathcal{R} \otimes \mathcal{R}$. Using the isomorphism between $\mathcal{R} \otimes \mathcal{R}$ and $\mathcal{R}$, we deduce that there is a family $(e_n)_n$ of mutually orthogonal non-zero projections of $\mathcal{R}$ such that $e_n \mathcal{R} e_n \cong \mathcal{R}$ and such that the w^*-closed $*$-subalgebra M of $\mathcal{R}$ generated by $\bigcup_n e_n \mathcal{R} e_n$ is isomorphic to $\ell_\infty(\mathcal{R})$. Then $L_p(M)$ is completely contractively complemented in $L_p(\mathcal{R})$ and

$$L_p(M) = \ell_p(L_p(\mathcal{R})).$$

Then for all $n \in \mathbb{N}$, we can find a copy $M_n \cong N_n \subset \mathcal{R}$. Moreover, the restriction of the trace τ to M_n is the normalized $\frac{tr}{n}$ on M_n and there is a conditional expectation $\mathcal{E}_n : \mathcal{R} \to M_n$. Let us note that in this case the operator space structure of $M_n.\tau \subset L_1(R,\tau)$ is given for a matrix $[x_{ij}]_{i,j=1}^m \subset M_n.\tau$ by

$$\begin{aligned} \|[x_{ij}]\|_{S_1^m \widehat{\otimes} L_1(R,\tau)} &= \|[x_{ij}]\|_{L_1(M_m \otimes R, tr_m \otimes \tau)} = \|[x_{ij}]\|_{L_1(M_m \otimes M_n, tr_m \otimes \frac{tr}{n})} \\ &= \frac{1}{n} \|[x_{ij}]\|_{L_1(M_m \otimes M_n, tr_m \otimes tr)} = \frac{1}{n} \|x_{ij}\|_{S_1^m \widehat{\otimes} S_1^n}. \end{aligned}$$

Hence by interpolation $u_n : S_p^n \to L_p(R, \tau)$, $u_n(x) = n^{\frac{1}{p}} x.\tau$ is a completely isometry. Again by interpolation $\mathscr{E}_n : L_p(\mathscr{R}) \to L_p(\mathscr{R})$ provides a completely contractive projection onto the range of u_n. Therefore, $\mathfrak{s}_p = (\sum_n \oplus S_p^n)_p \subset \ell_p(L_p(\mathscr{R}))$ is completely contractively complemented in $L_p(\mathscr{R}, \tau)$. $\square$

It is known that every non-type I von Neumann algebra N contains a copy of the hyperfinite II_1 factor together with a nice conditional expectation (see [Ma]). Hence $L_p(\mathscr{R})$ is completely isometrically isomorphic to a completely contractively complemented subspace of $L_p(N)$. Therefore, in combination with Example 2.3, the 'far out' property is never a problem for L_p spaces over a von Neumann algebra.

Lemma 2.16. *Let N be a von Neumann algebra and $1 \leqslant p \leqslant \infty$ with type decomposition $N = N_{\mathrm{I}} \oplus N_{\mathrm{II}} \oplus N_{\mathrm{III}}$. If $N_{\mathrm{II}} \neq \{0\}$ or $N_{\mathrm{III}} \neq \{0\}$, then $L_p(\mathscr{R})$ is completely isometric to a completely contractively complemented subspace of $L_p(N)$. In particular, $\mathfrak{s}_p$ is completely isometric to a completely contractively complemented subspace of $L_p(N)$ and $L_p(N)$ contains complemented S_p^n's far out with constant one.*

In the formulation of the following theorem we use the abstract Haagerup L_p spaces $L_p(N)$ and its natural operator space structure. For most of the applications in this paper it would be sufficient to consider the semifinite case but the proof is verbatim the same even if we include the more general setting of type III algebras.

Theorem 2.17. *Let N be a QWEP von Neumann algebra and $1 < p < \infty$. Then the following are equivalent*

(i) *$L_p(N)$ has the CBAP*;
(ii) *$L_p(N)$ is a $\mathscr{COL}_p$ space.*

Proof. Let $N = N_{\mathrm{I}} \oplus N_{\mathrm{II}} \oplus N_{\mathrm{III}}$ be the central decomposition of N into types I–III summands (see [Ta]). Then $L_p(N) = L_p(N_{\mathrm{I}}) \oplus_p L_p(N_{\mathrm{II}} \oplus N_{\mathrm{III}})$ and $L_p(N_{\mathrm{I}})$ is the ℓ_p-sum of L_p spaces $L_p(\Omega, \Sigma, \mu; L_p(B(H)))$. It is easy to see that $L_p(N_{\mathrm{I}})$ is $\mathscr{COL}_p$ (see e.g. Example 5.4 below). Thus it suffices to consider $N = N_{\mathrm{II}} \oplus N_{\mathrm{III}}$. Since N has the QWEP, we deduce from [Ju2] that $L_p(N)$ is completely contractively complemented in $\prod_{\mathscr{U}} S_p$. If $L_p(N)$ has the CBAP, then we deduce from Proposition 1.6, that X has the γ_p-AP. According to Lemma 2.16 $L_p(N)$ contains complemented S_p^n's far out. From Proposition 2.4 we infer that $L_p(N)$ is a $\mathscr{COL}_p$ space. Conversely, a $\mathscr{COL}_p$ space obviously has the CBAP. $\square$

Let us conclude this section with a list of open problems concerning $\mathscr{OL}_p$ and $\mathscr{COL}_p$ spaces. As mentioned in the introduction, the Kadec–Pełczyński dichotomy (see [KP]) provides the equivalence between the usual definition of $\mathscr{L}_p$ spaces and its complemented version. Although some deep work of Arazy [A2] is done in the discrete case $N = B(\ell_2)$, at the time of this writing it seems unclear how to prove a suitable substitute of the Kadec–Pełczyński dichotomy in the non-commutative setting. We refer to [RX] for further discussions and a possible formulation in the category of Banach spaces. Our main open problem is the equivalence between $\mathcal{OL}_p$ and $\mathcal{COL}_p$.

Problem 2.18. Let $1<p<\infty$. Is every $\mathcal{OL}_p$ space a $\mathcal{COL}_p$ space?

The missing link between these notions seems to be the CBAP. In contrast to the commutative L_p space theory, we will see that there are non-commutative L_p spaces without Grothendieck's approximation property and thus without the CBAP. In fact, our argument uses a deep result of Szankowski [Sz2] on the uniform approximation property of S_p.

Theorem 2.19. *Let $p>80$. Then there exists a finite von Neumann algebra N with separable predual such that $L_p(N,\tau)$ does not have Grothendieck's approximation property.*

Proof. Let $1<p<\infty$. We will prove the assertion by contradiction. Let us assume to the contrary that every finite M with separable predual $L_p(M,\tau)$ has the AP.

Claim I. *For every (semi-) finite von Neumann algebra N, $L_p(N,\tau)$ has the metric approximation property (MAP).* Indeed, for every finite M with separable predual, $L_p(M,\tau)$ is a separable dual space and thus by a result of Grothendieck (see [LT, Theorem 1.e.15]) the assumption implies that $L_p(M,\tau)$ even has the MAP. Since $L_p(N,\tau)$ is a direct limit of completely contractively complemented subspaces $L_p(M,\tau)$ with M finite and M_* separable, claim I is proved.

Claim II. *For every von Neumann algebra N with a normal faithful state, $L_p(N)$ has the MAP.* Indeed, according to an unpublished result of Haagerup [Ha1], $L_p(N)$ is completely contractively complemented in $L_p(\mathcal{M})$ such that $\mathcal{M}$ is the strong closure of complemented finite von Neumann algebras $\mathcal{M}_k \subset \mathcal{M}$ admitting in addition normal conditional expectations $E_k : \mathcal{M} \to \mathcal{M}_k$. Since $\mathcal{M}_k$ is finite, we have the *MAP* for all $L_p(\mathcal{M}_k)$ by claim I. Using the conditional expectation E_k, we see that $L_p(\mathcal{M}_k)$ is contained in $L_p(\mathcal{M})$. Then $L_p(\mathcal{M})$ has the *MAP* because $\bigcup_k L_p(\mathcal{M}_k)$ is norm dense. By complementation $L_p(N)$ has the *MAP*.

Claim III. *For every von Neumann algebra N, $L_p(N)$ has the MAP.* Indeed, let N be an arbitrary von Neumann algebra. Then N is a strong limit $N = \lim_i N_i$ of σ-finite von Neumann subalgebras N_i admitting normal conditional expectations, see the appendix in [GGMS]. Since σ-finite von Neumann algebras admit a normal faithful state, we deduce from claim II that for all i, $L_p(N_i)$ has the *MAP*. Thus the direct limit $L_p(N)$ also has the *MAP*.

Claim IV. *$\prod_{\mathcal{U}} S_p$ has the MAP for all ultrafilters $\mathcal{U}$.* Indeed, according to a result of Raynaud [Ra2] there exists a von Neumann algebra $N_{\mathcal{U}}$ such that $L_p(N_{\mathcal{U}}) = \prod_{\mathcal{U}} S_p$. Thus $\prod_{\mathcal{U}} S_p = L_p(N_{\mathcal{U}})$ has the *MAP* by Claim III.

Conclusion. According to a result of Heinrich [He], $\prod_{\mathcal{U}} S_p$ has the bounded approximation property (BAP) for some non-trivial $\mathcal{U}$ if and only if S_p has the uniform approximation property. However, Szankowski [Sz2] proved that this is not true for $p > 80$, a contradiction to claim IV. Therefore, there exists a finite von Neumann algebra with separable dual such that $L_p(M, \tau)$ does not has the approximation property. $\square$

The following problems are still open.

Problems 2.20. *Let $1 < p < \infty$.*

(i) *Do $\mathcal{OL}_p$ spaces have the CBAP?*
(ii) *Can the assumption CBAP be dropped in Corollary* 2.13(ii)?
(iii) *Does X $\mathcal{OL}_p$ and X CBAP imply that X is $\mathcal{COL}_p$?*
(iv) *Is $\prod_{\mathcal{U}} S_p$ an $\mathcal{OL}_p$ space?*

We note that a positive answer to (iv) would provide a negative answer to (i) for $p > 80$.

3. Duality for $\mathcal{COL}_p$-spaces

In this section, we follow Johnson et al. [JRZ] in order to find finite-dimensional decompositions of $\mathcal{COL}_p$ spaces and duality results. The notion of local reflexivity will be a crucial tool. Let us recall that an operator space X is *locally reflexive* (in the operator space sense) if there exists a constant $C > 0$ such that for every finite-dimensional operator space F, every finite-dimensional subspace $L \subset X^*$ and every linear map $u : F \to X^{**}$ there is a map $v : F \to X$ such that for all $f \in F$ and $y \in L$

$$\langle u(f), y\rangle = \langle v(f), y\rangle \quad \text{and} \quad \|v\|_{cb} \leqslant C\|u\|_{cb}.$$

Then $lcr(X) = \inf C$, where the infimum is taken over all constants above. In the operator space category this notion goes back to [AB]. In Banach space theory, every Banach space is locally reflexive (see [LR]). However, $B(\ell_2)$ is not locally reflexive in the operator space sense. We will start with an adaptation of a well-known application of local reflexivity in the category of operator spaces.

Lemma 3.1. *Let X be a locally reflexive operator space, $T : X^* \to X^*$ a finite rank map, $F \subset X^*$ a finite-dimensional subspace and $\varepsilon > 0$. Then there exists a finite rank* *map $S: X \to X$ such that*

$$Im(S^*) = Im(T), \quad \|S\|_{cb} \leqslant (1+\varepsilon) lcr(X) \|T\|_{cb} \quad \text{and} \quad S^*(f) = T(f)$$

for all $f \in F$. Moreover, if T is a projection, we can find a projection S with these properties.

Proof. Let $L = T(X^*)$ and $F' \supset F$ a finite-dimensional subspace of X^* such that $T(F') = L$. Consider $T: X^* \to L$ and $T^*: L^* \to X^{**}$. Applying the local reflexivity, we can find $S_1: L^* \to X$ such that

$$\langle S_1(l^*), f \rangle = \langle T^*(l^*), f \rangle$$

for all $l^* \in L^*$ and $f \in F'$, and satisfying the corresponding cb-norm estimate. Since S_1^* takes its values in the finite-dimensional subspace $L \subset X^*$, this implies $S_1^*|_{F'} = T|_{F'}$ and in particular, $S_1^*(X^*) = T(X^*) = L$. Let

$$q: X \to X/L_\perp \cong X^{**}/L^\perp \cong L^*$$

be the natural quotient map. Then $q^*: L \to X^*$ is the natural inclusion map and $S = S_1 q$ satisfies the assertion. If T is in addition a projection onto L, then $S_1^* q^* = Tq^* = id_L$ implies

$$S^* S^* = q^* S_1^* q^* S_1^* = q^* S_1^*.$$

Hence $(S^2)^* = S^*$ and therefore $S^2 = S$. $\square$

The following Lemmas 3.2, 3.3 and Corollary 3.4 are the operator space analogues of Lemmas 4.2, 4.3 and the corresponding Corollaries in [JRZ]. They provide the main technical tools for constructing bases in $\mathscr{COL}_p$ spaces.

Lemma 3.2. *Let X be a locally reflexive operator space, $Y \subset X^*$ a subspace and (F_i) a family of finite-dimensional operator spaces such that there are linear maps $r_i: F_i \to X$ and $s_i: X \to F_i$ with $s_i r_i = id_{F_i}$, $s_i^*(F_i^*) \subset Y$, $(r_i s_i)$ converging to the identity on X in the point-norm topology and*

$$\sup_i \|r_i\|_{cb} \|s_i\|_{cb} \leqslant C_1.$$

If $T_j: X^* \to Y$ is a net of finite rank maps such that $T_j|_Y$ converges in the point-norm topology to id_Y and

$$\sup_j \|T_j\|_{cb} \leqslant C_2,$$

then for every finite-dimensional subspace $E \subset X$, every finite-dimensional subspace $L \subset Y$ and $0 < \varepsilon < 1$, there exist an index $i \in I$, $r: F_i \to X$ and $s: X \to F_i$ such that

$sr = id_{F_i}$ and the projection $Q = rs : X \to X$ satisfies

(i) $Q(x) = x$ *and* $Q^*(y) = y$ *for all* $x \in E$, $y \in L$;
(ii) $Q^*(X^*) \subset Y$;
(iii) $\|Q\|_{cb} \leqslant \|r\|_{cb}\|s\|_{cb} \leqslant (1+\varepsilon)C_1(C_1 + C_2 lcr(X) + C_1C_2 lcr(X))$.

Proof. Let $n = \dim(L)$ and $y_1, \ldots, y_n$ be part of a biorthogonal system for L. Since T_j converges to id_Y in the point norm topology, we can find j such that

$$\|T_j(y_k) - y_k\| \leqslant \frac{\varepsilon}{2n}$$

for all $k = 1, \ldots, n$. According to Lemma 1.2, we can find $W : Y \to Y$ such that the finite rank map $T = WT_j : X^* \to Y$ satisfies $T|_L = id_L$ and $\|T\|_{cb} \leqslant (1+\varepsilon)C_2$. Applying Lemma 3.1, we can find a finite rank map $S : X \to X$ such that $S^*(X^*) = T(X^*) \subset Y$, $S^*(y) = T(y)$ for all $y \in T(X^*)$ and

$$\|S\|_{cb} \leqslant (1+\varepsilon)^2 lcr(X)C_2.$$

In particular, $S^*(y) = y$ for all $y \in L$. We now put $G = \operatorname{span}(E \cup S(X))$ and apply Corollary 1.3 to find an index $i \in I$ and $r : F_i \to X$, $s : X \to F_i$ such that $sr = id_{F_i}$, $s^*(F_i) \subset Y$,

$$\|r\|_{cb}\|s\|_{cb} \leqslant (1+\varepsilon)C_1$$

and $rs(x) = x$ for $x \in G$. Then, we obtain a projection $P = rs$ with

$$\|P\|_{cb} \leqslant \|s\|_{cb}\|r\|_{cb} \leqslant (1+\varepsilon)C_1.$$

Define $Q = S(id_X - P) + P$. Let us check that Q is a projection. Since $Im(S) \subset G$, we have $PS = S$ and thus

$$Q^2 = P(S(id_X - P) + P) = S(id_X - P) + P = Q.$$

Clearly, $Q(X) \subset P(X)$. Moreover, for $x \in P(X)$ we have $Q(x) = x$. This shows that $P(X) = Q(X)$ and hence $Qr = r$. Let us define $\tilde{s} = sQ$. Then we get

$$\tilde{s}r = sQr = sr = id_{F_i}.$$

Moreover, from $P(X) = Q(X)$ we deduce

$$r\tilde{s} = rsQ = PQ = Q.$$

For the norm estimate (iii), we observe

$$\begin{aligned}||r||_{cb}||\tilde{s}||_{cb} &\leqslant (1+\varepsilon)||r||_{cb}||s||_{cb}||S(id_X - P) + P||_{cb}\\ &\leqslant (1+\varepsilon)C_1(||S||_{cb}(1+||P||_{cb}) + ||P||_{cb})\\ &\leqslant (1+\varepsilon)^4 C_1(lcr(X)C_2(1+C_1)+C_1).\end{aligned}$$

To show (i), we consider $x \in E$, then

$$Q(x) = S(id_X - P)(x) + P(x) = P(x) = x.$$

Let $y \in L$, then $S^*(y) = y$ implies

$$Q^*(y) = S^*(y) + P^*(y) - P^*S^*(y) = y + P^*(y) - P^*(y) = y.$$

Since $P^* = s^*r^*$ has its image in Y, we also obtain $Q^*(X^*) \subset Y$. This shows (ii) and the assertion is proved. $\square$

The following analogue of [JRZ, Lemma 4.3] will be proved similarly.

Lemma 3.3. *Let X be a locally reflexive operator space and let $Y \subset X^*$ be a subspace. Assume that there exist a net of finite-dimensional operator spaces (F_i) and nets of maps $r_i : F_i \to Y$ and $s_i : X^* \to F_i$ such that $s_i r_i = id_{F_i}$ and $r_i s_i|_Y$ converges in the point-norm topology to id_Y and*

$$\sup_i ||s_i||_{cb}||r_i||_{cb} \leqslant C_1.$$

If $T_j : X \to X$ is a net of finite rank maps converging in the point-norm topology to id_X such that $T_j^*(X^*) \subset Y$ and

$$\sup_j ||T_j||_{cb} \leqslant C_2,$$

then for every finite-dimensional subspace $E \subset X$, every finite-dimensional subspace $L \subset Y$ and $0 < \varepsilon < 1$ there exist an index $i \in I$, $r : F_i^* \to X$ and $s : X \to F_i^*$ with $sr = id_{F_i^*}$ such that the projection $Q = rs : X \to X$ satisfies

(i) *$Q(x) = x$ and $Q^*(y) = y$ for all $x \in E$, $y \in L$;*
(ii) *$Q^*(X^*) \subset Y$;*
(iii) *$||Q||_{cb} \leqslant ||r||_{cb}||s||_{cb} \leqslant (1+\varepsilon)C_1 lcr(X)(C_1 lcr(X) + C_2 + C_1C_2 lcr(X))$.*

Proof. For a finite-dimensional subspace $E \subset X$, we apply Lemma 1.2 and the assumption to obtain a finite rank map $T : X \to X$ with $||T||_{cb} \leqslant (1+\varepsilon)C_2$, $T^*(X^*) \subset Y$ and $T|_E = id_E$. Let $G = \operatorname{span}(L \cup T^*(X^*)) \subset Y$ and apply Corollary 1.3

to find $r : F_i \to Y$, $s : X^* \to F_i$ such that $sr = id_{F_i}$, $rs|_G = id_G$,

$$||s||_{cb} \leqslant 1 \quad \text{and} \quad ||r||_{cb} \leqslant (1+\varepsilon)C_1.$$

Let us denote $G' = \text{span}(G \cup r(F))$ and apply the local reflexivity of X to find $v : F_i^* \to X$ such that $v^*(x^*) = s(x^*)$ for all $x^* \in G'$ and

$$||v||_{cb} \leqslant (1+\varepsilon) lcr(X).$$

In particular, $v^* r = id_{F_i}$. We define $P = vr^*|_X : X \to X$ and observe that $P^* = rv^*$ is a projection on X^*. Thus P is a projection on X. Let us note that

$$P^*|_G = rv^*|_G = rs|_G = id_G.$$

We define $Q = T + P - PT$ and deduce from $T^*(X^*) \subset G$

$$[Q^*]^2 = P^* T^*(id_{X^*} - P^*) + P^* P^* = Q^*.$$

Hence Q is a projection on X. Let $x \in E$, then

$$Q(x) = T(x) + P(x) - P(T(x)) = T(x) = x.$$

As above in the proof of Lemma 3.2, we obtain $Q^*(x^*) = x^*$ for all $x^* \in G$ and (i) is proved. In particular, we deduce $Q^* r = r$ and hence

$$v^* Q^* r = v^* r = id_{F_i}.$$

By duality we get $r^* Qv = id_{F_i^*}$. We put $w = Qv$ and observe from $T^*(X^*) \subset G$

$$(wr^*|_X)^* = rv^* Q^* = P^*[T^* + P^* - T^* P^*] = Q^*$$

and hence $wr^*|_X = Q$. Since $Q^*(X^*) \subset P^*(X^*) = r(F_i) \subset Y$, we have proved (ii). Finally, we deduce the norm estimate

$$\begin{aligned} ||r^*||_{cb} ||w||_{cb} &\leqslant (1+\varepsilon)^2 C_1 lcr(X)(||(id_X - P)T||_{cb} + ||P||_{cb}) \\ &\leqslant (1+\varepsilon)^4 C_1 lcr(X)(C_2(1 + lcr(X)C_1) + lcr(X)C_1). \qquad \square \end{aligned}$$

Corollary 3.4. *Let X be an operator space which satisfies the assumptions of Lemma* 3.2 (*respectively*, *Lemma* 3.3). *Then for every finite rank map $T : X \to X$ with $T(X^*) \subset Y$ and $\varepsilon > 0$ there exist a projection $Q : X \to X$ such that $QT = TQ = T$, $Q^*(X^*) \subset Y$, and an index i with $r : F_i \to X$, $s : X \to F_i$, (respectively, $r : F_i^* \to X$, $s : X \to F_i^*$) such that $Q = rs$ and*

$$||Q||_{cb} \leqslant ||r||_{cb} ||s||_{cb} \leqslant (1+\varepsilon) C_1 (C_1 + C_2 lcr(X) + C_1 C_2 lcr(X))$$

(*respectively*,

$$\|Q\|_{cb} \leqslant \|r\|_{cb}\|s\|_{cb} \leqslant (1+\varepsilon)C_1 lcr(X)(C_1C_2 lcr(X) + C_2 + C_1C_2 lcr(X))).$$

If in addition X is separable, we can find a sequence (Q_n) of projections converging in the point-norm topology to id_X satisfying the same norm estimates and

$$Q_nQ_{n+1} = Q_{n+1}Q_n = Q_n$$

for all $n \in \mathbb{N}$. If in addition Y is separable, (Q_n) can be chosen to satisfy also

$$\lim_n Q_n^*(y) = y$$

for all $y \in Y$.

Proof. Let $E = T(X)$ and $L = T^*(X^*) \subset Y$. According to Lemma 3.2 (respectively, Lemma 3.3), we can find a projection $Q : X \to X$ with $Q|_E = id_E$ and $Q^*|_L = id_L$ and the corresponding factorization properties. Then we have

$$QT = T \quad \text{and} \quad Q^*T^* = T^*.$$

Hence $TQ = (Q^*T^*)^* = T$. If X is separable, we may consider a dense sequence (x_n) in X and use Lemma 3.2 (respectively Lemma 3.3) in order to obtain an increasing sequence of finite-dimensional spaces $E_n \subset X$, $L_n \subset Y$ and $Q_n = r_ns_n$ such that $x_n \in E_n$ and

$$Q_n|_{E_n} = id_{E_n}, \quad Q_n^*|_{L_n} = id_{L_n}, \quad Q_n(X) \subset E_{n+1} \quad \text{and} \quad Q_{n+1}^*(X^*) \subset L_{n+1}.$$

Then, we deduce

$$Q_{n+1}Q_n = Q_n \quad \text{and} \quad Q_nQ_{n+1} = Q_n.$$

If in addition Y is separable, we may choose a dense sequence $(y_n) \subset Y$ and achieve $Q_n^*(y_n) = y_n$ for all n. The norm estimates follow from Lemma 3.2 (respectively Lemma 3.3). □

Theorem 3.5. *Let $1 \leqslant p, p' \leqslant \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and X a $\mathscr{COL}_p$ space. For $p = \infty$ assume in addition that X is locally reflexive. If X^* has the CBAP, then X^* is a $\mathscr{COL}_{p'}$ space and*

$$\mathscr{COL}_{p'}(X^*) \leqslant \mathscr{COL}_p(X)(\mathscr{COL}_p(X) + \Lambda(X^*)lcr(X) + \mathscr{COL}_p(X)\Lambda(X^*)lcr(X)).$$

Proof. Let X be a $\mathscr{COL}_p$ space. Note that for $1 < p < \infty$, we deduce from Lemma 1.5 that X is reflexive. For $p = 1$, we note that a $\mathscr{COL}_1$ space is completely isomorphic to a subspace of an ultraproduct $\prod_{\mathscr{U}} S_1$ (see the proof of Proposition 1.6). Since $(\prod_{\mathscr{U}} S_1)^*$ is a von Neumann algebra [Gr], we deduce from [EJR] that $\prod_{\mathscr{U}} S_1$ and thus X are both locally reflexive. Therefore, we can assume that X is locally reflexive for all $1 \leqslant p \leqslant \infty$. If X^* has the CBAP, we can apply Lemma 3.2 to $Y = X^*$ and find for every finite-dimensional subspace $L \subset X^*$ a finite-dimensional C^*-algebra A and $r = r_L : L_p(A) \to X$, $s = s_L : X \to L_p(A)$ such that $sr = id_{L_p(A)}$ and $s^*r^*|_L = id_L$. Hence we get $r^*s^* = id_{L_{p'}(A)}$. The net $((r_L), (s_L))_L$ of such maps indexed by the finite-dimensional subspaces of X^* yields the assertion. The estimate on $\mathscr{COL}_{p'}(X^*)$ follows from Lemma 3.2. $\square$

The next lemma is well-known in Banach space theory.

Lemma 3.6. *Let X be a locally reflexive operator space, $Y \subset X^*$ a subspace and (T_i) a net of finite rank maps $T_i : X^* \to Y$ such that $T_i|_Y$ converges in the point-norm topology to id_Y, T_i converges in the point-weak* topology to id_{X^*}, and*

$$\sup_i \|T_i\|_{cb} < C.$$

Then there exists a net of finite rank maps $S_i : X \to X$ such that S_i converges in the point-norm topology to id_X, $S_i^*|_Y$ converges in the point-norm topology to id_Y, $S_i^*(X^*) \subset Y$ for all $i \in I$, and

$$\sup_i \|S_i\|_{cb} \leqslant lcr(X)C.$$

Proof. Given $\varepsilon > 0$ and finite-dimensional subspaces $L \subset Y$, $F \subset X^*$, $E \subset X$, we can apply Lemma 3.1 to find $S = S_{L,F,E,\varepsilon} : X \to X$ such that $\|S\|_{cb} \leqslant lcr(X)C$, $S^*(X^*) \subset Y$ and

$$\|S^*(y) - y\| \leqslant \varepsilon \|y\| \quad \text{and} \quad |\langle x^*, S(x) - x \rangle| \leqslant \varepsilon \|x\| \, \|x^*\|$$

for all $y \in L$, $x^* \in F$ and $x \in E$. Then the new net $(S_{L,F,E,\varepsilon})$ converges in the point-weak topology to id_X and $(S^*_{L,F,E,\varepsilon}|_Y)$ converges in the point-norm topology to id_Y. A net (S_α) of convex combinations of these maps converges in the point-norm topology to id_X and the dual net $(S^*_\alpha|_Y)$ still converges in the point-norm topology to id_Y. $\square$

Corollary 3.7. *Let X be a locally reflexive operator space such that X^* has the CBAP, then X has the CBAP.*

Proof. Apply Lemma 3.6 to $Y = X^*$ and $C = \Lambda(X^*)$. $\square$

The case $p = 1$ in the following theorem seems to be particularly interesting.

Theorem 3.8. *Let $1 \leqslant p, p' \leqslant \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and X^* a $\mathscr{COL}_p$ space. If $p = 1$ assume in addition that X is locally reflexive. Then X is a $\mathscr{COL}_{p'}$ space and*

$$\mathscr{COL}_p(X) \leqslant \mathscr{COL}_{p'}(X^*)^2 lcr(X)^2 (2 + \mathscr{COL}_{p'}(X^*)).$$

Proof. Let us consider $1<p<\infty$ first. Then every $\mathscr{COL}_p$ is an $\mathscr{OL}_p$ space and hence reflexive according to Lemma 1.5. If $p=\infty$ and X^* is a $\mathscr{COL}_\infty$ space, we see that the identity on X^* factors through an ultraproduct $\prod_{\mathscr{U}} A_n$, where A_n are finite-dimensional C^*-algebras. Then the inclusion map $i: X \to X^{**}$ factors through $(\prod_{\mathscr{U}} A_n)^*$ which is the predual of a von Neumann algebra. Since preduals of von Neumann algebras are locally reflexive [EJR], we deduce that X is locally reflexive. Thus either by the argument above or by assumption, we may assume that X is locally reflexive. Obviously, X^* has the CBAP. Then X has the CBAP according to Corollary 3.7. Hence, Lemma 3.3 applies for $Y = X^*$ and yields the assertion. $\square$

Let us continue with immediate applications of Theorem 3.8.

Corollary 3.9. *Let $1<p,p'<\infty$ with $\frac{1}{p}+\frac{1}{p'}=1$ and X an operator space. Then X is a $\mathscr{COL}_p$ space if and only if X^* is a $\mathscr{COL}_{p'}$ space.*

Proof. According to Lemma 1.5 every $\mathscr{OL}_p$ and in particular every $\mathscr{COL}_p$ space is reflexive (and thus locally reflexive). Since X has the CBAP, we deduce that $X = X^{**}$ has the CBAP and therefore Corollary 3.7 implies that X^* has the CBAP. According to Theorem 3.5 X^* is a $\mathscr{COL}_{p'}$ space. Conversely, if X^* is a $\mathscr{COL}_{p'}$ space, we can apply what we just proved to deduce that X^* is reflexive and $X = X^{**}$ is a $\mathscr{COL}_p$ space. $\square$

Proposition 3.10. *Let N be a hyperfinite von Neumann algebra. Then N_* is a $\mathscr{COL}_1$ space with $\mathscr{COL}_1(N_*)=1$.*

Proof. First let us assume that M is a hyperfinite von Neumann algebra that acts on a separable Hilbert space. By the result in [ER1], we see that M_* is a rigid $\mathscr{OL}_1$ space, i.e. there exists a dense family of complete isometric copies of finite-dimensional non-commutative L_1 spaces. Now, we consider an arbitrary N. In the Appendix of [GGMS] Haagerup showed that for every separable subspace $F \subset N_*$, there is a countably generated von Neumann subalgebra $M \subset N$ with a normal conditional expectation $E: N \to N$ onto M such that the pre-adjoint map $E_*: N_* \to N_*$ satisfies $E_*(x) = x$ for all $x \in F$. In particular, for every finite-dimensional subspace $F \subset N_*$, we find M as above such that $F \subset E_*(M_*) \subset N_*$ and E_* is completely isometric. By the first part M_* is a rigid $\mathscr{OL}_1$ space and therefore F is arbitrarily close to a complete isometric copy of $L_1(A) \subset E_*(M_*)$ with A finite dimensional. Since F is arbitrary, we deduce that N_* is a rigid $\mathscr{OL}_1$ space. Finally, we apply the results from Ng and Ozawa (see [NO]) and obtain that completely isometric copies of $L_1(A)$ are automatically completely contractively complemented in N_*. Hence N_* is $\mathscr{COL}_1$ space with $\mathscr{COL}_1(N_*)=1$. $\square$

We will now improve the estimate $\mathscr{OL}_\infty(A) \leqslant 6$ (which improved former results of Kirchberg [Ki2]) for nuclear C^*-algebras in [JOR].

Theorem 3.11. *Let A be a nuclear C^*-algebra, then*

$$\mathcal{OL}_\infty(A) = \mathcal{COL}_\infty(A) \leqslant 3.$$

Proof. By the results in [EL, Theorem 6.4] A^{**} is hyperfinite and thus by Proposition 3.10 A^* satisfies $\mathcal{COL}_1(A^*) = 1$. According to [AB,EH] A is locally reflexive and thus we can apply Theorem 3.8 and deduce

$$\mathcal{COL}_\infty(A) \leqslant 1^2 1^2 (2+1) = 3. \qquad \square$$

Let us conclude this section by showing that the local reflexivity assumption in Theorem 3.8 and Corollary 3.7 is necessary.

Proposition 3.12. *For every $n \in \mathbb{N}$ there exist an operator space Y_n and a sequence $\mathbf{m}$ such that Y_n^* is completely isometric to $\ell_1^n \oplus_1 \ell_1(\mathfrak{s}_1(\mathbf{m}))$ but*

$$\Lambda(Y_n) \geqslant \frac{\sqrt{n}}{2}.$$

In particular, the (operator space) dual Y^* of the c_0 sum $Y = (\sum_n \oplus Y_n)_{c_0}$ is a $\mathcal{COL}_1$ space, but Y does not have the CBAP and thus is not an $\mathcal{OL}_\infty$ space.

Proof. The idea of this 'pull-back' construction Y_n goes back to Kirchberg [Ki2]. We will use the form presented in [OR, Lemma 4.5]. By [OR, Lemma 4.8], there exist a constant $c_n \in [\frac{\sqrt{n}}{2}, \sqrt{n}]$ and a sequence of hyperfinite maps $u_k : \ell_\infty^n \to M_{m_k}$ such that

$$||u_k \otimes id_{M_{m_{k-1}}}|| \leqslant 1,$$

$$||u_k^{-1} : u_k(\ell_\infty^n) \to \ell_\infty^n|| \leqslant \frac{k+1}{k} \leqslant 2,$$

$$\frac{k-1}{k} c_n \leqslant ||u_k \otimes id_{M_{m_k}}|| = ||u_k||_{cb} \leqslant c_n.$$

Let us recall that for $\mathbf{m} = (m_k)$ we use the notations $\mathfrak{s}_\infty(\mathbf{m})$ for the c_0-sum and $\mathfrak{b}(\mathbf{m})$ for ℓ_∞-product. Then, we consider the map $u = (u_k)_k : \ell_\infty^n \to \mathfrak{b}(\mathbf{m})$ and the image $F = u(\ell_\infty^n)$. We denote by $\pi : \mathfrak{b}(\mathbf{m}) \to \mathfrak{b}(\mathbf{m})/\mathfrak{s}_\infty(\mathbf{m})$ the quotient homomorphism. The interesting pull-back space is

$$Y_n = \pi^{-1}(\pi u(\ell_\infty^n)) = u(\ell_\infty^n) + \mathfrak{s}_\infty(\mathbf{m}) = F + \mathfrak{s}_\infty(\mathbf{m}).$$

Let us state some known facts. The quotient space $Y_n/\mathfrak{s}_\infty(\mathbf{m})$ is completely isometric to ℓ_∞^n [OR, Theorem 4.7]. Using the orthogonal decomposition of

$$\mathfrak{b}(\mathbf{m})^{**} = \mathfrak{b}(\mathbf{m}) \bigoplus_\infty [\mathfrak{b}(\mathbf{m})/\mathfrak{s}_\infty(\mathbf{m})]^{**}$$

it turns out that

$$Y_n^* \cong \ell_1^n \bigoplus_1 \mathfrak{s}_1(\mathbf{m}) \tag{3.1}$$

holds completely isometrically. We refer to [ER2, Theorem 14.5.6] for related details. Now, we want to show that Y_n has a bad CBAP constant. We will need a 'nice' projection onto F with a bad norm. Indeed, we can extend the maps $u_k^{-1} : u_k(\ell_\infty^n) \to \ell_\infty^n$ to maps $v_k : M_{m_k} \to \ell_\infty^n$ of norm less than 2. Then, we define $v : \mathfrak{b}(\mathbf{m}) \to \ell_\infty^n$ by

$$\langle v((x_k)), e_j \rangle = \lim_{k,\mathscr{U}} \langle v_k(x_k), e_j \rangle,$$

where $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$. Clearly, $vu(x) = x$ for every $x \in \ell_\infty^n$ and v vanishes for elements $x \in \mathfrak{s}_\infty(\mathbf{m})$. Hence $P = uv|_{Y_n} : Y_n \to F$ is a projection onto F (of cb-norm $\leqslant 2\sqrt{n}$ and $P|_{\mathfrak{s}_\infty(\mathbf{m})} = 0$). Furthermore, we will need the projections $Q_m : \mathfrak{b}(\mathbf{m}) \to \mathfrak{b}(\mathbf{m})$ defined by

$$(Q_m(x_k))_k = \begin{cases} x_k & \text{if } k > m, \\ 0 & \text{else.} \end{cases}$$

Note that $Q_m(\mathfrak{s}_\infty(\mathbf{m})) \subset \mathfrak{s}_\infty(\mathbf{m})$ and $vQ_m = v$. Let now $T : Y_n \to Y_n$ be a finite rank map such that $T|_F = id_F$. We consider the map $V = T|_{\mathfrak{s}_\infty(m)} : \mathfrak{s}_\infty(\mathbf{m}) \to \mathfrak{b}(\mathbf{m})$ and claim that for every $\varepsilon > 0$ there exists an l such that $||VQ_l|_{\mathfrak{s}_\infty(\mathbf{m})}||_{cb} \leqslant \varepsilon$. Indeed, let $G = Im(V)$ be the finite-dimensional range and consider the restriction $V : \mathfrak{s}_\infty(\mathbf{m}) \to G$. Then $V^*(G^*) \subset \mathfrak{s}_1(\mathbf{m})$ is contained in a finite-dimensional subspace. Thus, we can find an l such that for the projection P_l onto the l first coordinates we have

$$||P_l V^*(g) - V^*(g)|| \leqslant \frac{\varepsilon}{\dim(G)} ||g||.$$

Using (2.3), we deduce $||P_l V^* - V^*||_{cb} \leqslant \varepsilon$ and therefore

$$\begin{aligned} ||VQ_l|_{\mathfrak{s}_\infty(\mathbf{m})}||_{cb} &= |(V - VP_l)Q_l|_{\mathfrak{s}_\infty(\mathbf{m})} + VP_l Q_l|_{\mathfrak{s}_\infty(\mathbf{m})}||_{cb} \\ &\leqslant ||V - VP_l||_{cb} = ||V^* - P_l V^*||_{cb} \leqslant \varepsilon. \end{aligned}$$

Let us consider the subspace

$$Y_{n,l} = F + Q_l(\mathfrak{s}_\infty(\mathbf{m})) \subset Y_n.$$

Using

$$\begin{aligned} (id_{Y_n} - T) &= [(id_{Y_n} - T)P + (id_{Y_n} - T)(id_{Y_n} - P)] = (id_{Y_n} - T)(id_{Y_n} - P) \\ &= (id_{Y_n} - P) - T(id_{Y_n} - P). \end{aligned}$$

We deduce from $(id_{Y_n} - P)(Y_{n,l}) \subset Q_l(\mathfrak{s}_\infty(\mathbf{m})) \subset \mathfrak{s}_\infty(\mathbf{m})$ that

$$\begin{aligned}\|(id_{Y_n} - P)|_{Y_{n,l}}\|_{cb} &= \|(id_{Y_n} - T)|_{Y_{n,l}} + T(id_{Y_n} - P)|_{Y_{n,l}}\|_{cb} \\ &\leqslant \|(id_{Y_n} - T)|_{Y_{n,l}}\|_{cb} + \|(T|_{\mathfrak{s}_\infty(\mathbf{m})})Q_l(id_{Y_n} - P)|_{Y_{n,l}}\|_{cb} \\ &\leqslant \|(id_{Y_n} - T)\|_{cb} + \|VQ_l|_{\mathfrak{s}_\infty(\mathbf{m})}\|_{cb} \\ &\leqslant \|id_{Y_n} - T\|_{cb} + \varepsilon \leqslant 1 + \varepsilon + \|T\|_{cb}.\end{aligned}$$

Let us note that $(id_{Y_n} - P)|_{Y_{n,l}}$ leaves $Y_{n,l}$ (and $Q_l(F) + Q_l(\mathfrak{s}_\infty(\mathbf{m}))$) invariant and is a projection onto $Q_l(\mathfrak{s}_\infty(\mathbf{m}))$. However, the space $Y_{n,l} = F + Q_l(\mathfrak{s}_\infty(\mathbf{m}))$ (more precisely $Q_l(F) + Q_l(\mathfrak{s}_\infty(\mathbf{m}))$) also satisfies the conditions from [OR, Lemma 4.5] and therefore

$$\frac{\sqrt{n}}{2} \leqslant \|(I - P)|_{Y_{n,l}}\|_{cb} \leqslant 1 + \varepsilon + \|T\|_{cb}.$$

Since $\varepsilon > 0$ is arbitrary this implies

$$\frac{\sqrt{n}}{2} - 1 \leqslant \|T\|_{cb}.$$

In view of the perturbation Lemma 1.4, we obtain $\Lambda(Y_n) \geqslant \frac{\sqrt{n}}{2} - 1$. Using (3.1) and $\mathbf{m}_n$ for the sequence obtained at the n-step, we deduce that the dual space of $(\sum_n \oplus Y_n)_{c_0}$ is completely isometric to

$$\left(\sum_n \oplus (\ell_1^n \oplus_1 \mathfrak{s}_1(\mathbf{m}_n))\right)_1 \cong \ell_1 \oplus \mathfrak{s}_1(\tilde{\mathbf{m}}).$$

Here $\tilde{\mathbf{m}}$ is obtained as the union of the $\mathbf{m}_n$'s. Hence $(\sum_n \oplus Y_n)^*_{c_0}$ is completely isometric to the predual of an hyperfinite, semifinite von Neumann algebra. Since Y_n is completely contractively complemented in $(\sum_n \oplus Y_n)_{c_0}$, the second assertion is obvious. $\square$

Problems 3.13. (i) *Let* $1 < p < \infty$. *Does* $X^* \mathcal{OL}_p$ *imply that* X *is* $\mathcal{OL}_{p'}$?
(ii) *Is the constant* 3 *in Theorem* 3.11 *best possible*?
(iii) *Is every* $\mathcal{OL}_\infty$ *space locally reflexive*?

4. Basis for $\mathcal{COL}_p$ spaces

Let us outline how a basis for $\mathcal{L}_p$ spaces is obtained in the commutative case. Following the work of Johnson et al. [JRZ], Nielsen and Wojtaszczyk showed in [NW] that for every $\mathcal{L}_p$ space X there is an increasing sequence of integers (n_k)

such that

$$X \oplus_p \left(\sum_k \oplus \ell_p^{n_k} \right)_p$$

has a 'nice' FDD and therefore has a basis (see Proposition 4.2 below for the non-commutative counterpart). However, since every $\mathscr{L}_p$ space contains ℓ_p complemented, Pełczyński's decomposition trick can be used to prove that X is isomorphic to $X \oplus_p (\sum_k \oplus \ell_p^{n_k})$ and thus X itself has a basis.

Using the results from Raynaud and Xu [RX], we could apply similar techniques for arbitrary separable $\mathscr{C}\mathscr{O}\mathscr{L}_p$ spaces containing complemented S_p^n's. However, the results of [RX] do not hold for $p = \infty$ and this corresponds to the separate treatment of the case $p = \infty$ in the commutative case (see [JRZ,NW]). Moreover, in the commutative case the Pełczyński decomposition trick was used for $p < \infty$ only as a matter of convenience, because a complemented version of ℓ_p has anyway been available. Following a suggestion of W. B. Johnson, we will prove the existence of a nice FDD and a basis entirely relying on local properties of the underlying $\mathscr{C}\mathscr{O}\mathscr{L}_p$ space, see Lemma 4.5 below, and not relying on the results in [RX]. Even for Banach spaces this approach is new, although certainly known to specialists. This also covers the case $p = \infty$ and therefore provides bases for nuclear C^*-algebras. Moreover, using Lemma 4.5 unifies the approach to the basis problem for $\mathscr{L}_p$ spaces $(1 \leqslant p \leqslant \infty)$.

In analogy with theory of Banach spaces, we will need operator space FDDs. An operator space X has a *completely bounded finite-dimensional decomposition* (in short *cb-FDD*) if there exist a constant K and a sequence (F_n) of finite-dimensional subspaces such that every element $x \in X$ has a unique decomposition $x = \sum_n x_n$ with $x_n \in F_n$ and

$$\left\| \sum_{n \leqslant N} a_n \otimes x_n \right\|_{M_m(X)} \leqslant K \left\| \sum_{n \in \mathbb{N}} a_n \otimes x_n \right\|_{M_m(X)}$$

for all $N, m \in \mathbb{N}$ and $(a_n) \subset M_m$. If this holds for a particular K, we say that X has a *K*-*cb*-FDD. Equivalently, the projections $P_N : X \to \sum_{n \leqslant N} F_n$ satisfy $\|P_N\|_{cb} \leqslant K$.

Proposition 4.1. *Let $1 \leqslant p \leqslant \infty$ and X a separable $\mathscr{C}\mathscr{O}\mathscr{L}_p$ space (and locally reflexive for $p = \infty$) such that X^* has the CBAP, then X has a cb-FDD.*

Proof. Since a $\mathscr{C}\mathscr{O}\mathscr{L}_p$ space is locally reflexive for $1 \leqslant p < \infty$ (see Lemma 1.5 and the proof of Theorem 3.8 for $p = 1$), we can apply Corollary 3.4 and obtain an increasing sequence $(Q_n)_n$. The *cb*-FDD is then given by $X = Q_1(X) + \sum_{n=1}^{\infty} (Q_{n+1} - Q_n)(X)$. $\square$

The next Lemma provides the cb-FDD for a suitable enlargement of a $\mathscr{COL}_p$ space following the work of Nielsen and Wojtaszczyk [NW]. This approach provides better constants than the purely local techniques in Lemma 4.5 and is algebraically simpler.

Proposition 4.2. *Let $1 \leqslant p \leqslant \infty$ and X a separable operator space such that there exists an increasing sequence (Q_n) of projections $Q_n : X \to X$ satisfying*

$$Q_n Q_{n+1} = Q_{n+1} Q_n = Q_n \quad \textit{and} \quad \lim_n Q_n(x) = x$$

for all $x \in X$. If moreover, for all $n \in \mathbb{N}$ there exist finite-dimensional operator spaces F_n and maps $r_n : F_n \to X$, $s_n : X \to F_n$ satisfying $Q_n = r_n s_n$ and $s_n r_n = id_{F_n}$ with

$$||s_n||_{cb} ||r_n||_{cb} \leqslant C,$$

then the space $Z = X \oplus_p (\sum_{n=1}^{\infty} \oplus F_n)_p$ ($Z = X \oplus_\infty (\sum_{n=1}^{\infty} \oplus F_n)_{c_0}$ for $p = \infty$) admits a cb-FDD $Z = \sum_{n=1}^{\infty} Z_n$ with projections $P_N : Z \to \sum_{n=1}^{N} Z_n$ such that

$$||P_N||_{cb} \leqslant C, \tag{4.1}$$

$$d_{cb}(Z_n, F_n) \leqslant 4C^2, \tag{4.2}$$

$$d_{cb}\left(\sum_{n=1}^{N} Z_n, \left(\sum_{n=1}^{N} \oplus F_n \right)_p \right) \leqslant C. \tag{4.3}$$

Proof. Motivated by $Q_n(X) = Q_{n-1}(X) + (Q_n - Q_{n-1})(X)$, we define

$$Y_n = (Q_n - Q_{n-1})(X) \subset X \quad \text{and} \quad Z_n = Y_n \oplus_p F_{n-1}.$$

Here we use $Q_0 = \{0\}$ and $F_0 = \{0\}$. By scaling, we may assume that for every $n \in \mathbb{N}$ we have $||r_n||_{cb} \leqslant 1$ and $||s_n||_{cb} \leqslant C$. For any fixed $n \in \mathbb{N}$, we consider $T_n : Z_n \to F_n$ given by $T_n(y + x) = s_n(y) + s_n r_{n-1}(x)$ and $T_n^{-1}(f) = (Q_n - Q_{n-1}) r_n(f) + s_{n-1} r_n(f)$. Then

$$\begin{aligned} ||T||_{cb} ||T^{-1}||_{cb} &\leqslant (||s_n||_{cb}^{p'} + ||s_n r_{n-1}||^{p'})^{\frac{1}{p'}} (||(Q_n - Q_{n-1}) r_n||_{cb}^p + ||s_{n-1} r_n||_{cb}^p)^{\frac{1}{p}} \\ &\leqslant C 2^{\frac{1}{p'}} ((2C)^p + C^p)^{\frac{1}{p}}) \leqslant 4C^2. \end{aligned}$$

This proves (4.2). Let $\mathscr{F} = (\sum_{n=1}^{\infty} \oplus F_{n-1})_p$. For $p = \infty$ we use the c_0-sum. We want to show the cb-FDD of

$$Z = X \oplus_p \mathscr{F} = \sum_{n=1}^{\infty} Z_n.$$

By density we can assume that $z = x + f$ and there exists an $N \in \mathbb{N}$ such that

$$Q_N(x) = x \quad \text{and} \quad f = \sum_{n=1}^{N} f_{n-1} \text{ with } f_{n-1} \in F_{n-1}.$$

By definition $Q_0 = 0$, hence we deduce

$$z = x + f = Q_N(x) - 0 + f = \sum_{n=1}^{N} [(Q_n - Q_{n-1})(x) + f_{n-1}] \in \sum_{n=1}^{N} Z_n.$$

Clearly, we have $Y_n \subset Q_N(X)$ for all $n \leqslant N$. Therefore, this argument shows for all $N \in \mathbb{N}$

$$Q_N(X) \oplus_p \left(\sum_{n=1}^{N} \oplus F_{n-1} \right)_p = \sum_{n=1}^{N} Z_n.$$

On the other hand, we deduce

$$\begin{aligned} d_{cb}\left(\sum_{n=1}^{N} Z_n, \left(\sum_{n=1}^{N} \oplus F_n \right)_p \right) &= d_{cb}\left(Q_N(X) \oplus_p \left(\sum_{n=1}^{N-1} \oplus F_n \right)_p, F_N \oplus_p \left(\sum_{n=1}^{N-1} \oplus F_n \right)_p \right) \\ &\leqslant d_{cb}(Q_N(X), F_N) \leqslant \|r_n\|_{cb} \|s_n\|_{cb} \leqslant C. \end{aligned}$$

Moreover, we have a projection $P_N : Z \to \sum_{n=1}^{N} Z_n$ defined by

$$P_N(x, f) = Q_N(x) + \sum_{n=1}^{N} f_{n-1},$$

where f has components $f = \sum_n f_{n-1}$. It is obvious that $\|P_N\|_{cb} \leqslant \|Q_N\|_{cb} \leqslant C$. $\square$

Remark 4.3. In the proof above, the ℓ_p-sum of the F_n's can be replaced by a space with a cb-FDD $\mathscr{F} = \sum_n \oplus F_n$. This is interesting for rectangular versions of $\mathscr{COL}_p$ spaces.

The following elementary distance estimate is useful when dealing with orthogonal decompositions.

Lemma 4.4. *Let P and Q be projections on operator spaces X and Y, respectively. Then*

$$d_{cb}(X,Y) \leqslant (\|P\|_{cb} + \|id_X - P\|_{cb})(\|Q\|_{cb} + \|id_Y - Q\|_{cb})$$
$$\times \max\{d_{cb}(P(X), Q(Y)), d_{cb}((id_X - P)(X), (id_Y - Q)(Y))\}.$$

Proof. Let $u : P(X) \to Q(Y)$ be an isomorphism with $\|u\|_{cb} \leqslant 1$, and $v : (id_X - P)(X) \to (id_Y - Q)(Y)$ be an isomorphism with $\|v\|_{cb} \leqslant 1$. (By convention, the right-hand side is ∞ if either $P(X)$, $Q(Y)$ or $(id_X - P)(X)$, $id_Y - Q(Y)$ are not completely isomorphic.) We define $w = uP + v(id_X - P)$ and observe that w is invertible with $w^{-1} = u^{-1}Q + v^{-1}(id_Y - Q)$. The triangle inequality yields the norm estimate. $\square$

The following rather technical lemma allows us to construct nice FDD by using entirely local techniques suggested by W.B. Johnson. The idea of the proof is an application of the well-known principle "robbing Peter to pay Paul".

Lemma 4.5. *Let $1 \leqslant p \leqslant \infty$ and X a separable operator space together with a subspace $Y \subset X^*$ satisfying the assumptions of Lemma* 3.2 (*respectively Lemma* 3.3). *We assume in addition that for every finite rank map $T : X \to X$, every $\varepsilon > 0$, and every space $F = (F_{i_1} \oplus_p \cdots \oplus_p F_{i_n})$ (respectively $F = (F^*_{i_1} \oplus_p \cdots \oplus_p F^*_{i_n})$) there exist $\alpha : F \to X$, $\beta : X \to F$ such that $\beta\alpha = id_F$, $\beta^*(F^*) \subset Y$ and*

$$\|\alpha\|_{cb}\|\beta\|_{cb} \leqslant C_3 \quad \textit{and} \quad \|T\alpha\|_{cb} \leqslant \varepsilon.$$

Then X admits a cb-FDD

$$X = \sum_n Z_n$$

such that for every $n \in \mathbb{N}$, there exists an index i_n with

$$d_{cb}(Z_n, F_{i_n}) \leqslant C \quad (\textit{respect.}\ d_{cb}(Z_n, F^*_{i_n}) \leqslant C)$$

and for $N \in \mathbb{N}$

$$d_{cb}\left(\sum_{n=1}^N Z_n, \left(\sum_{n=1}^N \oplus F_{i_n}\right)_p\right) \leqslant C \quad \left(\textit{respect.}\ d_{cb}\left(\sum_{n=1}^N Z_n, \left(\sum_{n=1}^N \oplus F^*_{i_n}\right)_p\right) \leqslant C\right).$$

Here the constant C depends only on $lrc(X)$, C_1, C_2 in Lemma 3.2 (*respectively Lemma* 3.3) *and C_3 above.*

Proof. We will give the proof under the assumptions of Lemma 3.2. The modifications for the assumptions of Lemma 3.3 will then be obvious. Let (x_n) be a dense sequence in X. By induction we will construct a sequence (i_n) of indices in I, two sequences (Q_n) and (P_n) of projections on X such that for all $n \in \mathbb{N}$

(1) $\|Q_n\|_{cb} \leqslant c$;
(2) $Q_n Q_{n+1} = Q_{n+1} Q_n = Q_n$;
(3) $x_n \in Q_n(X)$;
(4) $d_{cb}(Q_n(X), F_{i_n}) \leqslant c$;
(5) $Im(Q_n^*) \subset Y$;
(6) $\|P_n\|_{cb} \leqslant d$;
(7) $P_n Q_n = Q_n P_n = 0$;
(8) $Im(P_n^*) \subset Y$;
(9) $P_n Q_{n+1} = Q_{n+1} P_n = P_n$;
(10) $d_{cb}(P_n(X), (\sum_{k=1}^{n-1} \oplus F_{i_k})_p) \leqslant d$.

Here the constants c and d only depend on $lrc(X)$, C_1, C_2 in Lemma 3.2 and C_3 above.

In the first step of the induction, we apply Corollary 3.4 in order to obtain $i_1 \in I$ and a projection Q_1 on X such that (1), (3)–(5) above hold for $n = 1$. We simply set $P_1 = 0$ and $Q_0 = 0$. Now we assume that $i_1, \ldots, i_n$ in I and projections $Q_1, \ldots, Q_n$, P_1, $\ldots$,P_n on X are found satisfying the required properties for all $k = 1, \ldots, n$. Then according to Corollary 3.4, we can find an index $i_{n+1} \in I$ and a projection Q_{n+1} such that (1), (3)–(5) hold and such that

$$Q_{n+1}(Q_n + P_n) = (Q_n + P_n)Q_{n+1} = Q_n + P_n.$$

Thus from (7) we deduce that (2) and (9) hold, too. In order to construct P_{n+1}, we consider

$$F = F_{i_1} \oplus_p \cdots \oplus_p F_{i_n}.$$

By assumption there exist $\alpha : F \to X$, $\beta : X \to F$ such that $\beta\alpha = id_F$, $\beta^*(F^*) \subset Y$ and

$$\|\alpha\|_{cb} \leqslant C_3, \quad \|\beta\|_{cb} \leqslant 1 \quad \textit{and} \quad \|Q_{n+1}\alpha\|_{cb} \leqslant \frac{1}{2C_3}.$$

Then we see that

$$\|id_F - \beta(id_X - Q_{n+1})\alpha\|_{cb} = \|\beta Q_{n+1}\alpha\|_{cb} \leqslant 1/2.$$

Thus $\beta(id_X - Q_{n+1})\alpha$ is invertible and its inverse $w : F \to F$ satisfies $\|w^{-1}\|_{cb} \leqslant 2$. Let us define

$$\tilde{\alpha} = (id_X - Q_{n+1})\alpha, \quad \tilde{\beta} = w\beta(id_X - Q_{n+1}).$$

Then $\tilde{\beta}\tilde{\alpha} = id_F$ and we obtain the norm estimates

$$\|\tilde{\alpha}\|_{cb} \leqslant (1 + c)C_3, \quad \|\tilde{\beta}\|_{cb} \leqslant 2(1 + c).$$

Therefore, $P_{n+1} = \tilde{\alpha}\tilde{\beta}$ is a projection on X with $P_{n+1}Q_{n+1} = Q_{n+1}P_{n+1} = 0$. This proves (7). Using $Im(P_{n+1}^*) \subset Im(Q_{n+1}^*) + Im(\beta^*) \subset Y$, we obtain conditions (6)–(10) at $(n+1)$th step with $d = 2(1+c)^2 C_3$. This concludes the inductive construction.

With the help of the two sequences (Q_n) and (P_n), we can easily construct the desired cb-FDD for X. Indeed, for $n \in \mathbb{N}$ we let (with $Q_0 = P_0 = 0$)

$$Z_n = (Q_n - Q_{n-1} - P_{n-1})(X) + P_n(X).$$

Then, we deduce for $R_n = Q_n - Q_{n-1} - P_{n-1}$ and by elementary calculation with $\oplus_1$ that

$$\begin{aligned}
d_{cb}(F_{i_n}, Z_n) &\leqslant d_{cb}(F_{i_n}, Q_n(X)) d_{cb}(Q_n(X), R_n(X) + P_n(X)) \\
&\leqslant c(\|R_n\|_{cb} + \|Q_{n-1} + P_{n-1}\|_{cb}) \\
&\quad \times d_{cb}(R_n(X) \oplus_1 (Q_{n-1} + P_{n-1})(X), R_n(X) + P_n(X)) \\
&\leqslant c(3c + 2d) d_{cb}(R_n(X) \oplus_1 P_n(X), R_n(X) + P_n(X)) \\
&\quad \times d_{cb}(P_n(X), (Q_{n-1} + P_{n-1})(X)) \\
&\leqslant c(3c + 2d)(\|R_n\|_{cb} + \|P_n\|_{cb}) \\
&\quad \times d_{cb}((Q_{n-1} + P_{n-1})(X), F_{i_{n-1}} \oplus_p F_{i_1} \oplus_p \cdots \oplus_p F_{i_{n-2}}) \\
&\quad \times d_{cb}(P_n(X), F_{i_1} \oplus_p \cdots \oplus_p F_{i_{n-1}}) \\
&\leqslant c(3c + 2d)(2c + 2d) 4(\max\{c, d\}) d.
\end{aligned}$$

In the last line we used Lemma 4.4 for $X = F_{i_{n-1}} \oplus_p (F_{i_1} \oplus_p \cdots \oplus_p F_{i_{n-2}})$. Moreover, we have

$$\sum_{n=1}^{N} Z_n = Q_N(X) + P_N(X).$$

By (7) $Q_N + P_N$ is a projection on X. Therefore, by the density of (x_n) in X and by (1), (3), (6), we deduce that $\sum_n Z_n$ is a cb-FDD of X with constant $\leqslant c + d$. Moreover, by Lemma 4.4 we deduce

$$\begin{aligned}
& d_{cb}\left(\sum_{n=1}^{N} Z_n, F_{i_1} \oplus_p \cdots \oplus_p F_{i_N} \right) \\
& \quad \leqslant 2(1 + 2\|Q_N\|_{cb})
\end{aligned}$$

$$\times \max\{d_{cb}(Q_N(X), F_{i_N}), d_{cb}(P_N(X), F_{i_1} \oplus_p \cdots \oplus_p F_{i_{N-1}})\}$$

$$\leqslant 2(1+2c)\max\{c,d\}.$$

The assertion is proved. $\square$

Let us indicate a natural *cb*-basis for the spaces $\mathfrak{s}_p(\mathbf{m})$. The following proposition is well-known. We add a proof for the convenience of the reader.

Proposition 4.6. *Let* $1 \leqslant p \leqslant \infty$. *Then* S_p *has a* 2-*cb-basis given by the enumeration* $e_{11}, e_{12}, e_{22}, e_{21}, e_{13}, e_{23}, e_{33}, e_{32}, e_{31}, \ldots$ *of the matrix units obtained by counting down the nth column and then returning on the nth row from the right to the left.*

Proof. Let us denote by (x_n) the natural basis of S_p indicated above. More precisely, for every $N \in \mathbb{N}$ the first N^2 basis elements are the matrix units $(e_{ij})_{i,j=1,\ldots,N}$. In particular, $x_1 = e_{11}$. The next $(N+1)^2 - N^2 = 2N+1$ basis elements are obtained by counting down the $(N+1)$th column

$$x_{N^2+1} = e_{1(N+1)}, x_{N^2+2} = e_{2(N+1)}, \ldots, x_{N^2+N+1} = e_{(N+1)(N+1)}.$$

Then, we return on the $(N+1)$th row from the right to the left

$$x_{N^2+N+2} = e_{(N+1)N}, x_{N^2+N+3} = e_{(N+1)(N-1)}, \ldots, x_{(N+1)^2} = e_{(N+1)1}.$$

For $i \in \mathbb{N}$, we denote by p_i be the projection onto the i-dimensional subspace $\ell_2^i \subset \ell_2$. Let $E_N(x) = p_N x p_N$ be the projection onto the matrices in the upper left $N \times N$ corner. Then $\Delta_N : S_p \to S_p$ defined by $\Delta_N = E_{N+1} - E_N$ is a projection. For convenience, we set $\Delta_0 = E_1$. Given $N^2 < n \leqslant (N+1)^2$, we define $i = n - N^2$. If $i \leqslant N$, the projection onto $\operatorname{span}\{x_k \mid k \leqslant n\}$ can be described by

$$P_n(x) = E_N(x) + p_i \Delta_N(x).$$

This can be written as a sum of two projections onto rectangular boxes. Hence $||P_n||_{cb} \leqslant 2$. Similarly, if $N < i < 2N+1$, we have

$$P_n(x) = E_N(x) + \Delta_N(x)(id_{\ell_2} - p_{2N+1-i}),$$

again the projection onto a large and very small rectangular box. For $i = (N+1)^2$ we use $E_{(N+1)}$. Since in any case P_n is the sum of two projections onto rectangular blocks, we obtain $\|P_n\|_{cb} \geqslant 2$. $\square$

In the following we call this the *natural basis* of S_p.

Corollary 4.7. *Let* $1 \leqslant p \leqslant \infty$ *and* $\mathbf{m} = (m(n))$ *a sequence of natural numbers, then* $\mathfrak{s}_p(\mathbf{m})$ *has a* (*natural*) *cb-basis* (x_n) *with constant* 2.

Proof. Let $\mathbf{m} = (m(n))_{n\in\mathbb{N}}$. For each n, we use the natural basis for $S_p^{m_n}$ constructed above and join them together in the ℓ_p-sum. $\square$

Definition 4.8. Let (b_k) be a cb-basis of an operator space X and $\mathbf{m}$ a sequence of natural numbers. We say that the basis (b_k) is initially equivalent to $\mathfrak{s}_p(\mathbf{m})$ if there exists a constant $C>0$ such that

$$d_{cb}(\mathrm{span}\{b_k \mid k\leqslant n\},\ \mathrm{span}\{x_k \mid k\leqslant n\})\leqslant C$$

for all $n\in\mathbb{N}$. Note that the basis depends on the ordering of $\mathbf{m}$.

The following observation is contained in [JRZ] in the context of Banach spaces.

Lemma 4.9. *Let X be an operator space with a FDD*

$$X = \sum_n Z_n.$$

Assume each Z_n has a cb-basis $\{b_1^n, \ldots, b_{k_n}^n\}$ with constant c_1 independent of n, then X has a cb-basis (b_n). Moreover, for every n there exist N and j such that

$$d_{cb}(\mathrm{span}\{b_1, \ldots, b_n\}, \left(\sum_{k\leqslant N} Z_k + \mathrm{span}\{b_1^{N+1}, \ldots, b_j^{N+1}\}\right)\leqslant c,$$

where c only depends on c_1 and the cb-FDD-constant in $X = \sum_k Z_k$.

Proof. Let us denote by $Q_N : X\to\sum_{n\leqslant N} Z_n$ the projection onto the first N blocks. The cb-basis is simply given by $b_1^1, \ldots, b_{k_1}^1, b_1^2, \ldots, b_{k_2}^2, \ldots$. We define $m(N) = \sum_{j=1}^N k_j$. Let $m(N)<n\leqslant m(N+1)$ and choose j such that $n = m(N)+j$. Let $\tilde{P}_j : Z_{N+1}\to\mathrm{span}\{b_1^{N+1}, \ldots, b_j^{N+1}\}$. Then the projection P_n onto the span of the first n elements is given by $P_n = Q_N + \tilde{P}_j(Q_{N+1} - Q_N)$. It is easily checked that $Q_N P_n = Q_N$ and $Q_{N+1}P_n = P_n$ and thus we have found a cb-basis. The second statement is obvious in view of Lemma 4.4. $\square$

Let us state our main result on bases for $\mathscr{COL}_p$ spaces (see the beginning of Section 2 for the technical notion 'with respect to Y').

Theorem 4.10. *Let $1\leqslant p\leqslant\infty$ and X a separable operator space such that one of the following conditions is satisfied*:

(i) *$1<p<\infty$, and X is a $\mathscr{COL}_p$ space;*
(ii) *$p=1$, X is a $\mathscr{COL}_1$ space and there is a subspace $Y\subset X^*$ with the CBAP such that X is a $\mathscr{COL}_1$ space with respect to Y;*
(iii) *$p=\infty$, X is a locally reflexive $\mathscr{COL}_\infty$ space and there is a subspace $Y\subset X^*$ with the CBAP such that X is a $\mathscr{COL}_\infty$ space with respect to Y.*

Then X has a cb-basis initially equivalent to some $\mathfrak{s}_p(\mathbf{m})$. Moreover, the basis constant can be estimated by a function depending only on $\mathscr{COL}_p(X)$, $\Lambda(Y^*)$ and $lcr(X)$.

Proof. Let X be a $\mathscr{COL}_p$ space (with respect to Y for $p=1$ or $p=\infty$ and being in addition locally reflexive). For $1<p<\infty$, we can apply Corollary 3.9 to deduce that X^* is a $\mathscr{COL}_{p'}$ space and in particular has the CBAP. We put $Y=X^*$ in this case and note that the assumptions of Lemma 3.2 are satisfied. Let

$$r_n : \ell_p^{k_1(n)}(S_p^1)\oplus_p\cdots\oplus_p\ell_p^{k_{l(n)}(n)}(S_p^{l(n)})\to X$$

and

$$s_n : X\to\ell_p^{k_1(n)}(S_p^1)\oplus_p\cdots\oplus_p\ell_p^{k_{l(n)}(n)}(S_p^{l(n)})$$

be such that $Im(s_n^*)\subset Y$, $s_nr_n=id$, r_ns_n tends to the identity in the point-norm topology and $\sup_n\|r_n\|_{cb}\|s_n\|_{cb}\leqslant c$. We define

$$k_m=\sup_n k_m(n).$$

Let us consider two cases.

(a) There exists a strictly increasing sequence of natural numbers m_j such that $k_{m_j}\geqslant 1$ for all $j\in\mathbb{N}$. Then X contains complemented $S_p^{m_j}$'s and thus complemented S_p^m's for all $m\in\mathbb{N}$. Using Lemma 2.9, we deduce that the additional assumption in Lemma 4.5 is satisfied for the family

$$\{L_p(A)\mid A \text{ finite-dimensional } C^*\text{-algebra}\}.$$

Therefore X admits an FDD

$$X=\sum_n Z_n$$

and there exists a sequence A_n such that

$$d_{cb}(Z_n,L_p(A_n))\leqslant c'$$

and

$$d_{cb}\left(\sum_{n\leqslant N}Z_n, L_p(A_1)\oplus_p\cdots\oplus_pL_p(A_N)\right)\leqslant c'.$$

Using Lemma 4.9, we deduce that X has a cb-basis which is initially equivalent to $\mathfrak{s}_p(\mathbf{m})$ with respect to suitable $\mathbf{m}$ formed using $L_p(A_n)=\ell_p^{k_1}(S_p^1)\oplus_p\cdots\oplus_p\ell_p^{k_{l(n)}}(S_p^{l(n)})$.

(b) There exists m_f such that for all $m>m_f$ $k_m=0$. Equivalently, $l(n)\leqslant m_f$ for all $n\in\mathbb{N}$. If $k(m)$ is finite for all $m\leqslant m_f$, then $rk(r_n)\leqslant\sum_{m=1}^{m_f}k(m)m^2$ and hence X is finite dimensional. Therefore, X is cb-isomorphic to $L_p(A)$ for some finite dimensional A. In the following, we assume X is infinite dimensional and denote by m_1 be the biggest m such that $k(m)$ is infinite. Unfortunately, we do not know that $m_1 = m_f$. We consider $k_f(n) = \sum_{m \leqslant m_f} k_m(n)$ and fix a free ultrafilter $\mathscr{U}$. Modifying r_n and s_n, we obtain a factorization

$$\iota_X : X \xrightarrow{(s'_n)} \prod_{n,\mathscr{U}} \ell_p^{k_f(n)}(S_p^{m_f}) \xrightarrow{(r'_n)} X^{**}.$$

Since $\prod_{n,\mathscr{U}} \ell_p^{k_f(n)}$ is an abstract $\mathscr{L}_p$ space, we may assume

$$\prod_{n,\mathscr{U}} \ell_p^{k_f(n)} = L_p(\Omega, \mathscr{B}, \mu)$$

for some measure space $(\Omega, \mathscr{B}, \mu)$. Then, we obtain completely bounded factorizations

$$\iota_X : X \xrightarrow{\alpha} L_p(\Omega, \mathscr{B}, \mu; S_p^{m_f}) \xrightarrow{\beta} X^{**} \quad \text{and} \quad id_{X^*} : X^* \xrightarrow{\beta^*} L_{p'}(\Omega, \mathscr{B}, \mu; S_{p'}^{m_f}) \xrightarrow{\alpha^*} X^*.$$

In particular, X^* has the CBAP. Passing to a subsequence, we may assume that $(k_{m_1}(n))$ is increasing and that r_n and s_n can be written as follows:

$$r_n : \ell_p^{k_1(n)}(S_p^1) \oplus_p \cdots \oplus_p \ell_p^{k_{m_1}(n)}(S_p^{m_1}) \oplus_p L_p(A) \to X$$

and

$$s_n : X \to \ell_p^{k_1(n)}(S_p^1) \oplus_p \cdots \oplus_p \ell_p^{k_{m_1}(n)}(S_p^{m_1}) \oplus_p L_p(A).$$

Here A is a fixed finite-dimensional von Neumann algebra. Let us define

$$r_n^1 = r_n|_{\ell_p^{k_1(n)}(S_p^1) \oplus_p \cdots \oplus_p \ell_p^{k_{m_1}(n)}(S_p^{m_1})}, \quad r_n^2 = r_n|_{L_p(A)}.$$

Similarly, we introduce s_n^1 and s_n^2. Now, we can pass to a further subsequence and assume that $s_n^2 : X \to L_p(A)$ converges in the point-norm topology to a completely bounded map $s^2 : X \to L_p(A)$. Let us denote by $E = (s^2)^*(L_{p'}(A))$ the finite-dimensional image. According to Lemma 3.2, we may find a projection $Q = rs : X \to X$ factorizing through $L_p(B)$ for some finite-dimensional C^*-algebra B such that

$$Q^*(s^2)^* = (s^2)^*.$$

This implies

$$s^2 Q = s^2.$$

Let us define $k_1(n) = \sum_{m \leqslant m_1} k_m(n)$. Using a free ultrafilter $\mathscr{U}$, we define the maps

$$s^1 = (s_n^1) : X \to \prod_{\mathscr{U}} \ell_p^{k_1(n)}(S_p^{m_1}),$$

and

$$r^1 : \prod_{\mathscr{U}} \ell_p^{k_1(n)}(S_p^{m_1}) \to X^{**} \quad \text{by} \quad \langle r^1((b_n)), x^* \rangle = \lim_{n,\mathscr{U}} \langle r_n^1(b_n), x^* \rangle$$
$$r^2 : L_p(A) \to X^{**} \text{ by } \langle r^2(b), x^* \rangle = \lim_{n,\mathscr{U}} \langle r_n^2(b), x^* \rangle.$$

Then, we deduce for $x \in X$ and $x^* \in X^*$ that

$$\begin{aligned} x^*(x) &= \lim_n x^*(r_n s_n(x)) = \lim_{n,\mathscr{U}} x^*(r_n^2 s_n^2(x)) + \lim_{n,\mathscr{U}} x^*(r_n^1 s_n^1(x)) \\ &= \langle r^2 s^2(x), x^* \rangle + \langle r^1 s^1(x), x^* \rangle. \end{aligned}$$

Hence, the inclusion map ι_X satisfies

$$\iota_X = r^2 s^2 + r^1 s^1.$$

Using $s^2 Q = s^2$ and $Z = (I - Q)(X)$, we deduce

$$\begin{aligned} \iota_Z &= (I - Q)^{**} \iota_X (I - Q) = (I - Q)^{**}(r^2 s^2 + r^1 s^1)(I - Q) \\ &= (I - Q)^{**} r^1 s^1 (I - Q). \end{aligned}$$

As above, we observe that

$$\prod_{n,\mathscr{U}} \ell_p^{k_1(n)}(S_p^{m_1}) \cong L_p(\Omega', \mathscr{B}', \mu'; S_p^{m_1})$$

for some measure space $(\Omega', \mathscr{B}', \mu')$. Using conditional expectations onto subalgebras with finitely many atoms, we see that the space $L_p(\Omega', \mathscr{B}'; \mu'; S_p^{m_1})$ has the γ_{p,m_1}^{ap}-approximation property, i.e. the γ_p^{ap} with respect to family of spaces $\ell_p^k(S_p^{m_1})$ (see Remark 2.6). Since, Z is locally reflexive, we deduce that Z also has the γ_{p,m_1}^{ap}-approximation property. On the other hand, Z contains complemented $\ell_p^n(S_p^{m_1})$'s far out. Indeed, since this is true for X, by Lemma 2.10, there is a *cb*-complemented subspace $F \subset X$ such that F is *cb*-isomorphic to $\ell_p^n(S_p^{m_1})$ and such that P vanishes on F. Thus $F \subset Z$. Therefore, by Theorem 2.2, Z is $\mathscr{COL}_p$ with respect to the family $(\ell_p^k(S_p^{m_1}))_{k \in \mathbb{N}}$. Finally, putting together this basis with that of $Q(X)$, we obtain a basis of X where the basis constant is controlled in terms of $f(\mathscr{COL}_p(X), lcr(X), \Lambda(Y))$. $\square$

Remark 4.11. Submitting this paper, we were only able to show in the case (b) that X has a basis without a control of the basis constant in terms of $\mathscr{COL}_p(X)$, $\Lambda(Y)$ and

$lcr(X)$. We are indebted to E. Ricard for his improvement in Lemma 2.9 and for pointing out a proof in case (b) for $1 < p < \infty$. His arguments lead to the complete answer given above.

5. Applications

We want to show the existence of a *cb*-bases for L_p spaces over a hyperfinite von Neumann algebra with a separable predual. The argument in Theorem 4.10 provides such a *cb*-basis, but the estimate of the basis constant is rather involved. Indeed, the proof yields a bad constant for type I von Neumann algebras and type I C^*-algebras. Since these examples are very important, we prefer to add a more direct argument. We will start with the most natural examples of L_p spaces over a hyperfinite semifinite von Neumann algebra and then establish the CBAP for L_p spaces of hyperfinite type III algebras (using modular theory). We refer to [Ha5,C1,C2] for general information on hyperfinite von Neumann algebra and to [JRX] for more details on the structure of the non-commutative L_p spaces associated to hyperfinite von Neumann algebras. Let us start with a simple remark. (The second assertion is certainly well-known and is stated in order to have a concrete estimate for the constant.)

Lemma 5.1. *Let N and M be von Neumann algebras with the QWEP. Let $1 \leqslant p < \infty$ and assume that $L_p(N)$ and $L_p(M)$ have cb-bases. Then $L_p(N \otimes M)$ has a cb-basis. In particular, $L_p([0,1]; S_p)(= L_p(L_\infty[0,1] \otimes B(\ell_2)))$ has a C-cb-basis* (*with* $C \leqslant 14$).

Proof. Let (x_n) and (y_n) be *cb*-bases of $L_p(N)$ and $L_p(M)$ with constants C_1 and C_2, respectively. Let us denote by $P_n : L_p(N) \to \mathrm{span}\{x_i \mid 1 \leqslant i \leqslant n\}$ respectively $Q_n : L_p(M) \to \mathrm{span}\{y_j \mid 1 \leqslant j \leqslant n\}$ the corresponding basis projections. As in Proposition 4.6, we may obtain a basis for $L_p(N \otimes M)$ by using the rectangular enumeration $z_1 = x_1 \otimes y_1,\ z_2 = x_1 \otimes y_2,\ z_3 = x_2 \otimes y_2,\ z_4 = x_2 \otimes y_1,\ z_5 = x_1 \otimes y_3$, etc. Indeed for $n = N^2$, the projection on $\mathrm{span}\{z_k \mid k \leqslant N^2\} = \mathrm{span}\{x_i \otimes y_j \mid 1 \leqslant i,j \leqslant N\}$ is given by $E_N = P_N \otimes Q_N$ which satisfies (see [Ju2])

$$\|E_N\|_{cb} \leqslant \|P_N\|_{cb} \|Q_N\|_{cb}.$$

Let $N^2 < n < (N+1)^2$ and $i = n - N^2$. If $i \leqslant N+1$, the projection $R_n : L_p(N \otimes M) \to \mathrm{span}\{z_k \mid k \leqslant n\}$ is given by

$$R_n = E_N + P_i \otimes (Q_{N+1} - Q_N)$$

and for $i > (N+1)$ is given by

$$R_n = E_N + P_{N+1} \otimes (Q_{N+1} - Q_N) + (P_{N+1} - P_N) \otimes (Q_N - Q_{2N+1-i}).$$

In both cases, we have $\|R_n\|_{cb} \leqslant 7C_1C_2$. Combining the Haar basis $(h_i)_{i\in\mathbb{N}}$ of $L_p([0,1])$ and the natural basis (x_n) of S_p, we obtain a basis for the space $L_p([0,1];S_p)$. $\square$

If N is a type I von Neumann algebra, we can decompose $L_p(N)$ as follows

$$L_p(N) = \left(\sum_n \oplus L_p(\Omega_n, \mathscr{B}_n, \mu_n; S_p^n)\right)_p. \tag{5.1}$$

Here for a given n, $(\Omega_n, \mathscr{B}_n, \mu_n)$ is a standard measure space or the empty set. (We might have some infinite cardinals n in the non-separable case.) Let us say that N *is subhomogeneous* (*as a von Neumann algebra*) if $N = \sum_{n\leqslant k} L_\infty(\Omega_n, \mathscr{B}_n, \mu_n; M_n)$ for some $k\in\mathbb{N}$. Due to the following remark, we will simply say that N is subhomogeneous.

Remark 5.2. N is subhomogeneous as a von Neumann algebra if and only if N is subhomogeneous as a C^*-algebra, and if and only if there exist a compact Hausdorff space K and $n\in\mathbb{N}$ such that N is a subalgebra of (or equivalently completely isomorphic to a subspace of) $C(K, M_n)$.

Proof. If N is subhomogeneous as a von Neumann algebra, then there exists an $n\in\mathbb{N}$ such that $N\subset C(K;M_n)$. This is also true if N is subhomogeneous as a C^*-algebra. On the other hand, if N is not subhomogeneous as a von Neumann algebra, N contains matrix algebras M_m for all $m\in\mathbb{N}$. Using Huruya's results [Hu], we see that a non-subhomogeneous C^*-algebra also contains a sequence (X_m) of subspaces such that $d_{cb}(X_m, M_m)\leqslant 2$. Therefore it suffices to prove that no subspace Y of $C(K, M_n)$ can contain a sequence (X_m) with $\sup_m d_{cb}(X_m, M_m) < \infty$. Indeed, given any subspace $Y\subset C(K, M_n)$, we deduce from a result of Smith [Sm] that for every operator space E and $v: E\to Y$, we have $\|v\|_{cb}\leqslant n\|v\|$. By Tomiyama's result (see e.g. [ER2]) the transposition map $T_m(x) = x^t$ on M_m satisfies $\|T_m\| = 1$ and $\|T\|_{cb} = m$. Hence Y cannot contain such a sequence (X_m). $\square$

For lack of a reference, we will give a proof of the following well-known observation.

Lemma 5.3. *Let* $1\leqslant p<\infty$ *and* N *a type I von Neumann algebra with separable predual, then* $L_p(N)$ *has a* C*-cb-basis* (*with* $C\leqslant 14$).

Proof. If N is of type I, we have

$$L_p(N) = \left(\sum_{n\in\mathbb{N}\cup\infty} \oplus L_p(\Omega_n, \mathscr{B}_n, \mu_n; S_p^n)\right)_p.$$

Since $(\Omega_n, \mathscr{B}_n, \mu_n)$ is a standard measure space and therefore admits a decomposition in atomic and non-atomic part

$$L_p(\Omega_n, \mathscr{B}_n, \mu_n) = L_p([0,1]) \oplus_p \ell_p(\Delta_n),$$

where Δ_n is a discrete, countable index set. Using the Haar basis for $L_p([0,1])$ and the unit vector basis in $\ell_p(\Delta_n)$, we obtain a *cb*-basis of $L_p(\Omega, \mathscr{B}_n, \mu_n)$ (with constant 1). According to Lemma 5.1, we get a 14-*cb* basis for $(\sum_{n\in\mathbb{N}\cup\{\infty\}} L_p([0,1]; S_p) \oplus_p \ell_p(\Delta_n; S_p))_p$. Using the special form of the basis, we easily obtain a 14-*cb*-basis for

$$\left(\sum_{n\in\mathbb{N}\cup\{\infty\}} \oplus L_p(\Omega, \mathscr{B}_n, \mu_n; S_p^n)\right)_p. \qquad \square$$

Example 5.4. Let N be a semifinite hyperfinite von Neumann algebra with a n.s.f. trace τ and $1 \leqslant p < \infty$. Then $L_p(N, \tau)$ is a $\mathscr{COL}_p$ space with constant 1.

Proof. According to [P5, Theorem 3.4.], there is an increasing net (E_α) of conditional expectations onto finite-dimensional subalgebras N_α of N such that $\alpha \leqslant \beta$ implies $E_\alpha = E_\alpha E_\beta = E_\beta E_\alpha$ for all $\alpha \leqslant \beta$. By interpolation E_α extends to complete contractions on $L_p(N, \tau)$. For $p < \infty$, we have that $\bigcup_\alpha L_p(N_\alpha, \tau)$ is norm dense in $L_p(N, \tau)$. Hence $L_p(N, \tau)$ is a $\mathscr{COL}_p$ space with constant 1. $\square$

Applying our abstract approach to this particular case, we obtain a basis in the separable hyperfinite case.

Theorem 5.5. *There exists a constant $C > 1$ with the following property. Let N be a semifinite hyperfinite von Neumann with a normal semifinite faithful trace τ and with separable predual. For $1 \leqslant p < \infty$ $L_p(N, \tau)$ has a C-cb-basis initially equivalent to some $\mathfrak{s}_p(\mathbf{m})$.*

Proof. We use the type decomposition $N = N_{\mathrm{I}} \oplus N_{\mathrm{II}}$. The assertion for $L_p(N_{\mathrm{I}})$ follows from Lemma 5.3. We note that by Example 5.4 $L_p(N_{\mathrm{II}})$ is a $\mathscr{COL}_p$ space with constant 1. By Lemma 2.16, we see that $L_p(N_{\mathrm{II}})$ contains $\mathfrak{s}_p$ completely complemented and therefore we are in the position to apply Theorem 4.10 and obtain a universal estimate for the *cb*-basis constant of $L_p(N_{\mathrm{II}})$. It is easy to combine the two bases for $L_p(N_{\mathrm{I}})$ and $L_p(N_{\mathrm{II}})$ and the assertion is proved. $\square$

In view of Theorem 2.17, the same proof applies to every QWEP von Neumann algebra with a separable predual and the CBAP. We are ready for the proof Theorem 0.4.

Proof of Theorem 0.4. Implication (i) $\Rightarrow$ (ii) is proved in [JR]. Equivalence (ii) $\Leftrightarrow$ (iii) is Theorem 2.17. By Theorem 4.10, we have (iii) $\Rightarrow$ (iv). (Note that as in the proof of Theorem 5.5 the *cb*-basis constant can be estimated as a function of the CBAP-constant. Moreover, by the results in [JR], the CBAP constant drops to 1 for $1<p<\infty$.) Since an operator space with a *cb*-basis has the CBAP, it also has the OAP. Hence, we have (iv) $\Rightarrow$ (i) and the proof is completed. $\square$

Remark 5.6. In the category of Banach spaces the same result is true. Indeed, the equivalence (i) $\Rightarrow$ (ii) is due to Grothendieck because $L_p(N)$ is a separable dual space (see e.g [LT, Theorem 1.e.15]). We refer to Remark 1.7 and the Banach space version of Proposition 2.4. The basis techniques can be directly deduced from the results in [JRZ].

Corollary 5.7. *Let $\mathbb{F}_n$ be the free group with n generators, $VN(\mathbb{F}_n)$ the von Neumann algebra generated by the left regular representation on $\ell_2(\mathbb{F}_n)$ with its canonical tracial state τ. If $1<p<\infty$, then $L_p(\mathbb{F}_n) = L_p(VN(\mathbb{F}_n),\tau)$ is a $\mathscr{COS}_p$ space with constant $\leqslant 9$ and has a cb-basis initially equivalent to some $\mathfrak{s}_p(\mathbf{m})$.*

Proof. According to Wassermann's construction [Wa], we see that $VN(\mathbb{F}_n)$ is completely contractively complemented in $\prod M_{n_k}/J_{\mathscr{U}}$, where $\mathscr{U}$ is an ultrafilter on $\mathbb{N}$ and

$$J_{\mathscr{U}} = \left\{ (x_k) | \lim_{k,\mathscr{U}} \tau_{n_k}(x_k^* x_k) = 0 \right\}.$$

Here τ_{n_k} is the normalized trace on M_{n_k}. This implies that $VN(\mathbb{F}_n)$ has the QWEP. This construction can also be used to prove directly that the space $L_p(VN(\mathbb{F}_n),\tau)$ is completely contractively complemented in $\prod_{\mathscr{U}} S_p^{n_k}$. By results of Haagerup (see [Ha3,DH]) it is known that $C^*_{red}(\mathbb{F}_n)$ has the CBAP with constant 1. Since the approximating finite rank maps are multipliers, we can use interpolation to show that $L_p(VN(\mathbb{F}_n),\tau)$ has the CBAP with constant 1. We refer to [JR] for more details. Hence the result follows from Theorem 4.10. $\square$

We will now discuss bases for preduals of hyperfinite non-semifinite von Neumann algebras.

Theorem 5.8. *Let N be a hyperfinite von Neumann algebra with $N_{\mathrm{I}} = \{0\}$ and N_* separable. Then N_* has a cb-basis initially equivalent to some $\mathfrak{s}_1(\mathbf{m})$.*

Proof. In view of Theorem 5.5 and the orthogonal decomposition $N_* = (N_{\mathrm{II}})_* \oplus_1 (N_{\mathrm{III}})_*$, we can assume $N_{\mathrm{II}} = 0$. In particular, we can and will assume that N is properly infinite. According to Lemma 2.16, $L_1(N)$ contains complemented S_1^n's. Then the result follows from Theorem 4.10 provided we can show that N_* is a $\mathscr{COL}_1$ space with respect to some $Y \subset N$ and Y has the CBAP. According to Proposition 2.4, it suffices to show that $\mathscr{N}_*$ has the γ_1-AP with respect to Y and contains complemented S_1^n with respect to Y. (Note that we know by Proposition 3.10 that N_* is $\mathscr{COL}_1$ but we need Y here!) Since N_* is separable, we can assume that N acts on a separable Hilbert space. According to [ElW, Theorem 3], we can find an increasing sequence N_k of I_{2^k} subalgebras of N such that $N = (\bigcup_k N_k)''$. Let Y be the norm closure of $\bigcup_k N_k$. Then Y is a nuclear, weak* dense subalgebra. In particular, Y has the CBAP. Since $N_k \cong M_{2^k}$ is a matrix algebra, it is hyperfinite and hence there exists a completely (positive) contraction $E_k : N \to N_k$ onto N_k. Let $F \subset N$ be a finite-dimensional subspace containing N_k. According to [EJR], we can apply local reflexivity and find a map $T_{k,F} : L_1(N_k) \to N_*$ such that $\|T_{k,F}\|_{cb} \leqslant (1+\varepsilon)$ and

$$|\langle T_{k,F}(x), y\rangle - \langle x, E_k(y)\rangle| \leqslant \frac{1}{2^{2k}}\varepsilon\|x\|\|y\|$$

for all $x \in L_1(N_k)$, $y \in F$. Let $\iota_k : N_k \to N$ be the inclusion map. In particular,

$$|\langle T_{k,N_k}(e_{ij}), e_{i'j'}\rangle - \delta_{ii'}\delta_{jj'}| \leqslant \frac{1}{2^{2k}}\varepsilon$$

holds for all the matrix units e_{ij}, $e_{i'j'}$, $i, i', j, j' \in \{1, \dots, 2^k\}$. As in Lemma 1.2, this implies that there is an isomorphism $w_k : N_k \to N_k$ such that $T^*_{k,N_k}\iota_k w_k = id_{N_k}$ and $\|w_k\|_{cb} \leqslant (1-\varepsilon)^{-1}$. Hence $w_k^* T_{k,N_k} = id_{L_1(N_k)}$ and from $\iota_k w_k(N_k) \subset N_k \subset Y$, we deduce that N_* contains S_1^n's with respect to Y. Moreover, the net $(T_{k,F})_{k\in\mathbb{N}, F\subset N}$ converges to the identity on N_* in the point-weak topology. Passing to a convex combination, we deduce that N_* has the γ_1-AP with respect to Y with constant 1. The proof is completed. $\square$

The next result involves modular theory and relies on the results in [EL]. This will be treated in more details in the subsequent paper [JRX]. In particular, we will improve on the constants.

Theorem 5.9. *Let N be a hyperfinite von Neumann algebra with a separable predual and $1 < p < \infty$. Then $L_p(N)$ has the CBAP with $\Lambda(L_p(N)) = 1$ and is a $\mathscr{COL}_p$ space with a cb-basis initially equivalent to some $\mathfrak{s}_p(\mathbf{m})$.*

Proof. First we note that a hyperfinite von Neumann algebra is injective and thus has the WEP, in particular is QWEP. Hence, we can apply Theorems 4.10 and 2.17. Thus it suffices to show that $L_p(N)$ has the CBAP with $\Lambda(L_p(N)) = 1$. Since N_* is separable, N admits a faithful normal state ϕ. We consider N as a von Neumann algebra on $L_2(N,\phi)$, where $L_2(N,\phi)$ is the Hilbert space obtained as the completion of N with respect to the norm

$$\|x\|_{L_2(N,\phi)} = \phi(x^*x)^{\frac{1}{2}}.$$

We denote by $\xi = \xi_\phi$ the image of 1 under the natural inclusion $N \subset L_2(N, \phi)$. Let $L = J\Delta^{1/2}$ be the polar decomposition of the densely defined conjugation map $L(x) = x^*$ on $L_2(N, \phi)$, i.e. J is an antilinear isometry and Δ is a positive selfadjoint (unbounded) operator. The automorphism group σ_t^ϕ is defined by $\sigma_t^\phi(x) = \Delta^{it} x \Delta^{-it}$ and leaves N invariant (see [KR]). We follow Kosaki [Ko] and consider for a fixed $x \in N$ the uniquely determined analytic function $f_x : \mathbb{C} \to N_*$ such that for all $t \in \mathbb{R}$ and $y \in N$

$$f_x(t)(y) = \phi(y\sigma_t^\phi(x)) \quad \text{and} \quad f_x(t-i)(y) = \phi(\sigma_t^\phi(x)y).$$

Using the density of the analytic elements of N (see [KR]) it turns out that the maps $I_z : N \to N_*,\ I_z(x) = f_x(z)$ are injective and therefore the interpolation space

$$E_p(N, \phi, \theta) = [I_{-i\theta}(N), N_*]_{\frac{1}{p}}$$

is well-defined for all $0 \leqslant \theta \leqslant 1$. Kosaki [Ko] showed that the space $E_p(N, \phi, \theta)$ is isometrically isomorphic to the Haagerup L_p space $L_p(N)$ and thus is independent of θ. Let us recall that the natural operator space structure on $E_1(N, \phi) = E_1(N, \phi, \frac{1}{2})$ is given by the map $\beta : E_1(N, \phi) \to N_*^{\mathrm{op}},\ \beta(\psi)(x) = \psi(x)$. Then the natural operator space structure on $E_p(N, \phi) = E_p(N, \phi, \frac{1}{2})$ is obtained by complex interpolation. Using Kosaki's isometric isomorphism between $L_p(N)$ and $E_p(N, \phi)$ we obtain the natural operator space structure on $L_p(N)$. We refer to [JRX] for a more detailed discussion. Therefore, it suffices to show that $E_p(N, \phi)$ has the CBAP. Let $\pi : N^{\mathrm{op}} \to N'$ be the *-isomorphism given by $\pi(x) = Jx^*J$. Then $\pi_*^{-1}\beta : E_1(N, \phi) \to N'_*$ is a complete isometry. Following [EL] there is a canonical embedding $\Psi : N \to N'_*$ given by $\Psi(x)(\pi(y)) = (\xi, xJyJ\xi)$ such that the map Ψ is a complete order isomorphism between N and the image $\Psi(N) \subset N'_*$. Let us show that

$$\Psi = \pi_*^{-1}\beta I_{-\frac{i}{2}}. \tag{5.2}$$

Indeed, let $x \in N$ be an analytic element and $y \in N \cong N^{\mathrm{op}}$. Then, we have

$$\begin{aligned} I_{-\frac{i}{2}}(x)(y) &= \phi(y\sigma_{-\frac{i}{2}}(x)) = (\xi, y\Delta^{\frac{1}{2}}x\Delta^{-\frac{1}{2}}\xi) = (y^*\xi, JJ\Delta^{\frac{1}{2}}xJ\xi) \\ &= (y^*\xi, Jx^*J\xi) = (\xi, yJx^*J\xi) = (\xi, Jx^*Jy\xi) = \Psi(y)(\pi(x)). \end{aligned}$$

This is not exactly what we claimed. However, $I_{-\frac{i}{2}} : N \to E_1(N, \phi) = N_*$ is formally symmetric, i.e.

$$I_{-\frac{i}{2}}(x)(y) = I_{-\frac{i}{2}}(y)(x).$$

It is easy to show that $(h, Jk) = (k, Jh)$ implies $(\xi, y\Delta x\xi) = (\xi, xy\xi)$ for any analytic elements x and y in N. Since $\Delta^{\pm\frac{1}{2}}\xi = \xi$, we deduce

$$\begin{aligned} I_{-\frac{i}{2}}(x)(y) &= (\xi, y\Delta^{\frac{1}{2}}x\xi) = (\Delta^{-\frac{1}{2}}\xi, \Delta^{\frac{1}{2}}y\Delta^{-\frac{1}{2}}\Delta x\xi) \\ &= (\xi, \sigma_{-\frac{i}{2}}(y)\Delta x\xi) = (\xi, x\sigma_{-\frac{i}{2}}(y)\xi) \\ &= \phi(x\sigma_{-\frac{i}{2}}(y)) = I_{-\frac{i}{2}}(y)(x) \end{aligned}$$

for all analytic elements. Therefore the symmetry of $I_{-\frac{i}{2}}$ follows by density of the analytic elements. Hence, we have

$$\Psi(x)(\pi(y)) = \Psi(y)(\pi(x))$$

for all $x, y \in N$ and (5.2) is proved. For a normal map $T : N \to N'_*$, we denote by $T' : N \to N'_*$ the map

$$T'(x)(\pi(y)) = T(y)(\pi(x)).$$

We say that T is symmetric of $T' = T$. Now, we want to apply the results in [EL] to show that we can approximate Ψ by a net R_v of completely positive symmetric finite rank maps. We will closely follow [EL] and indicate the modifications needed for this extra task. By the proof of [EL, Theorem 4.1], we see that Ψ can be approximated in the point-weak topology by an net (T_v) of completely positive normal finite rank contractions. Then the net T'_v also approximates $\Psi' = \Psi$ in the point-weak* topology. Passing to a convex combination, we can assume that (T_v) is a net of finite rank completely positive contractions such that (T_v) and (T'_v) converge in the point-norm topology to Ψ. Let us denote the state $\psi = \pi_*^{-1}\beta(\phi)$ and observe $\psi(\pi(x)) = \beta(\phi)(x) = \phi(x)$. Following [EL, Lemma 4.3], we can find for given $s_1, \dots, s_n$ and $\delta > 0$ an index v such that the perturbed map

$$S_v = T_v + \phi \otimes f + g \otimes \psi$$

is again completely positive. Here $f \in N'_*$, $g \in N_*$ are suitable positive functionals satisfying $||f|| \leqslant \frac{\delta}{2}$, $||g|| \leqslant \frac{\delta}{2}$ such that

$$||S_v|| \leqslant 1 + \delta, \quad S_v(1) \geqslant \psi \quad \text{and} \quad S'_v(1) \geqslant \psi,$$

and still

$$||S_v(s_i) - \Psi(s_i)|| \leqslant \delta \quad \text{and} \quad ||S'_v(s_i) - \Psi(s_i)|| \leqslant \delta.$$

Let us consider the symmetric map $R_v = \frac{S_v + S'_v}{2}$ and $\rho = R_v(1) = R'_v(1)$. Let $\delta + 2\delta^{\frac{1}{2}}(1+\delta)^{\frac{1}{2}}) < \varepsilon$. According to [EL, Lemma 4.4], we can find $0 \leqslant t \leqslant 1$ in N such that

$\rho(\pi(txt)) = \phi(x) = \Psi(1)(\pi(x))$. Let us define $\tilde{R}_v(x)(\pi(y)) = R_v(x)(\pi(tyt))$ and $M_t(x) = txt$. We consider $V_v = \tilde{R}_v M_t$, i.e.

$$V_v(x)(y) = R_v(txt)(\pi(tyt))$$

for all $x, y \in N$. Then we have again $V_v' = V_v$ and using the argument in [EL, Lemma 4.4], we deduce for $||r|| \leqslant 1$ and $||s_i|| \leqslant 1$

$$\begin{aligned}&|V_v(s_i)(\pi(r)) - R_v(s_i)(\pi(r))|\\ &\leqslant |R_v(ts_it)(\pi(trt)) - R_v(ts_it)(\pi(r))| + |(R_v(ts_it) - R_v(s_i))(\pi(r))|\\ &= |R_v(trt - r)(\pi(ts_it))| + |R_v(ts_it - s_i)(\pi(r))|\\ &\leqslant 2\delta^{\frac{1}{2}}(1 + (1+\delta)^{\frac{1}{2}}) \leqslant \varepsilon - \delta.\end{aligned}$$

Hence $||V_v(s_i)(r) - \Psi(s_i)|| \leqslant \varepsilon$ for all $i = 1, \ldots, n$ as in [EL, Proof Lemma 4.4]. This shows that, we can obtain a net (V_v) (indexed by finite subsets of N and positive real numbers) such that for every v, we have $V_v' = V_v$, there exists a $0 \leqslant t_v \leqslant 1$ with $V_v = \tilde{R}_v M_{t_v}$, $\tilde{R}_v(1) = \Psi(1)$ and (V_v) converges in the point-norm topology to Ψ. Let us consider

$$W_v = \Psi^{-1} V_v = \Psi^{-1} \tilde{R}_v M_{t_v}.$$

Since $\tilde{R}_v(1) = \Psi(1)$, we deduce that $W_v : N \to N$ is a completely positive normal contraction. Clearly, W_v also acts on N^{op} as a normal completely positive map with preadjoint $(W_v)_*$. Let us denote by $W_v' : N' \to N'$ the map defined by $W_v' = \pi W_v \pi^{-1}$ and denote by $(W_v')_* : N'_* \to N'_*$ the preadjoint map. Let $x, y \in N$, then we have

$$\begin{aligned}(W_v')_* \Psi(x)(\pi(y)) &= \Psi(x)(W_v'(\pi(y))) = \Psi(x)(\pi(W_v(y)))\\ &= \Psi(W_v(y))(\pi(x)) = V_v(y)(\pi(x))\\ &= V_v(x)(\pi(y)) = \Psi W_v(x)(\pi(y)).\end{aligned}$$

This implies

$$\pi_*^{-1}\beta(W_v)_* I_{-\frac{i}{2}} = (W_v')_* \pi_*^{-1} \beta I_{-\frac{i}{2}} = (W_v')_* \Psi = \Psi W_v = \pi_*^{-1}\beta I_{-\frac{i}{2}} W_v.$$

Since $\pi_*^{-1}\beta$ is an isomorphism, we deduce

$$(W_v)_* I_{-\frac{i}{2}} = I_{-\frac{i}{2}} W_v.$$

This shows that $U_v = ((W_v)_*, W_v)$ defines one operator on $I_{-\frac{i}{2}}(N) + N^*$ and $I_{-\frac{i}{2}}(N) \cap N^*$ and hence is compatible with the interpolation. In particular,

$$||U_v : E_p(N, \phi) \to E_p(N, \phi)||_{cb} \leqslant 1.$$

Passing to another convex combination if necessary, we can ensure that for every $x \in N$, the net $(U_\nu(x))$ converges in the strong* topology to x. This implies that $(I_{-\frac{i}{2}} U_\nu(x))$ converges in the norm topology to $I_{-\frac{i}{2}}(x)$, see for example [Ju1, Lemma 2.3]. By density of $I_{-\frac{i}{2}}(N)$ in $E_p(N, \phi)$, we deduce that U_ν converges to identity on $E_p(N, \phi)$ in the point-norm topology. Hence, $L_p(N) \cong E_p(N, \phi)$ has the CBAP (with constant 1). By Theorem 2.17 is a $\mathscr{COL}_p$ space. Theorem 4.10 yields the assertion for a universal constant. □

In our last application, we provide (a second proof for the existence of) a universal constant for the (*cb*-) basis constant of a nuclear C^*-algebra.

Theorem 5.10. *Let A be an infinite-dimensional, separable C^*-algebra. Then A is nuclear if and only if A has a cb-basis initially equivalent to some $\mathfrak{s}_\infty(\mathbf{m})$. Moreover, there exists an absolute constant C such that every separable nuclear C^*-algebra A has a C-cb-basis.*

Proof. If A has a *cb*-basis initially equivalent to some $\mathfrak{s}_\infty(\mathbf{m})$, we can find a constant $C > 0$ and an increasing sequence (X_n) of subspaces of A such that $d_{cb}(X_n, M_{m(1)} \oplus \cdots \oplus M_{m(n)}) \leqslant C$ and $\overline{\bigcup_n X_n} = A$. By the injectivity of $M_{m(1)} \oplus \cdots \oplus M_{m(n)}$, we deduce that A has the γ_∞-AP and hence A is nuclear by Pisier's result [P4, Remark after Theorem 2.9.]. Conversely, we assume that A is nuclear. Then A is locally reflexive. According to Theorem 3.11, A is a $\mathscr{COL}_\infty$ space with $\mathscr{COL}_\infty(A) \leqslant 3$. Since A^{**} is a hyperfinite von Neumann algebra and it is known that A^* has the CBAP with $\Lambda(A^*) = 1$. According to Theorem 4.10, A has a *cb*-basis. However, the estimates from Theorem 4.10 would give an estimate of the basis constant of the form $f(n)$ for subalgebras of $C(K, M_n)$ where $\lim_n f(n) = \infty$. We will now improve this estimate by proving the 'moreover part' of the assertion.

Let us first consider the case where A has an infinite-dimensional representation or a sequence $\pi_k : A \to M_{m_k}$ of finite-dimensional irreducible representations such that $\lim_k m_k = \infty$. Then there is a projection $p_k \in A^{**}$ such that $p_k A^{**} p_k \cong M_{m_k}$. Using the finite-dimensional spaces $\pi_k^*(M_{m_k}^*) \subset A^*$, we deduce from the locally reflexivity of A, see [AB], that A contains complemented M_{m_k}'s. According to Theorem 4.10, we deduce that A has a *cb*-basis initially equivalent to some $\mathfrak{s}_\infty(\mathbf{m})$ and this holds for a universal constant.

In the second case, we have to consider a C^*-algebra which has only finite-dimensional irreducible representations with a maximal degree d. In this case A is a type I C^*-algebra (see [Pe]) and the primitive spectrum $\check{A}$ coincides with the spectrum $\hat{A}$ of A (see [Pe, Theorem 6.1.5]). Moreover, the spectrum is locally compact [Pe, Theorem 6.1.11]. Let $d_1 \leqslant d$ be the largest integer such that A has infinitely many irreducible representations. We can remove the finite subset $X \subset \hat{A}$ of representations of dimension bigger than d_1. Indeed, using that for all m the set $\hat{A}_m$ of m-dimensional representations is Hausdorff [Pe, 4.4.10] and that the set of representations $> m$ are open, we deduce that X carries the discrete topology. According to the

Dauns-Hoffman Theorem [Pe, 4.4.8], we deduce

$$A = A_X \oplus_\infty B,$$

where A_X is a finite-dimensional C^*-algebra and B has infinitely many irreducible representations of dimension d_1 and all the other irreducible dimensions are of degree $< d_1$. Let us consider the countable open subset $\{\omega_n : n \in \mathbb{N}\} = X_{d_1} \subset \hat{A}$ of d_1 dimensional representations. Whether the sequence (ω_n) converges or not, we can find for every $n \in \mathbb{N}$, n disjoint open, non-empty sets $U_1, \ldots, U_n \subset X_{d_1}$. Thus we can find n points $\omega_1, \ldots, \omega_n$, functions $f_1, \ldots, f_n$ with support of f_j in U_j, $f_j : \hat{A} \to [0,1]$ and $f_j(\omega_j) = 1$. According to the Dauns–Hoffman Theorem [Pe, 4.4.8], we deduce that the subalgebra B_n generated by $\bigcup_{j=1,\ldots,n} f_j B_j$ contains $\ell_\infty^n(M_{d_1})$. Thus B contains complemented $\ell_\infty^n(M_{d_1})$'s and by Lemma 2.10 it contains $\ell_\infty^n(M_{d_1})$'s far out. According to Remark 2.6, we obtain that B is a $\mathscr{COL}_\infty$ space where the building blocks are $\ell_\infty^n(M_{d_1})$'s. Lemma 2.9 shows that the second assertion in Proposition 4.5 is satisfied for B and the family $(\ell_\infty^n(M_{d_1}))_{n\in\mathbb{N}}$. Adding the A_X part, we obtain a FDD $A = \sum_n Z_n$ such that

$$d_{cb}\left(\sum_{k\leqslant n} Z_n, (A_X \oplus_\infty \ell_\infty^{m_1+\cdots+m_n}(M_{d_1}))\right) \leqslant c,$$

and

$$d_{cb}(Z_1, A_X) \leqslant 1 \quad \text{and} \quad d_{cb}(Z_n, \ell_\infty^{m_n}(M_{d_1})) \leqslant c.$$

Thus Lemma 4.9 yields the assertion. □

Remark 5.11. As in the commutative theory, we can show that if X is locally reflexive and X^* has a *cb*-basis, then X has a *cb*-basis. This means we can alternatively deduce Theorem 5.10 from Theorem 5.8 by using the concept of ε-close finite-dimensional subspaces.

Acknowledgments

We want to express our special thanks to E. Ricard who contributed a crucial idea leading to a satisfactory version of Theorem 4.10 and permitted us to include his arguments in this paper.

References

[A1] J. Arazy, On large subspaces of the Schatten p-classes, Compositio Math. 41 (1980) 297–336.
[A2] J. Arazy, Basic sequences, embeddings, and the uniqueness of the symmetric structure in unitary matrix spaces, J. Funct. Anal. 40 (1981) 302–340.

[AB] R.J. Archbold, C.J. Batty, C^*-tensor norms and slice maps, J. London Math. Soc. 22 (1980) 127–138.
[BL] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, in: Grundlehren der Mathematischen Wissenschaften, Vol. 223, Springer, Berlin, New York, 1976.
[BP] D.P. Blecher, V.I. Paulsen, Tensor products of operator spaces, J. Funct. Anal. 99 (1991) 262–292.
[Bo] J. Bourgain, New Classes of $\mathscr{L}^p$-spaces, in: Lecture Notes in Mathematics, Vol. 889, Springer, Berlin, New York, 1981.
[BRS] J. Bourgain, H. Rosenthal, G. Schechtman, An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p, Ann. of Math. 114 (1981) 193–228.
[C1] A. Connes, Une classification des facteurs de type III, Ann. Sci. Ecole Norm. Sup. IV. Ser. 6 (1973) 133–252.
[C2] A. Connes, Classification of injective factors Cases II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. 104 (1976) 73–115.
[C3] A. Connes, Sur la theorie non commutative de l'integration, Algebres d'operateurs, Semin. Les Plans-sur-Bex 1978, Lecture Notes in Mathematics, Vol. 725, Springer, Berlin, 1979 pp. 19–143.
[CH] M. Cowling, U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. 96 (1989) 507–549.
[CS] E. Christensen, A. Sinclair, On von Neumann algebras which are complemented subspaces of $B(H)$, J. Funct. Anal. 122 (1994) 91–102.
[DH] J. De la Canniére, U. Haagerup, Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups, Amer. J. Math. 197 (1984) 455–500.
[D1] J. Dixmier, Formes linéaires sur un anneau d'opérateurs, Bull. Soc. Math. France 81 (1953) 9–39.
[D2] J. Dixmier, Von Neumann Algebras, North-Holland, Amsterdam, New York, Oxford, 1981.
[EH] E. Effros, U. Haagerup, Lifting problems and local reflexivity for C^*-algebras, Duke Math. J. 52 (1985) 103–128.
[EJR] E. Effros, M. Junge, Z.J. Ruan, Integral mappings and the principle of local reflexivity for noncommutative L^1-spaces, Ann. of Math. 151 (2000) 59–92.
[EL] E. Effros, E. Lance, Tensor products of operator algebras, Adv. in Math. 25 (1977) 1–34.
[ER1] E. Effros, Z.-J. Ruan, $\mathcal{OL}_p$ spaces, Contemp. Math. Amer. Math. Soc. 228 (1998) 51–77.
[ER2] E. Effros, Z.-J. Ruan, Operator Spaces, in: London Mathematical Society Monographs New Series, Vol. 23, Oxford University Press, Oxford, 2000.
[ElW] G.A. Elliot, E.J. Woods, The equivalence of various definitions for properly infinite von Neumann algebra to be approximately finite dimensional, Proc. Amer. Math. Soc. 60 (1976) 315–323.
[Fa] T. Fack, Type and cotype inequalities for non commutative L^p-spaces, J. Operator Theory 17 (1987) 255–279.
[FK] T. Fack, H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math. 123 (2) (1986) 269–300.
[Fi] F. Fidaleo, Canonical operator space structures on non-commutative L^p-spaces, J. Funct. Anal. 169 (1999) 226–250.
[GGMS] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, Some topological and geometric structures in Banach spaces Memo. Amer. Math. Soc. (1987) 379.
[GK] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Non-selfadjoint Operators, in: Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, RI, 1969.
[GL] Y. Gordon, D.R. Lewis, Absolutely summing operators and local unconditional structures, Acta Math. 133 (1974) 27–48.
[Gr] U. Groh, Uniform ergodic theorems for identity preserving Schwartz maps on W^*-algebras, J. Oper. Theory 11 (1984) 396–404.
[Ha1] U. Haagerup, L_p-spaces for von Neumann algebras of type III, unpublished preprint, 1978.

[Ha2] U. Haagerup, L^p-spaces associated with an arbitrary von Neumann algebra, Algébres d'opérateurs et leurs applications en physique mathématique (Proceedings of the Colloquium Marseille, 1977), Colloquium International CNRS, Vol. 274, CNRS, Paris, 1979, pp. 175–184.

[Ha3] U. Haagerup, The reduced C^*-algebra of the free group on two generator, 18th Scandinavian Congress of Mathematicians, Aarhus, 1980, Progress in Mathematics, Vol. 11, Birkhäuser, Boston, MA, 1981, pp. 321–335.

[Ha4] U. Haagerup, A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space, J. Funct. Anal. 62 (1985) 160–201.

[Ha5] U. Haagerup, Injectivity and decomposition of completely bounded maps, in: Operator Algebras and their Connection with Topology and Ergodic Theory, Springer Lecture Notes in Mathematics, Vol. 1132, Springer, Berlin, 1985, pp. 170–222.

[He] S. Heinrich, Ultraproducts in Banach spaces, J. Reine Angew. Math. 313 (1980) 104–702.

[Hu] T. Huruya, A norm property for spaces of completely bounded maps between C^*-algebras, Glasgow Math. J. 29 (1987) 181–184.

[JRZ] W.B. Johnson, H.P. Rosenthal, M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach space, Israel J. Math. 9 (1971) 488–506.

[Ju1] M. Junge, Doob's inequality for non-commutative martingales, J. Reine Angew. Math. 549 (2002) 149–190.

[Ju2] M. Junge, Applications of the Fubini theorem for non-commutative L_p spaces, Canad. J. Math., to appear.

[JOR] M. Junge, N. Ozawa, J-Z. Ruan, On $\mathcal{OL}_\infty$ structures of nuclear C^*-algebras, Math. Ann., to appear.

[JR] M. Junge, Z.J. Ruan, Approximation properties for non-commutative L_p spaces associated to discrete groups, Duke Math. J. 117 (2003) 313–341.

[JRX] M. Junge, Z. J. Ruan, Q. Xu, Rigid $\mathcal{OL}_p$ structures of non-commutative L_p spaces associated with hyperfinite von Neumann algebras, preprint.

[KP] M.I. Kadec, A. Pełczyńnski, Bases, lacunary sequences and complemented subspaces in the spaces L_p, Studia Math. 21 (1961/1962) 161–176.

[KR] R. Kadison, J. Ringrose, Fundamentals of the Theory of Operator Algebras I and II, in: Graduate Studies in Mathematics, Vols. 15, 16, American Mathematical Society, Providence, RI, 1997.

[Ki1] E. Kirchberg, On nonsemisplit extensions, tensor products and exactness of group C^*-algebras, Invent. Math. 112 (3) (1993).

[Ki2] E. Kirchberg, Commutants of unitaries in UHF algebras and functorial properties of exactness, J. Reine Angew. Math. 452 (1994) 39–77.

[Ko] H. Kosaki, Applications of the complex interpolation method to a von Neumann algebra: non-commutative L^p-spaces, J. Funct. Anal. 56 (1984) 29–78.

[Ku] R. Kunze, L^p-Fourier transforms in locally compact unimodular groups, Trans. Amer. Math. Soc. 89 (1958) 519–540.

[LP] J. Lindenstrauss, A. Pełczyński, Absolutely summing operators in L_p-spaces and their applications, Studia Math. 29 (1968) 275–326.

[LR] J. Lindenstrauss, H. Rosenthal, The $\mathcal{L}_p$ spaces, Israel J. Math. 7 (1969) 325–349.

[LT] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I,II, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 92, 97, Springer, Berlin, New York, 1977, 1979.

[Ma] J. Marcolino, La stabilité des espaces L^p non-commutatifs, Math. Scand. 81 (1997) 212–218.

[Ne] E. Nelson, Notes on non-commutative integration, J. Funct. Anal. 15 (1974) 103–116.

[NO] P-W. Ng, N. Ozawa, A characterization of completely 1-complemented subspaces of non-commutative L^1-spaces, Pacific J. Math 205 (2002) 171–195.

[NW] N.J. Nielsen, P. Wojtaszczyk, A remark on bases in $\mathcal{L}_p$-spaces with an application to complementably universal $\mathcal{L}_\infty$-spaces, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 21 (1973) 249–254.

[OR] T. Oikhberg, H.P. Rosenthal, Extension properties for the space of compact operators, J. Funct. Anal. 179 (2001) 251–308.

[Oz] N. Ozawa, Almost completely isometric embeddings between preduals of von Neumann algebras, J. Funct. Anal. 186 (2) (2001) 329–341.
[Pa] V.I. Paulsen, Completely Bounded Maps and Dilations, Pitman Research Notes in Mathematics, 1986.
[Pe] G.K. Pedersen, C^*-algebras and their Automorphism Groups, in: London Mathematical Society Monographs, Vol. 14, Academic Press, London, New York, 1979.
[P1] G. Pisier, Some results on Banach spaces without local unconditional structure, Compositio Math. 37 (1978) 3–19.
[P3] G. Pisier, Exact operator spaces (Orleans 1993), Astérisque 232 (1995) 159–186.
[P4] G. Pisier, The operator Hilbert space OH, complex interpolation and tensor norms, Memo. Amer. Math. Soc. 585 (1996).
[P5] G. Pisier, Non-commutative vector valued L_p spaces and completely p-summing maps, Astérisque 247 (1998).
[P6] G. Pisier, Introduction to operator space theory, London Math. Soc. Lect. Ser. 294, Cambridge University Press, 2003.
[Ra1] Y. Raynaud, On ultraproducts of non-commutative L_1-spaces, Colloq. del Dept. de Anal. Matematico (1997–1998), Univ. Complutense Madrid.
[Ra2] Y. Raynaud, On ultrapowers of non commutative L_p spaces, J. Operator Theory 48 (2002) 41–68.
[RX] Y. Raynaud, Q. Xu, On subspaces of non-commutative L_p-spaces, J. Funct. Anal. 203 (2003) 149–196.
[Se] I.E. Segal, A non-commutative extension of abstract integration, Ann. of Math. 57 (1953) 401–457.
[Sm] R. Smith, Completely contractive factorization of C^*-algebras, J. Funct. Anal. 64 (1985) 330–337.
[Su] F. Sukochev, Personal communication.
[Sz1] A. Szankowski, $B(\mathscr{H})$ does not have the approximation property, Acta Math. 147 (1981) 89–108.
[Sz2] A. Szankowski, On the uniform approximation property in Banach spaces, Israel J. Math. 49 (1984) 343–359.
[Ta] M. Takesaki, Theory of Operator Algebras I, Springer, New York, 1979.
[Te] M. Terp, L^p-spaces associated with von Neumann algebras, Copenhagen University, 1981.
[TJ] N. Tomczak-Jaegermann, The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p $(1<p<\infty)$, Studia Math. 50 (1974) 163–182.
[Wa] S. Wassermann, On tensor products of certain group C^*-algebras, J. Funct. Anal. 23 (1976) 239–254.