

A Banach space with a symmetric basis which is of weak cotype 2 but not of cotype 2

Peter G. Casazza*

Niels J. Nielsen†

Abstract

We prove that the symmetric convexified Tsirelson space is of weak cotype 2 but not of cotype 2.

Introduction

Weak type 2 and weak cotype 2 spaces were originally introduced and investigated by V.D. Milman and G. Pisier in [10] and weak Hilbert spaces by Pisier in [13]. A further detailed investigation can be found in Pisier's book [14]. The first example of a weak Hilbert space which is not isomorphic to a Hilbert space is the 2-convexified Tsirelson space (called the convexified Tsirelson space in this paper). This follows from the results of W.B. Johnson in [4]. For a detailed study of the original Tsirelson space we refer to [3].

Let X be a Banach space with a symmetric basis. It was proved in [14] that if X is a weak Hilbert space, then it is isomorphic to a Hilbert space and this has led to the belief that if X is just of weak cotype 2, then it is of cotype 2. However, this turns out not necessarily to be the case. The main result of this paper states that the symmetric convexified Tsirelson space is of weak cotype 2 but not of cotype 2.

We now wish to discuss the arrangement of this paper in greater detail.

In Section 1 we give some basic facts on properties related to weak type 2 and weak cotype 2 while Section 2 is devoted to a review of some results on the convexified Tsirelson space which we need for our main result. Most of these results are stated without proofs since they can be proved in a similar manner as the corresponding results for the original Tsirelson space.

In Section 3 we make the construction of the symmetric convexified Tsirelson space, investigate its basic properties and prove our main result stated above.

⁰2000 *Mathematics Subject Classification*: 46B03, 46B07.

*Supported by NSF grants DMS 9706108 and 0102686.

†Supported by the Danish Natural Science Research Council, grant 9801867.

Acknowledgement

The authors are indebted to Nigel Kalton for communicating Theorem 3.4 to us. Also, the authors are indebted to the referee for a very careful reading of the manuscript and many important suggestions for improvement.

1 Notation and Preliminaries

In this paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [8], [9] and [16]. B_X shall always denote the closed unit ball of the Banach space X and if X and Y are Banach spaces, then $B(X, Y)$ ($B(X) = B(X, X)$) denotes the space of all bounded linear operators from X to Y .

We let (g_n) denote a sequence of independent standard Gaussian variables on a fixed probability space $(\Omega, \mathcal{S}, \mu)$ and recall that a Banach space X is said to be of type p , $1 \leq p \leq 2$, (respectively cotype p , $2 \leq p < \infty$) if there is a constant $K \geq 1$ so that for all finite sets $\{x_1, x_2, \dots, x_n\} \subseteq X$ we have

$$\left(\int \left\| \sum_{j=1}^n g_j(t)x_j \right\|^p d\mu(t) \right)^{\frac{1}{p}} \leq K \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \quad (1.1)$$

(respectively

$$K \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \leq \left(\int \left\| \sum_{j=1}^n g_j(t)x_j \right\|^p d\mu(t) \right)^{\frac{1}{p}}. \quad (1.2)$$

The smallest constant K which can be used in (1.1) (respectively (1.2)) is denoted by $K^p(X)$ (respectively $K_p(X)$).

If L is a Banach lattice and $1 \leq p < \infty$, then L is said to be p -convex (respectively p -concave) if there is a constant $C \geq 1$ so that for all finite sets $\{x_1, x_2, \dots, x_n\} \subseteq L$ we have

$$\left\| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \right\| \leq C \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \quad (1.3)$$

(respectively

$$\left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \leq C \left\| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \right\|. \quad (1.4)$$

The smallest constant C which can be used in (1.3) (respectively (1.4)) is denoted by $C^p(L)$ (respectively $C_p(L)$).

It follows from [9, 1.d.6 (i)] that if L is of finite concavity (equivalently of finite cotype), then there is a constant $K \geq 1$ so that

$$\frac{1}{K} \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}} \right\| \leq \left(\int \left\| \sum_{j=1}^n g_j(t)x_j \right\|^2 d\mu(t) \right)^{\frac{1}{2}} \leq K \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}} \right\| \quad (1.5)$$

A Banach space X is said to be of weak type 2 if there is a constant C and a δ , $0 < \delta < 1$, so that whenever $E \subseteq X$ is a subspace, $n \in \mathbb{N}$ and $T \in B(E, \ell_2^n)$, then there is an orthogonal projection P on ℓ_2^n of rank larger than δn and an operator $S \in B(X, \ell_2^n)$ with $Sx = PTx$ for all $x \in E$ and $\|S\| \leq C\|T\|$.

Similarly X is called a weak cotype 2 if there is a constant C and a δ , $0 < \delta < 1$, so that whenever $E \subseteq X$ is a finite dimensional subspace, then there is a subspace $F \subseteq E$ so that $\dim F \geq \delta \dim E$ and $d(F, \ell_2^{\dim F}) \leq C$.

Our definitions of weak type 2 and weak cotype 2 space are not the original ones, but are chosen out of the many equivalent characterizations given by Pisier [14].

A weak Hilbert space is a space which is both of weak type 2 and weak cotype 2.

If A is a set we let $|A|$ denote the cardinality of A .

Definition 1.1 *If (x_n) and (y_n) are sequences in a Banach space X , we say that (x_n) is K -dominated by (y_n) if there is a constant $K > 0$ so that for all finitely non-zero sequences of scalars (a_n) we have*

$$\left\| \sum_n a_n x_n \right\| \leq K \left\| \sum_n a_n y_n \right\|.$$

The sequences (x_n) and (y_n) are K -equivalent if they K -dominate one another.

We will need some information about property (H) and related properties.

Definition 1.2 *A Banach space X has property (H_2) if there is a function $C(\cdot, \cdot)$ so that for every $0 < \delta < 1$ and for every normalized λ -unconditional basic sequence $(x_i)_{i=1}^n$ in X there is a subset $F \subseteq \mathbb{N}$ such that $|F| \geq \delta n$ and $(x_i)_{i \in F}$ is $C(\lambda, \delta)$ -equivalent to the unit vectors basis of $\ell_2^{|F|}$. If we only have that $(x_i)_{i \in F}$ is $C(\lambda, \delta)$ dominated by the unit vector basis of $\ell_2^{|F|}$, we say that X has property upper (H_2) . Similarly, we define property lower (H_2) .*

Definition 1.3 *A Banach space X is said to have property (H) if there is a function $f(\cdot)$ so that for every normalized λ -unconditional basic sequence $(x_i)_{i=1}^n$ in X , we have*

$$\frac{1}{f(\lambda)} n^{1/2} \leq \left\| \sum_{i=1}^n x_i \right\| \leq f(\lambda) n^{1/2}.$$

Similarly, we can define property upper (H) and property lower (H) .

The following is clear.

Proposition 1.4 *Property upper (resp. lower) (H_2) implies upper (resp. lower) (H) .*

We will see later that the converses of Proposition 1.4 fail.

The next result shows that any percentage of the basis will work in the definition of (H_2) . The proof follows from the argument of Pisier [14, Proposition 12.4, page 193].

Lemma 1.5 *For a Banach space X , the following are equivalent:*

- (1) X has property upper (resp. lower) (H_2) .
- (2) There exists one $0 < \delta < 1$ satisfying the conclusion of property upper (resp. lower) (H_2) .

The corresponding result for property (H) is in [3, Proposition Ae1, page 14].

Lemma 1.6 *For a Banach space X , the following are equivalent:*

- (1) X has property upper (resp. lower) (H) .
- (2) There is a $0 < \delta < 1$ so that for every λ -unconditional basic sequence $(x_i)_{i=1}^n$ in X there is a subset $F \subset \{1, 2, \dots, n\}$ with $|F| \geq \delta n$ and $(x_i)_{i \in F}$ has property upper (resp. lower) (H) .

The next theorem is due to Pisier [14, Proposition 12.4].

Proposition 1.7 *Every weak Hilbert space has property (H_2) .*

We also have from Pisier [14, Proposition 10.8, page 160 and Proposition 11.9, page 174]:

Proposition 1.8 *The following implications hold for a Banach space X :*

- (1) Weak cotype 2 implies property lower (H) .
- (2) Weak type 2 implies property upper (H) .

The converses of Proposition 1.8 are open questions. However, for Banach lattices it is known that property (H) , property (H_2) and being a weak Hilbert space are all equivalent. This is a result of Nielsen and Tomczak-Jaegermann [12].

2 Convexified Tsirelson Space

Since there is only a “partial theory” developed for the convexified Tsirelson space T^2 , we will review what we need here.

Notation 2.1 *If E, F are sets of natural numbers, we write $E < F$ if for every $n \in E$ and every $m \in F$, $n < m$. If $E = \{k\}$, we just write $k < F$ for $E < F$.*

Definition 2.2 *Let $(t_n)_{n=1}^\infty$ be the canonical unit vectors in $\mathbb{R}^\mathbb{N}$. The convexified Tsirelson space T^2 is (see [3]) the set of vectors $x = \sum_n a_n t_n$ for which the recursively defined norm below is finite.*

$$\|x\|_{T^2} = \max\{\sup|a_n|, 2^{-1/2} \sup \left(\sum_{j=1}^k \|E_j x\|_{T^2}^2 \right)^{1/2}\}, \quad (2.1)$$

where the second “sup” is taken over all choices

$$k \leq E_1 < E_2 < \dots < E_k,$$

and $E x = \sum_{n \in E} a_n t_n$.

Remark: It follows from Pisier [14, Chapter 13] ([3, Chapter 1] for the original Tsirelson space) that there exists a norm satisfying (2.1) above.

We will now list the known results for this space (which we will need) and where they can be found. Although many of the results we need have formally been proved for the original Tsirelson space T , using the fact that T^2 is the 2-convexification of T we can easily carry these results over to T^2 . The first result can be found in [3] and [14].

Proposition 2.3 *The unit vectors (t_n) form a 1-unconditional basis for T^2 . The space T^2 is of type 2 and weak cotype 2 but does not contain a Hilbert space. Also, the unit vector basis (e_n) of ℓ_2 1-dominates all subsequences of (t_n) . That is, for $x \in T^2$, $\|x\|_{T^2} \leq \|x\|_{\ell_2}$. Finally, if $I \subset \{n, n+1, n+2, \dots\}$ with $|I| \leq n$ and $x = \sum_{n \in I} a_n t_n$ then $\|x\|_{\ell_2} \leq 2\|x\|_{T^2}$.*

We also need Lemma II.1, page 19 of [3]. This lemma is stated in [3] for the original Tsirelson space but the proof works the same in T^2 .

Proposition 2.4 *Let $y_n = \sum_{i \in I_n} a_i t_i$ ($n = 1, 2, 3, \dots$) be a normalized disjointly supported sequence in (t_n) and let $p_n = \min I_n$. Then for every sequence of scalars (a_n) we have*

$$\left\| \sum_n a_n t_{p_n} \right\|_{T^2} \leq \left\| \sum_n a_n y_n \right\|_{T^2}.$$

Next we need to see which subsequences of the unit vector basis of T^2 are equivalent to the original basis. To do this we need:

Notation 2.5 *The fast growing hierarchy from logic is a family of functions on \mathbb{N} given by: $g_0(n) = n + 1$, and for $i \geq 0$, $g_{i+1}(n) = g_i^{(n)}(n)$, where for any function f , $f^{(n)}$ is the n -fold iteration of f . We also set $\text{exp}_0(n) = n$ and for $i \geq 1$ and n ,*

$$\text{exp}_i(n) = 2^{\text{exp}_{i-1}(n)}.$$

Finally we let $\log_0(n) = n$, and for any $i \geq 1$ and n large enough so that $\log_{i-1}(n) > 0$, let

$$\log_i(n) = \log(\log_{i-1}(n)).$$

The next result is due to Bellenot [1]. He does this result in the original Tsirelson's space T , but the proof works perfectly well in T^2 .

Proposition 2.6 *A subsequence (t_{k_n}) of (t_n) is equivalent to (t_n) if and only if there is a natural number i so that $k_n \leq g_i(n)$, for all large n . Moreover, (t_{k_n}) always 1-dominates (t_n) and there is a constant $K \geq 1$ so that the equivalence constant is K^i for the case $k_n = g_i(n)$.*

One important consequence is (see Pisier [14] or Casazza and Shura [3]).

Proposition 2.7 *For every natural number i , every $g_i(n)$ -dimensional subspace of $\text{span}(t_j)_{j \geq n}$ is K^i -isomorphic to a Hilbert space and K^i -complemented in T^2 .*

If X is a weak Hilbert space with an unconditional basis, then it follows from [12] that the conclusion of Proposition 2.7 remains true after a suitable permutation of the basis.

The next result comes from [3, Theorem IV.b.3, page 39]. The theorem there is proved for the regular Tsirelson space but the techniques easily adapt to convexified space. Also, although the theorem is stated for distances between subspaces, the proof actually checks the equivalence constant of normalized disjointly supported sequences with n -elements in (t_n) and the unit vector basis of ℓ_1^n . So we state this form here.

Proposition 2.8 *Every n -dimensional subspace of T^2 is $K^i \log_i(n)$ isomorphic to ℓ_2^n , for every $i \geq 2$ with $\log_i(n) \geq 1$. Moreover, every normalized disjointly supported sequence $(x_j)_{j=1}^n$ in T^2 is $K^i \log_i(n)$ -equivalent to the unit vector basis of ℓ_2^n , and if the support of the x_j 's lies in $(t_j)_{j=n}^\infty$, then $(x_j)_{j=1}^n$ is 2-equivalent to the unit vector basis of ℓ_2^n .*

We need one more result on convexified Tsirelson.

Proposition 2.9 *If $x = \sum_j a_j t_j \in T^2$, then for all $n \in \mathbb{N}$ and all $i \geq 2$ with $\log_i(n) \geq 1$,*

$$\left\| \sum_j a_j t_{nj} \right\|_{T^2} \leq 2K^i(\log_i n) \|x\|_{T^2}.$$

Proof: By Proposition 2.6 and Proposition 2.8 we have

$$\begin{aligned} \left\| \sum_j a_j t_{nj} \right\|_{T^2} &\leq \left\| \sum_{j=1}^n a_j t_{nj} \right\|_{T^2} + \left\| \sum_{j=n+1}^\infty a_j t_{nj} \right\|_{T^2} \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} + K^2 \left\| \sum_{j=n+1}^\infty a_j t_j \right\|_{T^2} \\ &\leq K^i(\log_i n) \left\| \sum_{j=1}^n a_j t_j \right\|_{T^2} + K^2 \left\| \sum_{j=n+1}^\infty a_j t_j \right\|_{T^2} \leq 2K^i(\log_i n) \|x\|_{T^2}, \end{aligned}$$

where in the second inequality above we use Propositions 2.3 and 2.6 and the fact that $nj \leq g_2(j)$ for all $j \geq n+1$; and in the third inequality we have used Proposition 2.8. \square

3 Symmetric Convexified Tsirelson Space

There is almost no existing theory for the symmetric convexified Tsirelson space. But there is a theory for the symmetric Tsirelson space. We will list the results we need on this topic. They can be found in Casazza and Shura [3, Chapter X.E].

Notation 3.1 *For T^2 or $(T^2)^*$ we will work with the non-decreasing rearrangement operator D . That is, if $x = \sum_n a_n t_n$ then $Dx = \sum_n a_n^* t_n$ where (a_n^*) is the non-decreasing re-arrangement of the non-zero a_n 's where by non-decreasing we mean the absolute values are non-decreasing.*

The construction of Chapter VIII of [3, Chapters VIII and X.B] shows

Proposition 3.2 *Let Π denote the group of all permutations of \mathbb{N} . There is a constant $K \geq 1$ so that for any $x = \sum_n a_n t_n^* \in (T^2)^*$ we have*

$$\|x\|_{S[(T^2)^*]} =: \sup_{\sigma \in \Pi} \left\| \sum_n a_{\sigma(n)} t_n^* \right\|_{(T^2)^*} \leq K \|Dx\|_{(T^2)^*} \leq K \sup_{\sigma \in \Pi} \left\| \sum_n a_{\sigma(n)} t_n^* \right\|_{S[(T^2)^*]}. \quad (3.1)$$

We will define the *dual space of the symmetric convexified Tsirelson space* first because it is natural in terms of the above.

Definition 3.3 *We let $S[(T^2)^*]$ be the family of all vectors for which $\|x\|_{S[(T^2)^*]}$ is finite. Then this is a Banach space with a natural symmetric basis, denoted (t_n^{s*}) , called the dual space of the symmetric convexified Tsirelson space.*

To define the *the symmetric convexified Tsirelson space* we need a result kindly communicated to us by N.J. Kalton.

Let X be a Banach sequence space with the standard unit vector basis (e_i) . Define the permutation operators $S_\sigma(\xi) = (\xi_{\sigma(n)})_{n=1}^\infty$ for $\sigma \in \Pi$ and let L_k^j to be the linear map such that $L_k^j(e_n) = e_{kn+j}$ for all $n \in \mathbb{N}$. Finally we let c_{00} denote the spaces of real sequences which are eventually 0.

Theorem 3.4 *Suppose X is a Banach sequence space which is p -convex and q -concave where $1 < p < q < \infty$. Suppose $\max_{0 \leq j < k} \|L_k^j\| \leq Ck^a$ where $a + p^{-1} < 1$. Then*

$$\|\xi\|_{X_{inf}} = \inf_{\sigma \in \Pi} \|S_\sigma \xi\|_X, \quad x \in c_{00}$$

defines a quasi-norm on c_{00} which is equivalent to a norm. The dual of X_{inf} is X_{sup}^ where*

$$\|\xi\|_{X_{sup}^*} = \sup_{\sigma \in \Pi} \|S_\sigma \xi\|_{X^*}.$$

Proof: Let us start by supposing $x_1, \dots, x_k \in c_{00}$ are disjointly supported and that $\sigma_1, \dots, \sigma_k \in \Pi$. Then

$$\begin{aligned} \|x_1 + \dots + x_k\|_{X_{inf}} &\leq \left\| \sum_{j=1}^k L_k^{j-1} S_{\sigma_j} x_j \right\|_X \\ &\leq \left(\sum_{j=1}^k \|L_k^{j-1} S_{\sigma_j} x_j\|_X^p \right)^{\frac{1}{p}} \\ &\leq Ck^a \left(\sum_{j=1}^k \|S_{\sigma_j} x_j\|_X^p \right)^{\frac{1}{p}}. \end{aligned}$$

Now taking an infimum over σ_j gives

$$\|x_1 + \dots + x_k\|_{X_{inf}} \leq Ck^a \left(\sum_{j=1}^k \|x_j\|_{X_{inf}}^p \right)^{\frac{1}{p}}. \quad (3.2)$$

Let us use (3.2) first to show that $\|\cdot\|_{X_{inf}}$ is a quasi-norm. Indeed if $x, y \in c_{00}$ then

$$\|x + y\|_{X_{inf}} \leq 2 \max(\|x\|, \|y\|) \leq 2^{a+1} C (\|x\|_{X_{inf}} + \|y\|_{X_{inf}}).$$

Next note that (3.2) implies

$$\|x_1 + \cdots + x_k\|_{X_{inf}} \leq C k^{a+\frac{1}{p}} \max_{1 \leq j \leq k} \|x_j\|_{X_{inf}}.$$

From this it follows easily that if $a + \frac{1}{p} < \frac{1}{r} < 1$ we have

$$\|x_1 + \cdots + x_k\|_{X_{inf}} \leq C_r \left(\sum_{j=1}^k \|x_j\|_{X_{inf}}^r \right)^{\frac{1}{r}}$$

for disjoint x_1, \dots, x_k . Thus we have an upper r -estimate for X_{inf} .

It is trivial to show X_{inf} has a lower q -estimate. Now by [5, Theorem 4.1] (a simpler proof is given in [6, Theorem 3.2]) it follows that X_{inf} is *lattice-convex* and this means that an upper r -estimate implies (lattice) s -convexity for all $s < r$ (Theorem 2.2 of [5]). Hence X_{inf} is r -convex for every r with $a + \frac{1}{p} < \frac{1}{r}$. In particular 1-convexity implies the quasi-norm is equivalent to a norm. In fact X_{inf}^* is a reflexive Banach space.

Now it is obvious that $X_{inf} \subset (X_{sup}^*)^*$ and $X_{sup}^* \subset (X_{inf})^*$. Hence it follows easily that $(X_{inf})^* = X_{sup}^*$. \square

Remark: We can apply the above result to the case of the weighted ℓ_p -space X , with $1 < p < \infty$ defined by the norm

$$\|\xi\|_X = \left(\sum_{n=1}^{\infty} |\xi_n|^p w_n \right)^{\frac{1}{p}}$$

where (w_n) is an increasing sequence satisfying an estimate of the form

$$w_{kn} \leq C k^a w_n$$

where $a < p - 1$. The X_{inf} is defined by the quasi-norm

$$\|\xi\|_{X_{inf}} = \left(\sum_{n=1}^{\infty} (\xi_n^*)^p w_n^p \right)^{\frac{1}{p}}$$

where (ξ_n^*) is the decreasing rearrangement of $(|\xi_n|)$. In this case X_{sup} is the Lorentz space $d((w_n)^{-q/p}, q)$.

This result can be rephrased. If (v_n) is a positive decreasing sequence satisfying an estimate $v_n \leq C k^b v_{kn}$ where $b < 1$ then $d((v_n), p)^*$ can be identified with the space of all sequences (ξ_n) so that

$$\left(\sum_{n=1}^{\infty} (\xi_n^*)^q v_n^{-q/p} \right)^{\frac{1}{q}} < \infty.$$

This result is a special case of results of Reisner [15].

Proposition VIII.a.8 of [3] states that the decreasing rearrangement operator D is a bounded non-linear operator on the original Tsirelson space T . This result then immediately carries over to the 2-convexification of T which is our convexified Tsirelson space T^2 . By Proposition 2.9 we have that Theorem 3.4 holds in this case. We summarize this in the following result:

Proposition 3.5 *There is a constant $K \geq 1$ so that for any $x = \sum_n a_n t_n \in T^2$ we have*

$$\inf_{\sigma \in \Pi} \left\| \sum_n a_{\sigma(n)} t_n \right\|_{T^2} \leq \|Dx\|_{T^2} \leq K \inf_{\sigma \in \Pi} \left\| \sum_n a_{\sigma(n)} t_n \right\|_{T^2}. \quad (3.3)$$

Moreover, there is a norm $\|\cdot\|_{S(T^2)}$ on the set of vectors for which $\|Dx\| < \infty$ satisfying

$$\frac{1}{K} \|x\|_{S(T^2)} \leq \|Dx\|_{T^2} \leq K \|x\|_{S(T^2)}. \quad (3.4)$$

Note that our operator D does not satisfy a triangle inequality, but does with the constant K on the sum side of the triangle inequality.

Definition 3.6 *The symmetric convexified Tsirelson space is the Banach space $S(T^2)$ of vectors for which $\|x\|_{S(T^2)} < \infty$ with natural unit vector basis (t_n^s) . By Theorem 3.4 this is a reflexive Banach space whose dual space is $S[(T^2)^*]$.*

It is known [3] that every infinite dimensional subspace of $S(T^2)$ contains a subspace which embeds into T^2 . In particular $S(T^2)$ is a Banach space with a natural symmetric basis which has no subspaces isomorphic to c_0 or ℓ_p for $1 \leq p < \infty$. Also T^2 embeds into $S(T^2)$. Since the unit vector basis of ℓ_2 uniformly dominates all block bases of (t_n) in T^2 , it follows that the unit vector basis of $S(T^2)$ is also dominated by the unit vector basis of ℓ_2 .

Proposition 3.7 *The space $S(T^2)$ fails property upper (H) (even for disjointly supported elements) and fails property lower (H_2) . Hence $S(T^2)$ is not of weak type 2 and not of cotype 2.*

Proof: First we check property lower (H_2) . Since (t_n^s) is symmetric and is dominated by the unit vector basis of ℓ_2 , it follows that if this family had subsets dominating the unit vector basis of ℓ_2 , then (t_n^s) would be equivalent to the unit vector basis of ℓ_2 which is impossible.

For property upper (H) , fix $M > 1$ and choose a decreasing sequence of non-zero scalars $(a_i)_{i=1}^n$ with $\|\sum_{i=1}^n a_i e_i\|_{\ell_2} \geq M$ and $\|\sum_{i=1}^n a_i t_i\|_{T^2} = 1$. This can be done by a obvious modification of the construction of [[3], Chapter IV]. For all $1 \leq j \leq 2n$ let

$$x_j = \sum_{i=nj+1}^{n(j+1)} a_{i-nj} t_i^s,$$

be vectors in $S(T^2)$. So $\|x_j\|_{S(T^2)} \leq K$ for all $1 \leq j \leq 2n$. Now, $D(\sum_{i=1}^{2n} x_i) = \sum_{i=1}^{2n} y_i$ where each y_i is an appropriate permutation of x_i . Since the y_i 's are disjoint, there is a subset $I \subset \{1, 2, \dots, 2n\}$ with $|I| \geq n$ and each y_i with $i \in I$ has its support in $(t_j^s)_{j=n}^\infty$. Hence,

$$\begin{aligned} \left\| \sum_{i=1}^{2n} x_i \right\|_{S(T^2)} &\geq K^{-1} \left\| D \left(\sum_{i=1}^{2n} x_i \right) \right\|_{T^2} \geq K^{-1} \left\| \sum_{i \in I} y_i \right\|_{T^2} \\ &\stackrel{(1)}{\geq} (2K)^{-1} \left(\sum_{i \in I} \|y_i\|_{T^2}^2 \right)^{1/2} \stackrel{(2)}{\geq} (4K)^{-1} \left(\sum_{i \in I} \|y_i\|_{\ell_2}^2 \right)^{1/2} \geq (4K)^{-1} M |I|^{1/2} = (4K)^{-1} M n^{1/2}. \end{aligned}$$

In inequality (1) above we use Proposition 2.8 and in inequality (2) we use Proposition 2.3.

Since M was arbitrarily large, it follows that $S(T^2)$ fails upper (H) - for disjoint elements.

□

We shall now need a result essentially due to S. Kwapien. In the form we present it is due to W.B. Johnson and it appeared in [7]

Proposition 3.8 *There is a function*

$$N(k, \epsilon) = \left\lceil \frac{2k^2}{\epsilon} \right\rceil^k$$

such that for any fixed $0 < \epsilon < 1$, every order complete Banach Lattice L , and every k -dimensional subspace F of L , there are $N = N(k, \epsilon)$ disjoint elements $(x_j)_{j=1}^N$ in L and a linear operator $V : F \rightarrow X = \text{span}(x_j)$ such that for all $x \in F$ we have

$$\|Vx - x\| \leq \epsilon \|x\|.$$

Proposition 3.9 *There is a constant $K > 1$ so that for every subspace E of $S(T^2)$ of dimension n , we have for all $i \in \mathbb{N}$ for which $\log_{i-1} n$ exists,*

$$d(E, \ell_2^n) \leq K^i \log_{i-2} n.$$

Moreover, any normalized disjointly supported sequence of vectors $(x_i)_{i=1}^n$ in $S(T^2)$ is $K^i \log_{i-2} n$ -equivalent to the unit vector basis of ℓ_2^n .

Proof: By giving up one level of logs we may assume by Proposition 3.8 that we are working with a normalized disjointly supported sequence of vectors $(x_j)_{j=1}^n$ in $S(T^2)$. We will show that this disjointly supported sequence is $K^i \log_{i-2} n$ -equivalent to the unit vector basis of ℓ_2 . Now there is a disjoint set of permutations y_j of the x_j so that

$$\begin{aligned} \left\| \sum_{j=1}^n a_j x_j \right\|_{S(T^2)} &\geq \frac{1}{K} \left\| \sum_{j=1}^n a_j y_j \right\|_{T^2} \\ &\geq \frac{1}{K} \left\| \sum_{j=1}^n a_j t_j \right\|_{T^2} \geq \frac{1}{K^{i+1} (\log_i n)} \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2}, \end{aligned}$$

where in the second inequality above we use Proposition 2.4 and the third uses Proposition 2.8. Also, let $Dx_j = z_j$ and

$$w_j = \sum_k z_j^{(k)} t_{n(k-1)+j},$$

We may assume that K is larger than the type 2 constant of T^2 . By Proposition 2.9 we have

$$\begin{aligned} \|D \sum_{j=1}^n a_j x_j\|_{ST^2} &\leq K \left\| \sum_{j=1}^n a_j w_j \right\|_{T^2} \leq K^2 \left(\sum_{j=1}^n |a_j|^2 \|w_j\|_{T^2}^2 \right)^{1/2} \\ &\leq K^2 \left(\sum_{j=1}^n |a_j|^2 [2K^i (\log_i n)]^2 \right)^{1/2} \leq 2K^{i+2} (\log_i n) \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2}, \end{aligned}$$

where in the second inequality above we used the fact that the type 2 constant of T^2 is less than or equal to K . Now,

$$d(E, \ell_2^n) \leq 2K^{2i+3} (\log_i n)^2 \leq K^i (\log_{i-1} n).$$

The $\log_{i-2} n$ in the statement of the theorem comes from the fact that we first applied Proposition 3.8. \square

Recall that the Maurey-Pisier Theorem (see e.g. [11], page 85) states that if X is a Banach space then ℓ_{p_X} and ℓ_{q_X} are finitely representable in X where

$$p_X = \sup \{p | X \text{ is of type } p\},$$

and

$$q_X = \inf \{q | X \text{ is of cotype } q\}.$$

However, Proposition 3.9 implies that the only ℓ_p -space which is finitely representable in $S(T^2)$ is ℓ_2 . This gives the following result.

Corollary 3.10 *The space $S(T^2)$ is of type p for all $1 \leq p < 2$ and of cotype q for all $2 < q$.*

Before we go on, we need a criterion for a Banach space to be of weak cotype 2. We shall say that a Banach space X has *property (P)* if there is a constant K so that whenever $\{x_1, x_2, \dots, x_n\} \subseteq X$ is a finite set with $\max_{1 \leq j \leq n} |s_j| \leq \|\sum_{j=1}^n s_j x_j\|$ for all $(s_j) \subseteq \mathbb{R}$, then

$$\sqrt{n} \leq K \left(\int \left\| \sum_{j=1}^n g_j(t) x_j \right\|^2 d\mu(t) \right)^{\frac{1}{2}} \quad (3.5)$$

It was proved by Pisier [14, Proposition 10.8] that if X is of weak cotype 2, then it has property (P). It turns out that (P) characterizes weak cotype 2 spaces. This fact might be known to specialists but we shall give a short proof here:

Theorem 3.11 *If X has property (P), then it is of weak cotype 2.*

Proof: Let $E \subseteq X$ be a finite dimensional subspace, say $\dim(E) = 2n$. By a result of Bourgain and Szarek [2, Theorem 2] there is a universal constant C and $\{x_1, x_2, \dots, x_n\} \subseteq X$ so that for all $(s_j) \subseteq \mathbb{R}$ we have

$$\max_{1 \leq j \leq n} |s_j| \leq \left\| \sum_{j=1}^n s_j x_j \right\| \leq C \left(\sum_{j=1}^n |s_j|^2 \right)^{\frac{1}{2}} \quad (3.6)$$

Using property (P) we get that

$$\sqrt{n} \leq K \left(\int \left\| \sum_{j=1}^n g_j(t) x_j \right\|^2 d\mu(t) \right)^{\frac{1}{2}} \quad (3.7)$$

where K is the constant of property (P). Now, (3.7) and the right inequality of (3.6) allow us to use the well-known formulation of Dvoretzky's theorem (see e.g. [11], Theorem 4.2, also [16], pages 25 and 81) to conclude that there is a universal constant η such that if $k \leq \eta K^{-2} C^{-2} n$, then there is a k -dimensional subspace $F \subseteq [x_j]$ with $d(F, \ell_2^k) \leq 2$. From [14, Theorem 10.2] it now follows that X is of weak cotype 2. \square

We shall say that a sequence $(x_j)_{j=1}^n$ in a Banach space X is 1-separated if $\|x_i - x_j\| \geq 1$ for all $1 \leq i, j \leq n, i \neq j$. It follows immediately from Theorem 3.11 that if every 1-separated sequence in X satisfies (3.5), then X is of weak cotype 2.

We are now ready to prove that the symmetric convexified Tsirelson space is a weak cotype 2 space with a symmetric basis which is not of cotype 2. Hence its dual space is a symmetric space which is of weak type 2 but fails to be of type 2.

Theorem 3.12 *The space $S(T^2)$ is a weak cotype 2 space.*

Proof: Let $(x_j)_{j=1}^n$ be a 1-separated sequence in $S(T^2)$. Without loss of generality we may assume that for all $1 \leq i \leq n$ we have $\|x_i\|_{S(T^2)} \geq 1$. We wish to show that (3.5) holds. If K is a constant which satisfies (1.5) for both T^2 and $S(T^2)$ and (3.4), then by definition we can find a $\sigma \in \Pi$ so that:

$$\left\| \left(\sum_{j=1}^n |S_\sigma x_j|^2 \right)^{\frac{1}{2}} \right\|_{T^2} = \left\| S_\sigma \left(\sum_{j=1}^n |x_j|^2 \right) \right\|_{T^2} \leq K \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}} \right\|_{S(T^2)} \quad (3.8)$$

Since S_σ is an isometry on $S(T^2)$, we can without loss of generality assume that actually $x_j = S_\sigma x_j$ for all $1 \leq j \leq n$.

Put $k = \log \log n$ and let P_k be the natural projection of T^2 onto the span of $(t_j)_{j=1}^k$. We now examine two cases.

Case I: There is a subset $I \subset \{1, 2, \dots, n\}$ with $|I| \geq \frac{n}{2}$ so that $\|P_k x_j\|_{\ell_2} \geq \log k$ for all $j \in I$.

Now we compute,

$$\begin{aligned}
\left(\int \left\| \sum_{j=1}^n g_j(t) x_j \right\|_{S(T^2)}^2 d\mu(t) \right)^{\frac{1}{2}} &\stackrel{(1)}{\geq} \frac{1}{K} \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_{S(T^2)} \geq \frac{1}{K^2} \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_{T^2} \quad (3.9) \\
&\stackrel{(2)}{\geq} \frac{1}{K^2} \left\| \left(\sum_{j=1}^n |P_k x_j|^2 \right)^{1/2} \right\|_{T^2} \stackrel{(3)}{\geq} \frac{1}{(\log k) K^3} \left\| \left(\sum_{j \in I} |P_k x_j|^2 \right)^{1/2} \right\|_{\ell^2} \\
&= \frac{1}{(\log k) K^3} \left(\sum_{j \in I} \|P_k x_j\|_{\ell^2}^2 \right)^{\frac{1}{2}} \stackrel{(4)}{\geq} \frac{1}{K^3 \sqrt{2}} \sqrt{n},
\end{aligned}$$

where inequality (1) follows from equation (1.5); (2) from the fact that $\|P_k\| \leq 1$; (3) from Proposition 2.7; and (4) from our assumption on case 1.

Case II: There is a subset $I \subset \{1, 2, \dots, n\}$ with $|I| \geq \frac{n}{2}$ so that $\|P_k x_j\|_{\ell^2} \leq \log k$ for all $j \in I$.

In this case we make the following claim:

Claim: There is a subset $J \subset I$ with $|J| \geq \frac{n}{4}$, so that for all $j \in J$,

$$\|(I - P_k)x_j\|_{T^2} \geq \frac{1}{8K}.$$

If not, there is a set J as above with

$$\|(I - P_k)x_j\|_{T^2} \leq \frac{1}{8K}.$$

By a standard volume comparison argument (see e.g. [11], Lemma 2.6, or [14], Lemma 4.16) the cardinality of a set of points which are $\frac{1}{4K}$ apart in a ball of radius $\log k$ in k -dimensional Hilbert space is at most $(1 + 8K \log k)^k$ which by our choice of k is less than or equal to $\frac{n}{4}$ (at least for large n). Hence there exist $i, j \in J, i \neq j$ so that

$$\|P_k(x_i - x_j)\|_{\ell^2} \leq \frac{1}{4K}.$$

Now we compute using our assumptions and Proposition 2.3:

$$\begin{aligned}
\|x_i - x_j\|_{S(T^2)} &\leq K \|x_i - x_j\|_{T^2} \leq K \|P_k(x_i - x_j)\|_{T^2} + K \|(I - P_k)x_i\|_{T^2} + K \|(I - P_k)x_j\|_{T^2} \\
&\leq K \|P_k(x_i - x_j)\|_{\ell^2} + K \frac{1}{8K} + K \frac{1}{8K} \leq K \frac{1}{4K} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

This contradicts our 1-separation assumption. So the claim holds.

Now by the claim, the beginning of the proof, (1.5) and Proposition 2.7 we get

$$\begin{aligned}
\left(\int \left\| \sum_{j=1}^n g_j(t)x_j \right\|_{S(T^2)}^2 d\mu(t) \right)^{\frac{1}{2}} &\stackrel{(1)}{\geq} \frac{1}{K^2} \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_{T^2} & (3.10) \\
&\stackrel{(2)}{\geq} \frac{1}{K^2} \left\| (I - P_k) \left(\sum_{j \in J} |x_j|^2 \right)^{1/2} \right\|_{T^2} \\
&\stackrel{(3)}{\geq} \frac{1}{K^3} \left(\int \left\| \sum_{j=1}^n g_j(t)(I - P_k)x_j \right\|_{T^2}^2 d\mu(t) \right)^{\frac{1}{2}} \\
&\stackrel{(4)}{\geq} \frac{1}{K^5} \left(\sum_{j \in J} \left\| (I - P_k)x_j \right\|_{T^2}^2 \right)^{1/2} \\
&\stackrel{(5)}{\geq} \frac{1}{K^5} \left(\sum_{j \in J} \left(\frac{1}{8K} \right)^2 \right)^{1/2} \geq \frac{|J|^{1/2}}{8K^6} \geq \frac{\sqrt{n}}{16K^6},
\end{aligned}$$

where inequality (1) follows from equation (1.5); (2) follows from the fact that $\|I - P_k\| = 1$; (3) holds because T^2 is type 2 with constant (we assume) less than or equal to K ; (4) follows from the fact that $k = \log \log n$ and $((I - P_k)x_n)$ is supported on $(t_i)_{i=k}^\infty$ and Proposition 2.7; and (5) is our Claim. This completes the proof. \square

As a corollary we obtain:

Corollary 3.13 *Even for Banach lattices property upper H and the weak type 2 property do not imply the upper H_2 property. Similarly, property lower H and the weak cotype 2 property do not imply the lower H_2 property.*

References

- [1] S. Bellenot, *The Banach space T and the fast growing hierarchy from logic*, Israel J. Math. **47** (1984) 305-313.
- [2] J. Bourgain and S.J. Szarek, *The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization*, Israel J. Math. **62** (1988), 169–180.
- [3] P.G. Casazza and T.J. Shura, *Tsirelson's Space*, Springer Lecture Notes No. 1363, Springer-Verlag, Berlin-Heidelberg-New York 1989.
- [4] W.B. Johnson, *A reflexive Banach space which is not sufficiently Euclidean*, Studia Math. **60** (1976), 187–204.
- [5] N.J. Kalton, *Convexity conditions for nonlocally convex lattices*, Glasgow Math. J. **25** (1984), 141–152.

- [6] N.J. Kalton and S.J. Montgomery-Smith, Set-functions and factorization. Arch. Math. (Basel) **61** (1993), 183–200.
- [7] J. Lindenstrauss and L. Tzafriri, *The uniform approximation property in Orlicz spaces*, Israel J. Math. **23** (1976), 142-155.
- [8] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, Sequence spaces*, Ergebnisse **92**, Berlin-Heidelberg-New York; Springer Verlag 1977.
- [9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II, Function spaces*, Ergebnisse **97**, Berlin-Heidelberg-New York; Springer Verlag 1979.
- [10] V.D. Milman and G. Pisier, *Banach spaces with a weak cotype 2 property*, Israel J. Math. **54** (1986), 139–158.
- [11] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. **1200**, Springer-Verlag, New York (1986).
- [12] N.J. Nielsen and N. Tomczak-Jaegermann, *Banach lattices with property (H) and weak Hilbert spaces*, Illinois J. Math. **36** (1992), 345-371.
- [13] G. Pisier, *Weak Hilbert spaces*, Proc. London Math. Soc. **56** (1988), 547–579.
- [14] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics **94**, Cambridge University Press, Cambridge 1989.
- [15] S. Reisner, *On the duals of Lorentz function and sequence spaces*, Indiana Univ. Math. J. **31** (1982), 65–72.
- [16] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics **38**, Longman Scientific and Technical 1989.

Department of Mathematics,
 University of Missouri,
 Columbia MO 65211,
 pete@casazza.math.missouri.edu

Department of Mathematics and Computer Science,
 University of Southern Denmark,
 Campusvej 55, DK-5230 Odense M, Denmark,
 njn@imada.sdu.dk