Brownian motion and stochastic integrals

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Introduction

These notes cover a series of lectures given at the University of Kiel in May 2011 in connection with an Erasmus project and is based on a regular course in stochastic differential equations given by me at the University of Southern Denmark. Some additional material which there was no time to cover in the lectures is included at the end. The notes follow the lectures quite closely since the source file is a slight modification of the file used for the lectures.

The construction of Brownian motion using the Haar system was originally carried out by P. Lévy in 1948 [2] and Z. Ciesielski in 1961 [1].

General results from functional analysis and probability theory used in the notes can be found in standard textbooks in these areas of mathematics.

1 Brownian motion

In the following we let $(\Omega, \mathcal{F}, P)$ be a fixed probability space.

We start with the following definition:

Definition 1.1 *A stochastic process in continuous time is a family* $(X_t)_{t\geq 0}$ *of real random variables defined on* $(\Omega, \mathcal{F}, P)$.

Given a stochastic process $(X_t)_{t\geq 0}$ we often only consider X_t for t in an interval $[0, R]$.

We shall also need the following definitions:

Definition 1.2 *Let*$(\mathcal{F}_t)_{t\geq 0}$ *be a family of sub-*σ*-algebras of* $\mathcal{F}$ *so that* $\mathcal{F}_s \subseteq \mathcal{F}_t$ *for all* $s \leq t$. *A stochastic process* $(X_t)_{t\geq 0}$ *is called adapted if* X_t *er* $\mathcal{F}_t$*-measurable for every* $t \geq 0$.

The following definition is important.

Definition 1.3 *Let $(\mathcal{F}_t)$ be as in Definition 1.2 and let $(X_t) \subseteq L_1(P)$ be an $(\mathcal{F}_t)$-adapted process. (X_t) is called a submartingale if*

$$X_s \leq E(X_t|\mathcal{F}_s) \quad \textit{for all } s < t. \tag{1.1}$$

If there for all $s < t$ is equality in (1.1), then (X_t) is called a martingale. (X_t) is said to be a supermartingale if $(-X_t)$ is a submartingale.

A process (X_t) on $(\Omega, \mathcal{F}, P)$ is called continuous if the function $t \to X_t(\omega)$ is continuous for a.a $\omega \in \Omega$.

A process (Y_t) is said to have a continuous version if there exists a continuous process (X_t) so that $P(X_t = Y_t) = 1$ for all $t \geq 0$. If (X_t) is a process on $(\Omega, \mathcal{F}, P)$, then the functions $t \to X_t(\omega)$, $\omega \in \Omega$ are called the paths of the process.

Now it is the time to define the Brownian motion.

Definition 1.4 *A real stochastic process (B_t) is called a Brownian motion starting at 0 with mean value ξ and variance σ^2 if the following conditions are satisfied:*

(i) $P(B_0 = 0) = 1$

(ii) $B_t - B_s$ *is normally distributed* $N((t-s)\xi, (t-s)\sigma^2)$ *for all* $0 \leq s < t$.

(iii) $B_{t_1}, B_{t_2} - B_{t_1}, \dots B_{t_n} - B_{t_{n-1}}$ *are (stochastically) independent for alle* $0 \leq t_1 < t_2 < t_3 < \cdots t_n$.

(B_t) is called a normalized Brownian motion if $\xi = 0$ and $\sigma^2 = 1$.

The essential task of this section is of course to prove the existence of the Brownian motion, i.e. we have to show that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a process (B_t) on that space so that the conditions in Definition 1.4 are satisfied. It is of course enough to show the existence of a normalized Brownian motion (B_t) for then $(\xi t + \sigma B_t)$ is a Brownian motion with mean value ξ and variance σ^2. We shall actually show a stronger result, namely that the Brownian motion has a continuous version. When we in the following talk about a Brownian motion we will always mean a normalized Brownian motion unless otherwise stated.

We will use Hilbert space theory for the construction so let us recall some of its basic facts.

In the following $(\cdot, \cdot)$, respectively $\|\cdot\|$ will denote the inner product, respectively the norm in an arbitrary Hilbert space H. If we consider several different Hilbert spaces at the same time it is of course a slight misuse of notation to use the same symbols for the inner products and norms in these spaces but it is customary and eases the notation.

Let us recall the following elementary theorem from Hilbert space theory:

Theorem 1.5 *Let H_1 be a Hilbert space with an orthonormal basis (e_n) and let (f_n) be an orthonormal sequence in a Hilbert space H_2. Then the map $T\colon H_1 \to H_2$ defined by*

$$Tx = \sum_{n=1}^{\infty} (x, e_n) f_n \quad \textit{for all } x \in H_1 \tag{1.2}$$

is an isometry of H_1 into H_2.

In the following we let (g_n) be a sequence of independent standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, P)$. Actually we can put $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, \boldsymbol{m})$ where $\boldsymbol{m}$ denotes the Lebesgue measure on $[0,1]$.

For the matter of convenience we shall in the sequel consider a constant k as normally distributed with mean k and variance 0.

We can now prove:

Theorem 1.6 *Put $\mathcal{H} = \overline{span}(g_n) \subseteq L_2(P)$. If $T\colon L_2(0,\infty) \to \mathcal{H}$ is an arbitrary isometry and we put*

$$B_t = T(1_{[0,t]}) \quad \textit{for all } t \in [0,\infty[, \tag{1.3}$$

then (B_t) is a Brownian motion.

Proof: Note that such isometries exist. Indeed, since $L_2(0,\infty)$ is a separable Hilbert space, it has an orthonormal basis (f_n) and we can e.g. define T by $Tf_n = g_n$ for all $n \in \mathbb{N}$.

Let us define (B_t) by (1.3). Since $0 = T(0) = B_0$, it is clear that (i) holds. Next let $0 \le s < t$. Since $B_t - B_s \in \mathcal{H}$, it is normally distributed with mean value 0 and furthermore we have:

$$\int_\Omega (B_t - B_s)^2 dP = \|B_t - B_s\|_2^2 = \|T(1_{]s,t]})\|_2^2 = \|1_{]s,t]}\|_2^2 = (t-s), \tag{1.4}$$

which shows that $B_t - B_s$ has variance $(t-s)$.

Let now $0 \le t_1 < t_2 < t_3 < \cdots < t_n$. Since $\{1_{[0,t_1]}, 1_{]t_1,t_2]}, \ldots, 1_{]t_{n-1},t_n]}\}$ is an orthogonal set also $\{T(1_{[0,t_1]}), T(1_{]t_1,t_2]}), \ldots, T(1_{]t_{n-1},t_n]})\} = \{B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\}$ is an orthogonal set. Since in addition all linear combitions of $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are normally distributed, they are independent. □

The following corollary put our results so far together but gives also new information.

Corollary 1.7 *If (f_n) is an arbitrary orthonormal basis for $L_2(0,\infty)$, then the series*

$$B_t = \sum_{n=1}^\infty \int_0^t f_n(s) ds \, g_n \quad t \ge 0 \tag{1.5}$$

converges in $L_2(P)$ and almost surely for all $t \ge 0$. (B_t) is a Brownian motion on $(\Omega, \mathcal{F}, P)$.

Proof: If we define the isometry $T\colon L_2(0,\infty) \to L_2(P)$ by $Tf_n = g_n$, it is given by

$$Tf = \sum_{n=1}^\infty \int_0^\infty f(s) f_n(s) ds \, g_n, \tag{1.6}$$

where the series converges in $L_2(P)$. It follows from the above that $B_t = T(1_{[0,t]})$ is a Brownian motion and equation (1.6) gives that

$$B_t = T(1_{[0,t]}) = \sum_{n=1}^{\infty} \int_0^t f_n(s)ds\, g_n \quad \text{for alle } t \geq 0. \tag{1.7}$$

Since the terms in this sum are independent, have mean value 0, and

$$\sum_{n=1}^{\infty} E(\int_0^t f_n(s)ds\, g_n)^2 = \|B_t\|_2^2 = t < \infty, \tag{1.8}$$

it follows from classical results in probability theory that the series (1.7) converges almost surely for every $t \geq 0$. □

We shall now prove that there is a continuous version of the Brownian motion and then we do not as so far have a free choice of the orthonormal basis (f_n) for $L_2(0, \infty)$. We construct an orthonormal basis (f_n) with the property that there is an $A \in \mathcal{F}$ with $P(A) = 1$ so that if $\omega \in A$, then the series in (1.5) converges to $B_t(\omega)$ uniformly in t on every compact subinterval of $[0, \infty[$. Since every term of the series is continuous in t, this will give that $t \to B_t(\omega)$ is continuous for all $\omega \in A$. The construction of (f_n) is based on the Haar system (an orthonormal basis for $L_2(0, 1)$ explained below) with the aid of the Borel-Cantelli lemma.

In the following we let $(\tilde{h}_m)$ denote the (non-normalized) be the Haar system, defined as follows (make a picture!!):

$$\tilde{h}_1(t) = 1 \quad \text{for alle } t \in [0, 1]. \tag{1.9}$$

For all $k = 0, 1, 2, \dots$ og $\ell = 1, 2, \dots, 2^k$ we put

$$\tilde{h}_{2^k+\ell}(t) = \begin{cases} 1 & \text{if} \quad t \in [(2\ell-2)2^{-k-1}, (2\ell-1)2^{-k-1}[\\ -1 & \text{if} \quad t \in [(2\ell-1)2^{-k-1}, 2\ell \cdot 2^{-k-1}[\\ 0 & \text{else.} \end{cases}$$

We norm this system in $L_2(0, 1)$ and define

$$h_1 = \tilde{h}_1 \qquad h_{2^k+\ell} = 2^{k/2}\tilde{h}_{2^k+\ell} \quad \text{for alle } k = 0, 1, 2, \dots \text{ og } \ell = 1, 2, 3, \dots, 2^k. \tag{1.10}$$

By direct computation we check that it is an orthonormal system and since it is easy to see that every indicator function of a dyadic interval belongs to $span(h_m)$, it follows that $span(h_m)$ is dense in $L_2(0, 1)$. Therefore (h_m) is an orthonormal basis for $L_2(0, 1)$. It follows from Theorem 1.7 that

$$B_t = \sum_{m=1}^{\infty} \int_0^t h_m(s)ds\, g_m \qquad 0 \leq t \leq 1 \tag{1.11}$$

is a Brownian motion for $t \in [0, 1]$. The series converges in $L_2(P)$ and almost surely and the same is the case if we permute the terms. We should however note that the set with measure

1 on which the series converges pointwise depends on the permutation. In order not to get into difficulties with zero sets we shall fix the order of the terms in the sum. We define for all $0 \le t \le 1$

$$B_t = \int_0^t h_1(s)ds\, g_1 + \sum_{k=0}^{\infty} \sum_{m=2^k+1}^{2^{k+1}} \int_0^t h_m(s)ds\, g_m \stackrel{def}{=} \sum_m * \int_0^t h_m(s)ds\, g_m \tag{1.12}$$

and can now prove:

Theorem 1.8 *$(B_t)_{0\le t\le 1}$ given by* (1.12) *is a continuous Brownian motion (on $[0,1]$).*

In the proof of the theorem we need the following lemmas:

Lemma 1.9 *For all $k \ge 0$ we have $0 \le \sum_{m=2^k+1}^{2^{k+1}} \int_0^t h_m(s)ds \le 2^{-k/2-1}$.*

Proof: For every $2^k < m \le 2^{k+1}$ we put $S_m(t) = \int_0^t h_m(s)ds$ for all $0 \le t \le 1$. If $m = 2^k + \ell$, $1 \le \ell \le 2^k$, then it follows directly from the definition of h_m, that the graph of S_m is an triangle centered in $(2\ell - 1)2^{-k-1}$ and with highth $2^{-k/2-1}$. For different ℓ's these triangles do not overlap. This shows the statement. □

Lemma 1.10 *For all $k \ge 0$ we put*

$$G_k(\omega) = \max\{|g_m(\omega)| \mid 2^k < m \le 2^{k+1}\} \quad \text{for all } \omega \in \Omega. \tag{1.13}$$

There is a subset $\tilde{\Omega} \subseteq \Omega$ with $P(\tilde{\Omega}) = 1$ so that there to every $\omega \in \tilde{\Omega}$ exists a $k(\omega)$ with the property that $G_k(\omega) \le k$ for all $k \ge k(\omega)$.

Proof: For every $x > 0$ we find

$$P(|g_m| > x\} = \sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-u^2/2} du \le \sqrt{\frac{2}{\pi}} \int_x^{\infty} \frac{u}{x} e^{-u^2/2} du = \sqrt{\frac{2}{\pi}} x^{-1} e^{-x^2/2}, \tag{1.14}$$

which gives:

$$P(G_k > k) = P(\bigcup_{m=2^k+1}^{2^k+1} (|g_m| > k) \le 2^k P(|g_1| > k) \le \sqrt{\frac{2}{\pi}\frac{1}{k}} \cdot 2^k e^{-k^2/2}. \tag{1.15}$$

Since

$$\sum_{k=1}^{\infty} P(G_k > k) \le \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} k^{-1} 2^k e^{-k^2/2} < \infty,$$

it follows from the Borel-Cantelli lemma that $P(G_k \le k \text{ from a certain step}) = 1$. Choosing $\tilde{\Omega}$ as this set the statement follows. □

Proof of Theorem 1.8: Let $\tilde{\Omega}$ be as in Lemma 1.10 and let $\omega \in \tilde{\Omega}$. Then there exists a $k(\omega) \geq 1$ so that $G_k(\omega) \leq k$ for alle $k \geq k(\omega)$. If $k \geq k(\omega)$ is now fixed, we find

$$\sum_{m=2^k+1}^{2^{k+1}} |\int_0^t h_m(s)ds \cdot g_m(\omega)| \leq \sum_{m=2^k+1}^{2^{k+1}} \int_0^t h_m(s)ds \cdot G_k(\omega) \leq k\, 2^{-k/2-1}. \tag{1.16}$$

for all $0 \leq t \leq 1$.

Since $\sum_{k=1}^{\infty} k\, 2^{-k/2-1} < \infty$, it follows from Weierstrass' M-test that the series $\sum_{k=k(\omega)}^{\infty} \sum_{m=2^k+1}^{2^{k+1}} \int_0^t h_m(s)dsg_m(\omega)$ converges uniformly for $t \in [0,1]$. This gives that the series

$$B_t(\omega) = \sum_m * \int_0^t h_m(s)dsg_m(\omega) \tag{1.17}$$

also converges uniformly for $t \in [0,1]$ and hence $t \to B_t(\omega)$ is continuous. □

In order to find a continuous Brownian motion on $[0,\infty[$ we define the functions $h_{nm} \in L_2(0+,\infty)$ by

$$h_{nm}(t) = \begin{cases} h_m(t-n) & \text{for } t \in [n-1,n] \\ 0 & \text{else} \end{cases} \quad n \in \mathbb{N}, m \in \mathbb{N} \tag{1.18}$$

and note that $(h_{nm})_{m=1}^{\infty}$ is an orthonormal basis for $L_2(n-1,n)$ for all $n \in \mathbb{N}$ which implies that (h_{nm}) is an orthonormal basis for $L_2(0,\infty)$.

The following theorem easily follows from the above:

Theorem 1.11 *Let $(\Omega, \mathcal{F}, P)$ be a probability space on which there exists a sequence of $N(0,1)$-distributed random variables and let (g_{nm}) be such a sequence. Define:*

$$B_t = \sum_{n=1}^{\infty} \sum_m * \int_0^t h_{nm}(s)ds\, g_{nm} \quad \textit{for allt} \geq 0. \tag{1.19}$$

Then $(B_t)_{t\geq 0}$ is a continuous Brownian motion.

Let now (B_t) be a Brownian motion and let for every $t \geq 0$ $\mathcal{F}_t$ denote the σ-algebra generated by $\{B_s \mid 0 \leq s \leq t\}$ and the set $\mathcal{N}$ of zero–sets.

Theorem 1.12 *$(B_t, \mathcal{F}_t)$ is a martingale.*

Proof: Let $0 \leq s < t$. It follows directly from the definition that $B_t - B_s$ is independent of $\{B_u \mid u \leq s\}$ and therefore also independent of $\mathcal{F}_s$. Hence we find

$$E(B_t|\mathcal{F}_s) = E(B_s|\mathcal{F}_s) + E(B_t - B_s|\mathcal{F}_s) = B_s + E(B_t - B_s) = B_s \tag{1.20}$$

□

2 The Ito integral

In this section we let $(\Omega, \mathcal{F}, P)$ denote a probability space on which we have a Brownian motion (B_t) and we shall always assume that it is continuous. Further, $\mathcal{B}$ denotes the set of Borel subsets of $\mathbb{R}$, $\boldsymbol{m_n}$ denotes the Lebesgue measure on $\mathbb{R}^n$ ($\boldsymbol{m_1} = \boldsymbol{m}$) and $(\mathcal{F}_t)$ denotes the family of σ–algebras defined above. Also we let $0 < T < \infty$ be a fixed real number. We wish to determine a subspace of functions f of $L_2([0,T] \times \Omega . \boldsymbol{m} \otimes P)$ so that we can define $\int_0^T f dB$ as a stochastic variable. Since it can be proved that for a.a. $\omega \in \Omega$ the function $\omega \to B_t(\omega)$ is not of finite variation, the Riemann–Stiltjes construction will not work. However, since (B_t) is a martingale, we have other means which we are now going to explore.

For every $n \in \mathbb{N}$ we define the sequence (t_k^n) by

$$t_k^n = \begin{cases} k2^{-n} & \text{if} \quad 0 \leq k2^{-n} \leq T \\ T & \text{if} \quad k2^{-n} > T \end{cases}$$

If n is fixed we shall often write t_k instead of t_k^n.

We let $\mathcal{E} \subseteq L_2([0,T] \times \Omega, \boldsymbol{m} \otimes P)$ consist of all functions ϕ of the form

$$\phi(t, \omega) = \sum_k e_k(\omega) 1_{[t_k^n, t_{k+1}^n[}(t)$$

where $n \in \mathbb{N}$ and every $e_k \in L_2(P)$ and is $\mathcal{F}_{t_k^n}$–measurable. The elements of $\mathcal{E}$ are called elementary functions.

If $\phi \in \mathcal{E}$ is of the form above we define the *Ito integral* by:

$$\int_0^T \phi dB = \sum_k e_k (B_{t_{k+1}} - B_{t_k})$$

It is straightforward that the map $\phi \to \int_0^T \phi dB$ is linear.

The following theorem is called the Ito isometry for elementary functions.

Theorem 2.1 *If $\phi \in \mathcal{E}$, then*

$$E(\int_0^T \phi dB)^2 = E(\int_0^T \phi^2 d\boldsymbol{m}).$$

Proof: Let ϕ be written as above. If $j < k$, then $e_j e_k (B_{t_{j+1}} - B_{t_j})$ is $\mathcal{F}_{t_k}$–measurable and therefore independent of $(B_{t_{k+1}} - B_{t_k})$. Hence

$$E(e_j e_k (B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})) = E(e_j e_k (B_{t_{j+1}} - B_{t_j}) E(B_{t_{k+1}} - B_{t_k}) = 0.$$

If $j = k$, e_k is independent of $B_{t_{k+1}} - B_{t_k}$ and hence

$$E[(e_k^2 (B_{t_{k+1}} - B_{t_k})^2] = E(e_k^2) E(B_{t_{k+1}} - B_{t_k})^2 = E(e_k^2)(t_{k+1} - t_k)$$

This clearly gives that

$$E(\int_0^T \phi dB)^2 = E(\int_0^T \phi^2 d\boldsymbol{m})$$

□

This means that the map $\phi \to \int_0^T \phi dB$ is a linear isometry from $\mathcal{E} \to L_2(P)$ and can therefore be extended to a linear isometry from $\overline{\mathcal{E}}$ to $L_2(P)$. Hence we can define $\int_0^T \phi dB$ for all $f \in \overline{\mathcal{E}}$ and it is clear that $E(\int_0^T \phi dB) = 0$.

Theorem 2.2 *If $f \in \overline{\mathcal{E}}$, then $(\int_0^T f dB)_{0 \le t \le T}$ is a martingale.*

Proof: Let first $\phi \in \mathcal{E}$ be written as above. Then

$$\int_0^T \phi dB = \sum_k e_k(B_{t_{k+1}} - B_{t_k})$$

and if $0 \le t < T$, say $t_m \le t < t_{m+1}$, then for $k > m$ we get

$$E(e_k(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_{t_k}) = e_k E(B_{t_{k+1}} - B_{t_k} \mid \mathcal{F}_{t_k}) = 0$$

and hence also

$$E(e_k(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_t) = E(E(e_k(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_{t_k}) \mid \mathcal{F}_t) = 0.$$

If $k < m$

$$E(e_k(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_t) = e_k(B_{t_{k+1}} - B_{t_k}).$$

Finally we get:

$$E(e_m(B_{t_{m+1}} - B_{t_m}) \mid \mathcal{F}_t) = e_m(B_t - B_{t_m}).$$

Summing up we get that $E(\int_0^T \phi dB \mid \mathcal{F}_t) = \int_0^t \phi dB$. Since $E(\cdot \mid \mathcal{F}_t)$ is an orthogonal projection on $L_2(P)$, it follows that $E(\int_0^T f dB \mid \mathcal{F}_t) = \int_0^t f dB$ for all $f \in \overline{\mathcal{E}}$. □

Using Doob's martingale inequality and the Borel–Cantelli lemma, the following can be proved:

Theorem 2.3 *The Ito integral has a continuous version, meaning that we can achieve that for every $f \in \overline{\mathcal{E}}$ that map $t \to \int_0^t f dB$ is continuous almost surely.*

Our next task is to determine $\overline{\mathcal{E}}$. A $\mathcal{B} \otimes \mathcal{F}$–measurable function f is called *progressively measurable* if for all $0 \le t \le T$ $f : [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B} \otimes \mathcal{F}_t$–measurable. We let $\mathcal{P}_2(0, T)$ denote the closed subspace of $L_2([0, T] \times \Omega, \boldsymbol{m} \otimes P)$ which are progressively measurable.

We can now prove:

Theorem 2.4 $\overline{\mathcal{E}} = \mathcal{P}_2(0, T)$.

Proof: Let first $g \in \mathcal{P}_2(0,T)$ be bounded so that $g(\cdot,\omega)$ is continuous for almost all $\omega \in \Omega$ and define ϕ_n by

$$\phi_n(t,\omega) = \sum_k g(t_k^n,\omega) 1_{[t_k^n, t_{k+1}^n[}(t)$$

Clearly $\phi_n \in \mathcal{E}$ for all $n \in \mathbb{N}$. Let now $\varepsilon > 0$ and let $\omega \in \Omega$ be fixed. By uniform continuity we can find a $\delta > 0$ so that $|t-s| < \delta \Rightarrow |g(t,\omega) - g(s,\omega)| < \varepsilon$. Determine n_0 so that $2^{-n_0} < \delta$ and let $n \geq n_0$. Then $|g(t,\omega) - g(t_k,\omega)| < \varepsilon$ for all $t_k \leq t \leq t_{k+1}$ and therefore

$$\int_0^T (g(t,\omega) - \phi_n(t,\omega))^2 dt < \varepsilon^2 T$$

so that $\int_0^T (g(t,\omega) - \phi_n(t,\omega))^2 dt \to 0$. Since g is bounded, majorized convergence gives that also $E(\int_0^T (g-\phi_n)^2 d\boldsymbol{m}) \to 0$ as well. Hence $g \in \overline{\mathcal{E}}$.

The next step is the tricky one where progressive measurability is used. Let $h \in \mathcal{P}_2(0,T)$ be a bounded function, say $|h| \leq M$ a.s. We wish to show that there is a sequence $(g_n) \subseteq \mathcal{P}_2(0,T)$ so that for evey n and a.a. ω the function $t \to g_n(t,\omega)$ is continuous and so that $g_n \to h$ in $L_2(\boldsymbol{m} \otimes P)$. Together with the above this will give that $h \in \overline{\mathcal{E}}$. Let for each n ψ_n be the non–negative continuous function which is zero on the intervals $]-\infty, -\frac{1}{n}]$ and $]0,\infty[$ and so that $\int_{-\infty}^{\infty} \psi_n(x)dx = 1$. We can e.q choose ψ_n so that its graph is a triangle. (g_n) is now defined by:

$$g_n(t,\omega) = \int_0^t \psi_n(s-t)h(s,\omega)ds \quad \text{for all } \omega \text{ and all } 0 \leq t \leq T.$$

The properties of the sequence (ψ_n) readily give that each g_n is continuous in t and $|g_n| \leq M$ a.s. For fixed t the function $(s,u,\omega) \to \psi_n(s-u)h(s,\omega)$ is integrable over $[0,t] \times [0,t] \times \Omega$ and since $h \in \mathcal{P}_2(0,T)$, it is $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{F}_t$–measurable. An application of Fubini's theorem now gives that g_n is progressively measurable for every n.

Since (ψ_n) constitutes an approximative identity with respect to convolution, it follows that $\int_0^T (h-g_n)^2 d\boldsymbol{m} \to 0$ and an application of majorized convergence gives $g_n \to h$ in $L_2(\boldsymbol{m} \otimes P)$.

Let now $f \in \mathcal{P}_2(0,T)$ be arbitrary. For every $n \in \mathbb{N}$ we define

$$h_n(t,\omega) = \begin{cases} -n & \text{if} \quad f(t,\omega) < -n \\ f(t,\omega) & \text{if} \quad -n \leq f(t,\omega) \leq n \\ n & \text{if} \quad f(t,\omega) > n \end{cases}$$

By the above $(h_n) \subseteq \overline{\mathcal{E}}$ and it clearly converges to f in $L_2(\boldsymbol{m} \otimes P)$. □

It is worthwhile to note that if $h \in L_2([0,T])$, then $\int_0^t h dB$ is normally distributed with mean 0 and variance $\int_0^T h^2 d\boldsymbol{m}$.

We say that a measurable function $f; [0,T \times \Omega \to \mathbb{R}$ is *adapted* to the filtration$(\mathcal{F}_t)$, if for fixed $0 \leq t \leq T$ the function $\omega \to f(t,\omega)$ is $\mathcal{F}_t$–measurable. A lengthy and quite demanding proof gives the following result.

Theorem 2.5 *(Meyer) If f is adapted to a filtration $(\mathcal{F}_t)$, then it has a progressively measurable modification g, that is for every $0 \leq t \leq T$ $f(t,\cdot) = g(t,\cdot)$ a.s.*

We let $\mathcal{A}_2(0,T)$ consist of those functions in $L_2(\boldsymbol{m} \otimes P)$ which are adapted to $(\mathcal{F}_t)$. Using Meyer's theorem we can now define the Ito integral of an $f \in \mathcal{A}_2$. We choose namly a progressively measurable modification g of f and simply define $\int_0^T f dB = \int_0^T g dB$. We shall not go into details.

The Ito integral can be defined for a larger class of integrants. If $f \in L_2([0,T] \times \Omega)$ so that there is an increasing family $(\mathcal{H}_t)$ of σ–algebras so that

(i) $\mathcal{F}_t \subseteq \mathcal{H}_t$ for all $0 \leq t \leq T$.

(ii) For all $0 \leq s < t \leq T$ $B_t - B_s$ is independent of $\mathcal{H}_s$.

(iii) f is $(\mathcal{H}_t)$–adapted.

The arguments are similar to the ones given above. Note that (ii) implies that (B_t) is a martingale with respect to $(\mathcal{H}_t)$. It also follows $(\int_0^t f dB)$ is a martingale.

Let $f : [0,T] \times \Omega \to \mathbb{R}$ be a function satisfying (i)–(iii) and so that

$$P(\{\omega \in \Omega \mid \int_0^T f(t,\omega)^2 dt < \infty\}) = 1. \tag{2.1}$$

In that case it can be proved that there is a sequence (f_n) of elementary functions so that $\int_0^T (f - f_n)^2 d\boldsymbol{m} \to 0$ in probability. It turns out that then also $(\int_0^T f_n dB)$ will converge in probability and we can therefore define

$$\int_0^T f dB = \lim_n \int_0^T f_n dB \quad \text{in probability.}$$

Note however that since conditional expectations are not continuous in probability, this extended Ito integral will in general not be a martingale.

Let $n \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, P)$ be a probability space on which we can find n independent Brownian motions, $B_1(t), B_2(t), \cdots, B_n(t))$. We can then put $B(t) = (B_j(t)$ to get an n–dimensional Brownian motion. As before we let for every $t \geq 0$ $\mathcal{F}_t$ denote the σ–algebra generated by $\{B(s) \mid 0 \leq s \leq t\}$ and the zero sets $\mathcal{N}$.

If $A(t,\omega)$ is an $m \times n$ stochastic matrix which is $(\mathcal{F}_t)$–adapted and so that all entries satisfy the equation (2.1) above, we can define the m–dimensional Ito integral $\int_0^T A dB$ by writing the dB as a column "vector" and perform matrix multiplication, e.g. the kth coordinate of $\int_0^T A dB$ will be

$$\sum_{j=1}^n \int_0^T A_{kj} dB_j.$$

It follows that if each entry of A is square integrable in both variables, this Ito integral will be an m–dimensional martingale.

3 Ito's formula

We consider the one dimensional case and let (B_t) be an one dimensional Brownian motion.

Definition 3.1 *An Ito process is a stochastic process of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)dt + \int_0^t v(s,\omega)dB_t(\omega) \quad t \geq 0,$$

where u and v are so that the integrals make sense for all $t \geq 0$.

If X is an Ito process of the form above, we shall often write

$$dX_t = udt + vdB_t$$

Theorem 3.2 *Let $dX_t = udt + vdB_t$ be an Ito process and let $g \in C^2([0,\infty[\times\mathbb{R})$. Then $Y_t = g(t, X_t)$ is again an Ito process and*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2.$$

The "multiplication rules" here are

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$

We shall not prove it here. It is based on Taylor expansions and the difficult part is to show that that the remainer tends to zero in L_2.

There is also an Ito formula in higher dimensions.

As an example of the use of Ito's formular let us compute $\int_0^t B_s dB_s$.

Ito's formular used with the function x^2 gives

$$d(B_t)^2 = 2B_t dB_t + (B_t)^2 = 2B_t dB_t + t,$$

so that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$$

In particular it follows that $(B_t^2 - t)$ is a martingale, a fact we shall prove directly later. The next theorem is called the *Ito representation theorem*

Theorem 3.3 *Let $0 < T < \infty$ and let $F \in L_2(\Omega, \mathcal{F}_T, P)$. Then there is a unique $f \in \mathcal{A}_2(0,T)$ so that*

$$F = E(F) + \int_0^T f dB.$$

We shall only prove it in the one–dimensional case. We need however two lemmas.

Lemma 3.4 *The set*

$$\{\phi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) \mid n \in \mathbb{N}, \quad \phi \in C_0^\infty(\mathbb{R}^n), \quad (t_i) \subseteq [0,T]\}$$

is dense in $L_2(\Omega, \mathcal{F}_T, P)$

Proof: Let (t_i) be a dense sequence in $[0,T]$ and let for each n $\mathcal{H}_n$ be the σ–algebra generated by $\{B_{t_i} \mid 0 \leq i \leq n\}$ and the zero sets. Clearly $\mathcal{F}_T$ is the smallest σ–algebra containing all of the $\mathcal{H}_n$'s. Let now $g \in L_2(\Omega, \mathcal{F}_T, P)$ be arbitrary. By the martingale convergence theorem we get that $g = E(g \mid \mathcal{F}_T) = \lim_n E(g \mid \mathcal{H}_n)$, where the limit is in L_2 and a.s.

A result of Doob and Dynkin gives the existence of a Borel function $g_n : \mathbb{R}^n \to \mathbb{R}$ so that for every n:

$$E(g \mid \mathcal{H}_n) = g_n(B_{t_1}, B_{t_2}, \cdots, B_{t_n}).$$

Let μ denote the distribution Borel measure on $\mathbb{R}^n$ of $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, i.e $\mu = (B_{t_1}, B_{t_2}, \cdots, B_{t_n})(P)$. Note that μ has a normal density which implies that $C_0^\infty(\mathbb{R}^n)$ is dense in $L_2(\mu)$.

From the above we get:

$$\int_{\mathbb{R}^n} g_n^2 d\mu = \int_\Omega g_n(B_{t_1}, B_{t_2}, \cdots, B_{t_n})^2 dP \leq \int_\Omega g^2 dP$$

so that $g_n \in L_2(\mu)$. Hence g_n can be approximated well in $L_2(\mu)$ by a function $\phi_n \in C_0^\infty(\mathbb{R}^n)$ and hence $g_n(B_{t_1}, B_{t_2}, \cdots, B_{t_n}$ can be approximated well in $L^2(P)$ by $\phi_n(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, Combining this with the above we get the result. $\square$

Lemma 3.5 *Put* $\mathcal{M} = \{\exp(\int_0^T hdB - \frac{1}{2}\int_0^T h(s)^2 ds) \mid h \in L_2(0,T)\}$. *Then* $span(\mathcal{M})$ *is dense in* $L_2(\Omega, \mathcal{F}^T, P)$.

Proof: Note that $\int_0^T hdB$ is normally distributed and therefore $\exp(\int_0^T hdB) \in L_2(P)$. Note also that the term $\frac{1}{2}\int_0^T h(s)^2 ds$ is actually not needed since we take the span.

Let now $g \in L_2(\Omega, \mathcal{F}_T, P)$ be orthogonal to $\mathcal{M}$. We have to prove that $g = 0$ a.s.

Let $n \in \mathbb{N}$ and let $\{t_j \mid 1 \leq j \leq n\}$ For all $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ we get that

$$\int_\Omega \exp(\sum_{j=1}^n \lambda_j B_{t_j}) g dP = 0$$

For every $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$ we define

$$G(z) = \int_\Omega \exp(\sum_{j=1}^n z_k B_{t_k}) g dP$$

G is seen be be a holomorphic function of n variables (use majorized convergence). Since $G = 0$ on $\mathbb{R}^n$, we must have $G(z) = 0$ for all $z \in \mathbb{C}^n$. In particular

$$G(iy) = 0 \quad \text{for all } y \in \mathbb{R}^n$$

We wish to show that g is orthogonal to the set from the previous lemma, so let $\phi \in C_0^\infty(\mathbb{R}^n)$. By the inverse Fourier transform theorem we have

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\phi}(y) \exp(i < x, y >) d\boldsymbol{m}_{\mathbf{n}}$$

for all $x \in \mathbb{R}^n$. We have:

$$\int_\Omega \phi(B_{t_1,t_2}, \cdots, B_{t_n}) g dP = (2\pi)^{-\frac{n}{2}} \int_\Omega g \int_{\mathbb{R}^n} \hat{\phi}(y) \exp(\sum_{k=1}^n y_k B_{t_k}) d\boldsymbol{m}_{\mathbf{n}}(y) dP =$$

$$2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\phi}(y) G(iy) d\boldsymbol{m}_{\mathbf{n}}(y) = 0$$

By the previous lemma we get that $g = 0$ a.s. □

Proof of the Ito representation theorem:

By the above lemma and the Ito isometry it follows that it is enough to prove it for $F \in \mathcal{M}$. Hence we assume that F has the form

$$F = \exp(\int_0^T h dB - \frac{1}{2} \int_0^T h(s)^2 ds)$$

where $h \in L_2(0, T)$

Let $Y_t = \exp(\int_0^t h dB - \frac{1}{2} \int_0^t h(s)^2 ds)$ for all $0 \le t \le T$

Ito's formula gives

$$dY_t = Y_t(h(t)dB_t - \frac{1}{2} h(t)^2 dt) + \frac{1}{2} Y_t(h(t)dB_t)^2 = Y_t h(t) dB_t.$$

Hence written in integral form:

$$Y_t = 1 + \int_0^t Y_s h(s) dB_s$$

In particular

$$F = 1 + \int_0^T Y_s h(s) dB_s$$

Clearly the function $(t, \omega) \to Y_t(\omega) h(t)$ is $(\mathcal{F}_t)$–adapted so we need to verify that it belongs to $L_2(\boldsymbol{m} \otimes P)$.

We note that for fixed t $\int_0^t h dB$ is normally distributed with mean 0 and variance $\sigma_t^2 = \int_0^t h(s)^2 ds$ and hence

$$E(Y_t^2) = \frac{1}{\sigma_t \sqrt{2\pi}} \int_{-\infty}^\infty \exp(2x - \sigma_t^2 - \frac{x^2}{2\sigma_t^2}) dx =$$
$$\exp(\sigma_t^2) \frac{1}{\sigma_t \sqrt{2\pi}} \int_{-\infty}^\infty \exp(-\frac{(x - 2\sigma_t^2)^2}{2\sigma_t^2}) dx = \exp(\sigma_t^2) \quad .$$

Therefore

$$\int_0^T h(t)^2 E(Y_t^2)dt = \int_0^T h(t)^2 \exp(\int_0^t h(s)^2 ds)dt < \infty.$$

Hence (Y_t) is a martingale and in particular $E(F) = 1$

The uniqueness follows from Ito isometry. $\square$

We can now prove the *martingale representation theorem*

Theorem 3.6 *Let (B_t) be an n–dimensional Brownian motion. If $(M_t) \subseteq L_2(P)$ is an $(\mathcal{F}_t)$–martingale, then there is a unique stochastic process g so that $f \in \mathcal{A}_2(0,t)$ for all $t \geq 0$ so that*

$$M_t = E(M_0) + \int_0^t f dB \quad \textit{for all } t \geq 0.$$

Proof: We shall only prove it for $n = 1$. Let $0 \leq t < \infty$. The representation theorem give us a unique $f^t \in \mathcal{A}_2(0,t)$ so that

$$M_t = E(M_0) + \int_0^t f^t dB.$$

If $0 \leq t_1 < t_2$, then

$$M_{t_1} = E(M_{t_2} \mid \mathcal{F}t_1) = E(M_0) + \int_0^{t_1} f_{t_2} dB.$$

But

$$M_{t_1} = E(M_0) + \int_0^{t_1} f^{t_1} dB$$

so by uniqueness $f^{t_1}(t,\omega) = f^{t_2}(t,\omega)$ for almost all $(t,\omega) \in [0,t_1] \times \Omega$. If we now put $f(t,\omega) = f^N(t,\omega)$ for almost all $0 \leq t \leq N$ and almost all ω, then f is well–defined and is clearly the one we need. $\square$

4 Stochastic differential equations

Let (X_t) be an $(\mathcal{F}_t)$–adapted process. We say that (X_t) satisfies the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

or in differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where b and σ are so that the integrals make sense.

As an example we can consider the equation

$$dX_t = rX_t dt + \alpha X_t dB_t$$

where r and α are constants.

Assume that (X_t) is a solution so that $X_t > 0$ a.s for all $t \geq 0$. An application of Ito's formula then gives

$$\begin{aligned} d\log(X_t) = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2 &= \\ \frac{1}{X_t}(rX_t dt + \alpha X_t dB_t) - \frac{1}{2X_t^2}\alpha^2 X_t^2 dt &= (r - \frac{1}{2}\alpha^2)dt + \alpha dB_t. \end{aligned}$$

. Hence

$$\log(X_t) = \log(X_0) + (r - \frac{1}{2}\alpha^2)t + \alpha B_t$$

or

$$X_t = X_0 \exp((r - \frac{1}{2}\alpha^2)t + \alpha B_t).$$

A test shows that (X_t) is actually a solution. We shall later see that given X_0 it is the only one. (X_t) is called a *geometric Brownian motion*. It can be shown that:

- If $r > \frac{1}{2}\alpha^2$, then $X_t \to \infty$ for $t \to \infty$.
- If $r < \frac{1}{2}\alpha^2$, then $X_t \to 0$ for $t \to \infty$.
- If $r = \frac{1}{2}$, then X_t fluctuates between arbitrary large and arbitrary small values when $t \to \infty$.

The law of *iterated logarithm* is used to prove these statements. It says that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log(\log t)}} = 1 \quad \text{a.s.}$$

Let $n, m \in \mathbb{N}$ and let M_{nm} denote the space of all $n \times m$–matrices. Further let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to M_{nm}$ be measurable functions so that there exists constants C and D with

- $\|b(t,x)\| + \|\sigma(t,x)\| \leq C(1 + \|x\|)$
- $\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq D\|x - y\|$

for all $x, y \in \mathbb{R}^n$ and all $t \in [0,T]$. Here $\|\cdot\|$ denotes the norm in the Euclidian space (we identify here M_{nm} with $\mathbb{R}^{nm}$. Further we let B be an m–dimensional Brownian motion.

We have the following existence and uniqueness theorem:

Theorem 4.1 *Let $Z \in L_2(P)$ so that Z is independent of $\{\mathcal{F}_t \mid 0 \le t \le T$. The equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \quad 0 \le t \le T \quad X_0 = Z$$

has a unique solution with $X \in L_2(\boldsymbol{m} \otimes P)$ so that X is adapted to the σ–algebra $\mathcal{F}_t^Z$ generated by Z and $\mathcal{F}_t$.

We shall not prove the theorem here. The uniqueness is based on the assumptions above and Ito isometry. The existence is based on Picard iteration.

In fact, we put $Y_t^{(0)} = Z$ and define $Y_t^{(k)}$ inductively by

$$Y_t^{(k+1)} = Z + \int_0^t B(s, Y_s^{(k)})ds + \int_0^t \sigma(s, Y_s^{(k)})dB_s.$$

We then use our assumptions to prove that the sequence $(Y^{(k)})$ has a limit in $L_2(\boldsymbol{m} \otimes P)$. This limit is our solution. The uniqueness involves Ito isometry.

Definition 4.2 *A time homogeneous Ito diffusion (X_t) is a process that satisfies an equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad 0 \le s \le t \quad X_s = x \in \mathbb{R}^n$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to M_{nm}$ satisfy

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \le D\|x - y\| \quad \textit{for all } x, y \in \mathbb{R}^n.$$

5 Lévy's characterization of Brownian motion

For every $n \in \mathbb{N}$ $\mathcal{B}_n$ denotes the Borel algebra on $\mathbb{R}^n$ and if $X\colon \Omega \to \mathbb{R}^n$ a random variable, then we let $X(P)$ denote the distribution measure (the image measure) on $\mathbb{R}^n$ of X, e.g.

$$X(P)(A) = P(X^{-1}(A)) \quad \text{for all } A \in \mathcal{B}_n. \tag{5.1}$$

If $n \in \mathbb{N}$, we let $\langle \cdot, \cdot \rangle$ denote the canonical inner product on $\mathbb{R}^n$. Hence for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ og alle $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ we have

$$\langle x, y \rangle = \sum_{j=1}^n x_j y_j. \tag{5.2}$$

Let $(\mathcal{F}_t)_{t \ge 0}$ be an increasing family of sub-σ-algebras so that $\mathcal{F}_t$ contains all sets of measure 0 for all $t \ge 0$ (it need not be generated by any Brownian motion). We start with the following easy result.

Theorem 5.1 *Let (B_t) be a one–dimensional normalized Brownian motion, adapted to $(\mathcal{F})$ and so that $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \le s < t$ (this ensures that (B_t) is a martingale with respect to $(\mathcal{F}_t)$). Then $(B_t^2 - t)$ is a martingale with respect to $(\mathcal{F}_t)$.*

Proof: If $0 \le s < t$, then $B_t^2 = (B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s)$ and hence

$$E(B_t^2 \mid \mathcal{F}_s) = E((B_t - B_s)^2 \mid \mathcal{F}_s) + B_s^2 + 2B_s E((B_t - B_s) \mid \mathcal{F}_s) = (t-s) + B_s^2$$

where we have used that $B_t - B_s$ and hence also $(B_t - B_s)^2$ are independent of $\mathcal{F}_s$. $\square$

The main result of this section is to prove that the converse is also true for continuous processes, namely:

Theorem 5.2 *Let (X_t) be a continous process adapted to $(\mathcal{F}_t)$ so that $X_0 = 0$ and*

(i) (X_t) is a martingale with respect to $(\mathcal{F}_t)$.

(ii) $(X_t^2 - t)$ is a martingale with respect to $(\mathcal{F}_t)$.

Then (X_t) is a (normalized) Brownian motion.

Before we can prove it, we need yet another theorem which is a bit like Ito's formula and a lemma.

Theorem 5.3 *Let (X_t) be as in Theorem 5.2 and let $f \in C^2(\mathbb{R})$ so that f, f' and f'' are bounded. For all $0 \le s < t$ we have*

$$E(f(X_t) \mid \mathcal{F}_s) = X_s + \frac{1}{2}\int_s^t E(f''(X_u) \mid \mathcal{F}_s)du. \tag{5.3}$$

Proof: Let $\Pi = (t_k)_{k=0}^n$ be a partition of the interval $[s,t]$ so that $s = t_0, t_1 < t_2 < \cdots, < t_n = t$. By Taylor's formula we get

$$\begin{aligned} f(X_t) &= f(X_s) + \sum_{k=1}^n (f(X_{t_k}) - f(X_{t_{k-1}})) \qquad (5.4)\\ &= f(X_s) + \sum_{k=1}^n f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^n f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 + R_\Pi \end{aligned}$$

Taking conditional expectations on each side we obtain:

$$\begin{aligned} E(f(X_t) \mid \mathcal{F}_s) = f(X_s) + \sum_{k=1}^n E(E(f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) \mid \mathcal{F}_{k-1}) \mid \mathcal{F}_s) \quad + \\ \frac{1}{2}\sum_{k=1}^n E(E(f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}) \mid \mathcal{F}_s) + E(R_\Pi \mid \mathcal{F}_s) = f(X_s) \quad + \\ \frac{1}{2}\sum_{k=1}^n E(f''(X_{t_{k-1}}) \mid \mathcal{F}_s)(t_k - t_{k-1}) + E(R_\Pi \mid \mathcal{F}_s). \qquad (5.5) \end{aligned}$$

Using the continuity of the (X_t) it can be shown that $R_\Pi \to 0$ in $L_2(P)$, when the length $|\Pi|$ of Π tends to 0. Hence also $E(R_\Pi \mid \mathcal{F}_s) \to 0$ in $L_2(P)$as $|\Pi| \to 0$. Since the function $u \to E(f''(X_u) \mid \mathcal{F}_s))$ is continuous a.s., we get that

$$\sum_{k=1}^{n} E(f''(X_{t_{k-1}}) \mid \mathcal{F}_s)(t_k - t_{k-1}) \to \int_s^t E(f''(X_u) \mid \mathcal{F}_s)du \quad \text{a.s.} \tag{5.6}$$

when $|\Pi| \to 0$ and since f'' is bounded, the bounded convergence theorem gives that the convergence in (5.6) is also in $L_2(P)$. Combining the above we get formula (5.3). □

Let us recall the following defintion:

Definition 5.4 *If $X : \Omega \to \mathbb{R}^n$, then its characteristic function $\phi : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$\phi(y) = \int_\Omega \exp(i < y, X >)dP = \int_{\mathbb{R}^n} \exp(i < y, x >)dX(P).$$

Lemma 5.5 *Let $n \in \mathbb{N}$, let $Y_j : \Omega \to \mathbb{R}$, $1 \le j \le n$ be stochastic variables, and put $Y = (Y_1, Y_2, \cdots, Y_n) : \Omega \to \mathbb{R}^n$. Further, let ϕ_{Y_j} denote the characteristic function of Y_j for $1 \le j \le n$ and ϕ_Y the characteristic function of Y. Then $Y_1, Y_2, \ldots, Y_n$ are independent if and only if*

$$\phi_Y(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} \phi_{Y_j}(x_j) \tag{5.7}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof: It follows from the definition of independence that $Y_1, Y_2, \ldots, Y_n$ are independent if and only if $Y(P) = \otimes_{j=1}^n Y_j(P)$ Noting that the right hand side of (5.7) is the characteristic function of $\otimes_{k=1}^n Y_j(P)$, the statement of the lemma follows from the above and the uniqueness theorem for characteristic functions. □

Proof of Theorem 5.2: The main part of the proof will be to prove that for all $0 \le s \le t$ we have the formula

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = \exp(-\frac{1}{2}u^2(t-s)) \quad \text{for all } u \in \mathbb{R}. \tag{5.8}$$

To prove (5.8) fix an s with $0 \le s < \infty$, a $u \in \mathbb{R}$, and apply Theorem 5.3 to the function $f(x) = \exp(iux)$ for all $x \in \mathbb{R}$. For all $s \le t$ we then obtain:

$$E(\exp(iuX_t) \mid \mathcal{F}_s) = \exp(iuX_s) - \frac{1}{2}u^2 \int_s^t E(\exp(iuX_v) \mid \mathcal{F}_s)dv$$

or

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = 1 - \frac{1}{2}u^2 \int_s^t E(\exp(iu(X_v - X_s)) \mid \mathcal{F}_s)dv. \tag{5.9}$$

Since the integrand on the right side of (5.9) is continuous in v, the left hand side is differentiable with respect to t and

$$\frac{d}{dt}E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = -\frac{1}{2}u^2 E(\exp(iu(X_t - X_s) \mid \mathcal{F}_s).$$

This shows that on $[s, \infty[$ $E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s)$ is the solution to the differential equation

$$g'(t) = -\frac{1}{2}u^2 g(t)$$

with the initial condition $g(s) = 1$. Hence

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = \exp(-\frac{1}{2}u^2(t-s)) \quad \text{for all } 0 \le s \le t$$

and equation (5.8) is established.

Let now $0 \le s < t$. By (5.8) the characteristic function of $X_t - X_s$ is given by:

$$E(\exp(iu(X_t - X_s)) = E(E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s)) = \exp(-\frac{1}{2}u^2(t-s))$$

and hence $X_t - X_s$ is normally distributed with mean 0 and variance $t - s$.

Let now $0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty$ and put $Y = (X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$. If ϕ_Y denotes the characteristic function of Y, then we get for all $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}$:

$$\phi_Y(u) = \exp(i < u, Y >) = E(\prod_{k=1}^{n} \exp(iu_k(X_{t_k} - X_{t_{k-1}})) =$$

$$E(E(\prod_{k=1}^{n} \exp(iu_k(X_{t_k} - X_{tk-1})) \mid \mathcal{F}_{t_{n-1}}) =$$

$$\exp(-\frac{1}{2}u_n^2(t_n - t_{n-1}))E(\prod_{k=1}^{n-1} \exp(iu_k(X_{t_k} - X_{t_{k-1}}))$$

Continuing in this way we obtain:

$$\phi_Y(u) = \prod_{k=1}^{n} \exp(-\frac{1}{2}u_k^2(t_k - t_{k-1}) = \prod_{k=1}^{n} E(\exp(iu_k(X_{t_k} - X_{t_{k-1}}))$$

which together with Lemma 5.5 shows that $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

Thus we have proved that (X_t) is a normalized Brownian motion. □

In many cases where Theorem 5.2 is used, $\mathcal{F}_t$ is for each t the σ–algebra generated by $\{X_s \mid 0 \le s \le t\}$ and the sets of measure 0. However, the theorem is often applied to cases where the $\mathcal{F}_t$'s are bigger.

We end this note by showing that the continuity assumption in Theorem 5.2 can not be omitted. Let us give the following definition:

Definition 5.6 *An $(\mathcal{F}_t)$–adapted process (N_t) is called a Poisson process with intensity 1 if $N_0 = 0$ a.s. and for $0 \le s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$ and Poisson distributed with parameter $t - s$.*

Hence if (N_t) is a Poisson process with intensity 1, then $N_t - N_s$ takes values in $\mathbb{N} \cup \{0\}$ for all $0 \le s < t$ and

$$P(N_t - N_s = k) = \frac{(t-s)^k}{k!} \exp(-(t-s)) \quad \text{for all } k \in \mathbb{N} \cup \{0\}$$

It can be proved that such processes exist.

Easy calculations show that $E(N_t - N_s) = t - s = V(N_t - N_s)$. The process (M_t), where $M_t = N_t - t$ for all $t \in [0, \infty[$, is called *the compensated Poisson process* with intensity 1. Note that (M_t) is not continuous. We have however:

Theorem 5.7 *If (M_t) is a compensated Poisson process with intensity 1, then it satisfies the conditions (i) and (ii) in Theorem 5.2.*

Proof: Let $0 \le s < t$. Since $M_t - M_s$ is independent of $\mathcal{F}_s$, we get

$$E(M_t \mid \mathcal{F}_s) = M_s + E(M_t - M_s) = M_s.$$

Since $M_t^2 = M_s^2 + (M_t - M_s)^2 + 2M_s(M_t - M_s)$, we also get

$$E(M_t^2 \mid \mathcal{F}_s) = M_s^2 + E((M_t - M_s) \mid \mathcal{F}_s) + 2M_s E(M_t - M_s \mid \mathcal{F}_s) = (t - s) + M_s^2.$$

□

6 Girsanov's theorem

In this section we let again (B_t) denote a one–dimensional Brownian motion, let $0 < T < \infty$, and let $(\mathcal{F}_t)$ be defined as before. Before we can formulate the main theorem of this section we need a little preparation. Let us recall that if Q is another probability measure on $(\Omega, \mathcal{F})$, then Q is said to be *absolutely continuous* with respect to P, written $Q << P$, if $P(A) = 0 \Rightarrow Q(A) = 0$ for all $A \in \mathcal{F}$. A famous result of Radon and Nikodym says that in that case there is a unique $h \in L_1(P)$ so that

$$Q(A) = \int_A h dP \quad \text{for all } A \in \mathcal{F}.$$

We often write this as $dQ = hdP$. In this situation we have:

Theorem 6.1 *If $f \in L_1(Q)$ and $\mathcal{H}$ is a sub–σ–algebra of $\mathcal{F}$, then*

$$E_Q(f \mid \mathcal{H}) E_P(h \mid \mathcal{H}) = E_P(fh \mid \mathcal{H})$$

Proof: Let $A \in \mathcal{H}$ be arbitrary. On one hand we have:

$$\int_A E_Q(f \mid \mathcal{H}) h dP = \int_A E_Q(f \mid \mathcal{H}) dQ = \int_A f dQ =$$
$$\int_A f h dP = \int_A E_P(fh \mid \mathcal{H}) dP \quad .$$

On the other hand we have

$$\int_A E_Q(f \mid \mathcal{H}) h dP = \int_A E_P(E_Q(f \mid \mathcal{H}) h \mid \mathcal{H}) dP =$$
$$\int_A E_Q(f \mid \mathcal{H}) E_P(h \mid \mathcal{H}) dP.$$

which gives the formula. □

P and Q are called equivalent if both $Q << P$ and $P << Q$.

In the rest of this section we let $a : [0, \infty] \times \Omega \to \mathbb{R}$ be a measurable, $(\mathcal{F}_t)$–adapted function which satisfies

$$P(\{\omega \in \Omega \mid \int_0^t a(s,\omega)^2 ds < \infty\}) = 1 \quad \text{for all } 0 \le t < \infty \tag{6.1}$$

Since in particular (6.1) holds for all $n \in \mathbb{N}$, we have also

$$P(\{\omega \in \Omega \mid \int_0^t a(s,\omega)^2 ds < \infty \quad \text{for all } 0 \le t\}) = 1.$$

We recall that a function $\tau : \Omega \to [0, \infty]$ is called a stopping time if $\{\omega \in \Omega \mid \tau(\omega) \le t\} \in \mathcal{F}_t$ for all $0 \le t < \infty$. It is not difficult to prove that if (X_t) is a continuous $(\mathcal{F}_t)$–adapted n–dimensional process, $G \subseteq \mathbb{R}^n$ is open, and

$$\tau_G(\omega) = \inf\{t > 0 \mid \tau(\omega) \notin G\} \quad \inf \emptyset = \infty,$$

then τ_G is a stopping time.

We can now formulate the one–dimensional Girsanov theorem.

Theorem 6.2 *Let Y_t be the Ito process given by*

$$Y_t = \int_0^t a(s,\cdot) ds + B_t \quad \textit{for all } 0 \le t \le T$$

and put

$$M_t = \exp(-\int_0^t a dB - \frac{1}{2}\int_0^t a(s,\cdot)^2 ds) \quad \textit{for all } 0 \le t \le T. \tag{6.2}$$

Assume that $(M_t)_{0 \le t \le T}$ is a martingale. If we define the measure Q on $\mathcal{F}_T$ by $dQ = M_T dP$, then Q is a probability measure and (Y_t) i a Brownian motion with respect to Q.

Before we can prove the theorem in a special case, we will investigate when expressions like the ones in (6.2) form a martingale. Hence let us look on

$$M_t = \exp(\int_0^t adB - \frac{1}{2}\int_0^t a(s,\omega)^2 ds) \quad \text{for all } 0 \le t. \tag{6.3}$$

(we write a instead of $-a$, since the sign does not matter for our investigation)

Ito's formula gives that

$$dM_t = M_t(a(t,\cdot)dB_t - \frac{1}{2}a(t,\cdot)^2 dt) + \frac{1}{2}M_t a(t,\cdot)^2 dt = M_t a(t,\cdot)dB_t,$$

so that

$$M_t = 1 + \int_0^t aMdB \quad \text{for alle } t \ge 0. \tag{6.4}$$

Our first result states:

Theorem 6.3 *(i)* (M_t) *is a supermartingale with* $\mathbb{E}M_t \le 1$ *for all* $t \ge 0$.

(ii) (M_t) *is a martingale if and only if* $\mathbb{E}M_t = 1$ *for all* $t \ge 0$.

Proof: For every $n \in \mathbb{N}$ we put

$$\tau_n = \inf\{t > 0 \mid \int_0^t M_s^2 a(s,\cdot)^2 \ge n\}$$

(remember that $\inf \emptyset = \infty$).

τ_n is a stopping time for all $n \in \mathbb{N}$ and let us show that $\tau_n \to \infty$ a.s. for $n \to \infty$. To see this let $t > 0$ and let $\omega \in \Omega$ so that $s \to M_s(\omega)$ is continuous. Hence there is a constant $K(\omega)$ with $|M_s(\omega)| \le K(\omega)$ for all $0 \le s \le t$. (6.1) gives that except for ω in a zero–set we can find an n_0 so that

$$K(\omega)^2 \int_0^t a(s,\omega)^2 ds < n_0.$$

If $n \ge n_0$, we get for all $0 \le u \le t$ that

$$\int_0^u M_s^2 a(s,\omega)^2 ds < n,$$

which shows that $\tau_n(\omega) > t$ for all $n \ge n_0$. Hence $\tau_n(\omega) \to \infty$.

If $0 < T < \infty$ and we only consider the situation on $[0,T]$, a similar argument shows that for almost all $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for n sufficiently large.

If $n \in \mathbb{N}$ and $t \ge 0$, then

$$M_{t\wedge\tau_n} = 1 + \int_0^t 1_{[0,\tau_n]}(s) M_s a(s,\cdot) dB_s. \tag{6.5}$$

Since τ_n is a stopping time, $1_{[0,\tau_n]}aM \in \mathcal{A}_2(0,t)$ and hence $(M_{t\wedge\tau_n})$ is a martingale with $\mathbb{E}M_{t\wedge\tau_n} = 1$ for all $n \in \mathbb{N}$. Since $M_{t\wedge\tau_n} \geq 0$ the Fatou lemma gives that

$$\mathbb{E}M_t \leq \liminf \mathbb{E}M_{t\wedge\tau_n} = 1 \quad \text{for all } t \geq 0.$$

If we apply Fatou's lemma for conditional expectations we get for all $0 \leq s < t$ that

$$\mathbb{E}(M_t \mid \mathcal{F}_s) \leq \liminf \mathbb{E}(M_{t\wedge\tau_n} \mid \mathcal{F}_s) = \lim M_{s\wedge\tau_n} = M_s,$$

which shows that (M_t) is a supermartingale.

Let us now show (ii). If (M_t) is a martingale, then $\mathbb{E}M_t = \mathbb{E}M_0 = 1$.

Assume next that $\mathbb{E}M_t = 1$ for all $t \geq 0$ and let $0 \leq s < t$. If we put

$$A = \{\omega \in \Omega \mid \mathbb{E}(M_t \mid \mathcal{F}_s)(\omega) < M_s(\omega)\},$$

then we need to show that $P(A) = 0$. The assumption $P(A) > 0$ gives that

$$\begin{aligned} 1 = \mathbb{E}M_t &= \int_\Omega \mathbb{E}(M_t \mid \mathcal{F}_s)dP = \\ \int_A \mathbb{E}(M_t \mid \mathcal{F}_s) + \int_{\Omega\setminus A} \mathbb{E}(M_t \mid \mathcal{F}_s)dP &< \int_A M_s dP + \int_{\Omega\setminus A} M_s dP = \\ \mathbb{E}M_s &= 1 \end{aligned}$$

which is a contradiction. Hence $P(A) = 0$ and (M_t) is a martingale. □

In connection with applications of the Girsanov theorem it is of course important to find sufficient conditions for (M_t) being a martingale, often only in the interval $[0, T]$. One of the most important sufficient conditions is the **Novikov condition**:

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T a(s,\cdot)^2 ds\right) < \infty \quad \text{where } 0 < T < \infty. \tag{6.6}$$

If (6.6) holds for a fixed T, then $\{M_t \mid 0 \leq t \leq T\}$ is a martingale and if (6.6) holds for all $0 \leq T < \infty$, then $\{M_t \mid 0 \leq t\}$ is a martingale. It lies outside the scope of these lectures to show this and we shall therefore do something simpler which covers most cases that appear in practice: Since aM is adapted, it follows from (6.4) that if $aM \in L_2([0,t] \times \Omega)$ for every $0 \leq t < \infty$ (respectively for every $0 \leq t \leq T < \infty$), then $\{M_t \mid 0 \leq t\}$ is a martingale (respectively $\{M_t \mid 0 \leq t \leq T\}$ is a martingale). The next theorem gives a sufficient condition for this:

Theorem 6.4 *Let $f\colon [0,\infty[\to [0,\infty[$ be a measurable function and $0 < T < \infty$. If*

$$f \in L_2[0,T] \tag{6.7}$$

and

$$|a(t,w)| \leq f(t) \quad \textit{for all } 0 \leq t \leq T \textit{ and a.a. } s \in \Omega, \tag{6.8}$$

then:

(i) For all $1 \le p < \infty$ and all $0 \le t \le T$ we have $M_t \in L_p(P)$ with

$$\mathbb{E}M_t^p \le \exp\big(\frac{p^2-p}{2}\int_0^t f(s)^2 ds\big). \tag{6.9}$$

(ii) $\{M_t \mid 0 \le t \le T\}$ is a martingale.

(iii) . If (6.7) og (6.8) holds for every $0 \le T < \infty$, then $\{M_t \mid 0 \le t\}$ is a martingale.

Proof: To show (i) we let $1 \le p < \infty$ and let $0 \le t \le T$. We find:

$$\begin{aligned} M_t^p = \exp\big(p\int_0^t a dB - \frac{p}{2}\int_0^t a(s,\cdot)^2 ds\big) &= \quad (6.10)\\ \exp\big(\int_0^t padB - \frac{1}{2}\int_0^t (pa(s,\cdot))^2 ds\big)\exp\big(\frac{p^2-p}{2}\int_0^t a(s,\cdot)^2 ds\big) &\le \\ \exp\big(\int_0^t padB - \frac{1}{2}\int_0^t (pa(s,\cdot))^2 ds\big)\exp\big(\frac{p^2-p}{2}\int_0^t f(s)^2 ds\big). \end{aligned}$$

Since pa satisfies (6.1), Theorem 6.3 (i) gives that

$$\mathbb{E}M_t^p \le \exp\big(\frac{p^2-p}{2}\int_0^t f(s)^2 ds\big),$$

which shows (i).

To prove (ii) we show that $aM \in L_2([0,T]\times\Omega)$. From (6.9) with $p=2$ we get:

$$\begin{aligned} \int_0^T \mathbb{E}(a(t,\cdot)^2 M_t^2)dt \le \int_0^T f(t)^2 \exp\big(\int_0^t f(s)^2 ds\big)dt &\le \\ \exp\big(\int_0^T f(t)^2 dt\big)\int_0^T f(s)^2 ds < \infty &, \end{aligned}$$

which shows that $aM \in L_2([0,T]\times\Omega)$.

(iii) follows directly from (ii) □

Note that in particular Theorem 6.4 is applicable in the important case where a is bounded.

Before we go on, we wish to make a small detour and apply the above to geometric Brownian motions. Hence let (X_t) be a geometric Brownian motion starting in a point $x \in \mathbb{R}$, say

$$X_t = x\exp\big((r - \frac{1}{2}\alpha^2)t + \alpha B_t\big) = x\exp(rt)\exp(\alpha B_t - \frac{1}{2}\alpha^2 t) \quad \text{for all } t \ge 0,$$

where $r, \alpha \in \mathbb{R}$. By the above $(\exp(\alpha B_t - \frac{1}{2}\alpha^2 t))$ is a martingale and therefore $\mathbb{E}(X_t) = x\exp(rt)$ for all $t \ge 0$. This can of course also be obtained using that B_t is normally distributed.

We also need:

Lemma 6.5 *Let $0 \leq T < \infty$ and assume that $\{M_t \mid 0 \leq t \leq T\}$ is a martingale.Let Q be the measure on $\mathcal{F}_T$ defined by $dQ = M_T dP$. If $(X_t) \subseteq L_1(Q)$ is $(\mathcal{F}_t)$–adapted, then (X_t) is a Q–martingale if and only if $(X_t M_t)$ er en P–martingale.*

Proof:

This is an immediate consequence of the following formula which follows directly from Theorem 6.1 together with our assumptions. For all $0 \leq s < t$ we have:

$$\mathbb{E}_P(X_t M_t \mid \mathcal{F}_s) = \mathbb{E}_Q(X_t \mid \mathcal{F}_s))\mathbb{E}_P(M_t \mathcal{F}_s) = \mathbb{E}_Q(X_t \mid \mathcal{F}_s)M_s.$$

□

Proof of Girsanov's Theorem in a special case:

We will show Theorem 6.2 under the assumption that a satisfies the conditions of Theorem 6.4. According to Lévy's Theorem we have to show that (Y_t) and $(Y_t^2 - t)$ are Q–martingales. Note that from (6.4) we get that $dM_t = -aMdB_t$. To see that (Y_t) is a Q–martingale we need to show that $(M_t Y_t)$ is a P–martingale and get by Ito's formula:

$$\begin{aligned} d(M_t Y_t) = M_t dY_t + Y_t dM_t + dM_t dY_t = \\ M_t(a(t)dt + dB_t) - Y_t a(t) M_t dB_t - a(t) M_t dt = \\ M_t(1 - Y_t a) dB_t \end{aligned} \quad ,$$

or

$$M_t Y_t = \int_0^t M_s(1 - Y_t a(s, \cdot))dB_s \quad \text{for all } 0 \leq t \leq T.$$

Hence we can finish by proving that the integrand belongs to $L_2([0,T] \times \Omega)$. We note that

$$M_t|1 - Y_t a(t)| \leq M_t + M_t|Y_t|f(t)$$

and since $\mathbb{E}(M_t) = 1$ for all $0 \leq t \leq T$, $M \in L_2([0,T] \times \Omega)$. Further

$$|Y_t| \leq \int_0^t f(s)ds + |B_t|$$

so that

$$M_t|Y_t|f(t) \leq f(t)M_t \int_0^T f(s)ds + f(t)M_t|B_t| \tag{6.11}$$

For $p = 2$ Theorem 6.4 gives:

$$\begin{aligned} \int_0^T f(t)^2 \mathbb{E}(M_t^2)dt &\leq \\ \int_0^T f(t)^2 \exp(\int_0^t f(s)^2 ds)dt &\leq \\ \exp(\int_0^T f(s)^2 ds) \int_0^T f(t)^2 dt &< \infty \end{aligned}$$

which takes care of the first term in (6.11).

To take care of the second term we use Theorem 6.4 and the Cauchy–Schwartz's inequality to get:

$$\mathbb{E}(M_t^2 B_t^2) \leq \mathbb{E}(M_t^4)^{\frac{1}{2}} \mathbb{E}(B_t^4)^{\frac{1}{2}} \leq \sqrt{3} T \exp(3 \int_0^T f(s)^2 ds)$$

where we have used that $\mathbb{E}(B_t^4) = 3t^2$. Finally

$$\int_0^T f(t)^2 \mathbb{E}(M_t^2 B_t^2) dt \leq \sqrt{3} T \exp(3 \int_0^T f(s)^2 ds) \int_0^T f(t)^2 dt < \infty,$$

which shows what we wanted. Hence (X_t) is a Q–martingale.

Similar arguments and estimates will show that $((Y_t^2 - t) M_t)$ is a P–martingale and hence that $(Y_t^2 - t)$ is a Q– martingale. □

Girsanov's Theorem has the following corollary

Corollary 6.6 *Let $0 < T < \infty$ and let (X_t) be an Ito process of the form*

$$X_t = X_0 + \int_0^t u d\boldsymbol{m} + \int_0^t v dB,$$

where u and v are such that the integrals make sense.

Assume further that $v \neq 0$ a.s and put $a = \frac{u}{v}$ and that a satisfies (6.1) *and define (M_t) and Q as in Theorem 6.2. If (M_t) is a martingale, then the process*

$$\tilde{B}_t = \int_0^t a d\boldsymbol{m} + B_t \quad 0 \leq t \leq T,$$

is a Q– Brownian motion and

$$X_t = X_0 + \int_0^t v d\tilde{B}.$$

Proof: It follows from Theorem 6.2 that Q is a probability measure on $\mathcal{F}_T$ and that $\tilde{B}$ is a Q–Brownian motion. Further we get:

$$dX_t = u(t)dt + v(t)(d\tilde{B}_t - a(t)dt) = v(t)d\tilde{B}_t.$$

□

Theorem 6.2 and its corollary can be generalized to higher dimensions. In that case the a in Theorem 6.2 will take values in $\mathbb{R}^n$ and if we interpret a^2 as $\|a\|^2$, then (M_t) and Q are defined as before and the result carries over using a multi–dimensional form of Lévy's result. In the corollary (B_t) will be an m–dimensional Brownian motion, u will take values in $\mathbb{R}^n$ and v will take values in the space of $n \times m$ matrices. The requirement is then that the matrix equation $va = u$ has a solution satisfying the requirements of the corollary. If our process (X_t) there represents a financial market, then the mathematical conditions reflex to some extend the behaviour in practice of the financial market. In other words, Theorem 6.2 and Corollary 6.6 has a lot of applications in practice.

At the end we will discuss under which conditions Theorem 6.2 can extended to the case where $T = \infty$. Hence we let $a : [0, \infty[\times \Omega \to \mathbb{R}$ satisfy (6.1) and define M_t as in (6.3), that is

$$M_t = \exp(\int_0^t adB - \frac{1}{2}\int_0^t a^2 d\boldsymbol{m}) \quad t \geq 0.$$

For convenience we shall assume that $\mathcal{F}$ **is equal to the σ–algebra generated by** $\{\mathcal{F}_t \mid 0 \leq t\}$**.**

If (M_t) is a martingale, we can for every $t \geq 0$ define a probability measure Q_t on $\mathcal{F}_t$ by $dQ_t = M_t dP$ and the question is now whether there is a probability measure Q on $\mathcal{F}$ so that $Q|\mathcal{F}_t = Q_t$ for all $0 \leq t < \infty$. The next theorem gives a necessary and sufficient condition for this to happen.

Theorem 6.7 *Assume that $\{M_t \mid 0 \leq t\}$ is a martingale. Then $M_\infty = \lim_{t\to\infty} M_t$ exists a.s.*

The following statements are equivalent:

(i) There exists a probability measure Q on $\mathcal{F}$ with $Q << P$ og $Q \mid \mathcal{F}_t = Q_t$ for all $t \geq 0$.

(ii) (M_t) is uniformly integrable.

If (i) (or equivalently (ii)) holds, then $dQ = M_\infty dP$.

Proof: Since $\mathbb{E}M_t = 1$ for alle $0 \leq t$, the martingale convergence theorem gives us the existence of M_∞ a.e.

Assume first that (i) and determine $f \in L_1(P)$ so that $dQ = fdP$. Since $Q \mid \mathcal{F}_t = Q_t$, it clearly follows that $\mathbb{E}(f \mid \mathcal{F}_t) = M_t$ for alle $0 \leq t$. Let us show that this implies that $\{M_t \mid t \geq 0\}$ is uniformly integrable. Since $(M_t(\omega))$ is convergent for a.a ω, $\sup_{t\geq 0} M_t(\omega) < \infty$ for a.a ω. If $0 \leq t < \infty$ and $x > 0$, then

$$\begin{aligned}\int_{(M_t>x)} M_t dP &= \int_{(M_t>x)} \mathbb{E}(f \mid \mathcal{F}_t) dP = \\ \int_{(M_t>x)} f dP &\leq \int_{(\sup M_s > x)} f dP,\end{aligned}$$

where we have used that $(M_t > x) \in \mathcal{F}_t$

Hence

$$\begin{aligned}\lim_{x\to\infty} \sup_{t\geq 0} \int_{(M_t>0)} M_t dP &\leq \lim_{x\to\infty} \int_{(\sup M_t > x)} f dP = \\ \int_{(\sup M_t = \infty)} f dP &= 0,\end{aligned}$$

which shows that (M_t) is uniformly integrable.

Assume next that (ii) holds. Then $M_t \to M_\infty$ in $L_1(P)$ which implies that $\mathbb{E}M_\infty = \lim \mathbb{E}M_t = 1$, and that $\mathbb{E}(M_\infty \mid \mathcal{F}_t) = \lim_s \mathbb{E}(M_s \mid \mathcal{F}_t) = M_t$ for all $t \geq 0$. If we put $dQ = M_\infty dP$, then Q is a probability measure and if $t \geq 0$ and $A \in \mathcal{F}_t$, then:

$$\begin{aligned} Q(A) = \int_A M_\infty dP &= \int_A \mathbb{E}(M_\infty \mid \mathcal{F}_t) dP = \\ \int_A M_t dP &= Q_t(A), \end{aligned}$$

which shows that $Q \mid \mathcal{F}_t = Q_t$ so that (i) holds.

Hence we have proved that (i) and (ii) are equivalent.

Let again (i) hold. From the proof of $(ii) \Rightarrow (i)$ we get that if we put $dQ_1 = M_\infty dP$, then $Q_1(A) = Q(A)$ for all $A \in \bigcup_{0\leq t} \mathcal{F}_t$ and since this class constitutes a $\cap$-stable generator system for $\mathcal{F}$, $Q_1(A) = Q(A)$ for alle $A \in \mathcal{F}$; hence $dQ = dQ_1 = M_\infty dP$. $\square$

If we combine Theorem 6.3 with Theorem 6.7 we get the following corollary.

Corollary 6.8 *Let* $f\colon [0,\infty[\to [0,\infty[$ *be a measurable function so that*

$$f \in L_2([0,\infty[) \tag{6.12}$$

$$|a(t,\omega)| \leq f(t) \quad \text{for all } 0 \leq t \text{ og n.a. } \omega \in \Omega. \tag{6.13}$$

Then (M_t) *is a uniformly integrable martingale and hence Theorem 6.7 can be applied*

Proof: It is immediate that (6.7) og (6.8) of Theorem 6.4 are satisfied so that (M_t) is a martingale. If we apply (6.9) med $p = 2$, we get:

$$\mathbb{E}M_t^2 \leq \exp\Big(\int_0^t f(s)^2 ds\Big) \leq \exp\Big(\int_0^\infty f(s)^2 ds\Big),$$

which shows that (M_t) is bounded in $L_2(P)$ and therefore uniformly integrable. This proves the corollary. $\square$

Girsanov's Theorem 6.2 holds on the interval $[0,\infty[$, if we assume that the (M_t) there is a uniformly integrable martingale. If a satisfies the conditions in Corollary 6.8 small modifications of our proof of Theorem 6.2 will give a proof of this.

Let us end this section with the following example-

Example Lad a være konstant, $a \neq 0$. (6.7)) and (6.8) are clearly satisfied so that (M_t) is a martingale. In fact, $M_t = \exp(aB_t - \frac{1}{2}a^2 t)$ for all $t \geq 0$. The martingale convergence theorem shows that $M_\infty = \lim_{t\to\infty} M_t$ exists a.e. Since however $M_\infty = 0$ a.e. (see below) and $\mathbb{E}M_t = 1$ for all $0 \leq t$, (M_t) cannot be uniformly integrable.

That $M_\infty = 0$ can be seen as follows: It is sufficient to show that $M_t \to 0$ in probability because then there is a subsequence (t_n) with $M_{t_n} \to 0$ a.s.

Hence let $\varepsilon > 0$ and determine t_0 so that $\frac{1}{2}a^2 t_0 + \log\varepsilon > 0$ and put $b_t = a^{-1}(\frac{1}{2}a^2 t + \log\varepsilon)$. If $a > 0$, then for all $t \geq t_0$ we get:

$$\begin{aligned} P(M_t \geq \varepsilon) &= P(B_t \geq b_t) = \frac{1}{\sqrt{2t\pi}}\int_{b_t}^{\infty} \exp(-\frac{1}{2t}x^2)dx \leq \\ \frac{1}{\sqrt{2t\pi}}\frac{1}{b_t}\int_{b_t}^{\infty} x\exp(\frac{1}{2t}x^2)dx &= \sqrt{t}\frac{1}{\sqrt{2\pi}}\big[-\exp(-\frac{1}{2t}x^2)\big]_{x=b_t}^{\infty} = \\ \sqrt{t}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2t}b_t^2) &\to 0 \quad \text{for } t \to \infty \end{aligned}$$

Similar calculations show that also $P(M_t \geq \varepsilon) \to 0$ for $t \to \infty$ in case $a < 0$.

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