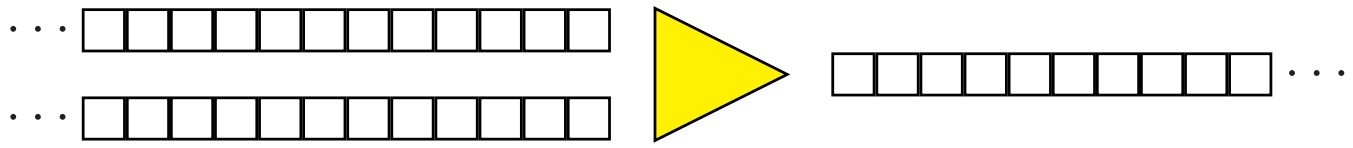
# I/O Efficient Sorting

# Upper and Lower bounds

- Aggarwal and Vitter, *The Input/Output Complexity of Sorting and Related Problems*. Communications of the ACM, 31(9), p. 1116-1127, 1988.

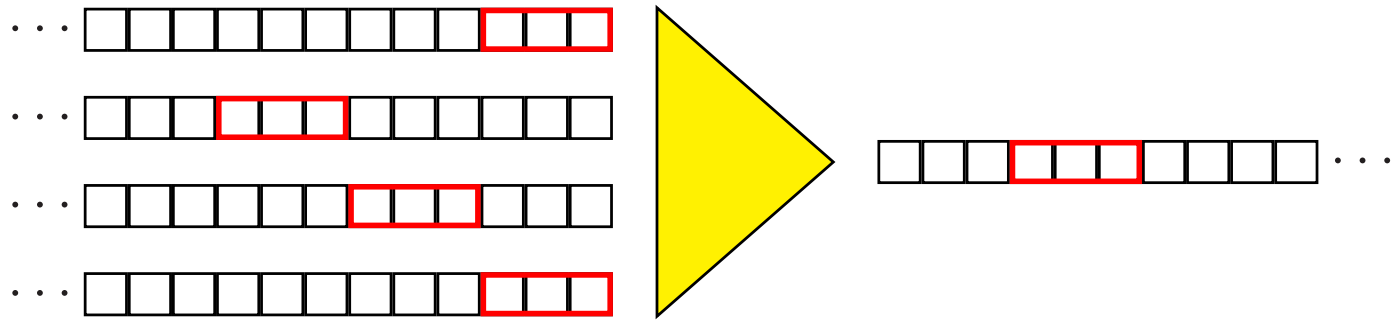
# Standard MergeSort

Merge of two sorted sequences $\sim$ sequential access



MergeSort: $O(N \log_2(N/M)/B)$ I/Os

# Multiway Merge



- For I/O-efficient $k$-way merge of sorted lists we need:

$$M \geq B(k+1) \Leftrightarrow M/B - 1 \geq k$$

- Number of I/Os: $2N/B$.

# Multiway MergeSort

- $N/M$ times sort M elements internally $\Rightarrow$ $N/M$ sorted *runs* of length $M$.
- Merge $k$ runs at at time, to produce $(N/M)/k$ sorted runs of length $kM$.
- Repeat: Merge $k$ runs at at time, to produce $(N/M)/k^2$ sorted runs of length $k^2M$, ...

At most $\log_k N/M$ phases, each using $2N/B$ I/Os.

Best $k$: M/B-1.

$$O(N/B \log_{M/B}(N/M)) \text{ I/Os}$$

# Multiway MergeSort

$$1 + \log_{M/B}(x) = \log_{M/B}(M/B) + \log_{M/B}(x) = \log_{M/B}(x \cdot M/B)$$

$$\Downarrow$$

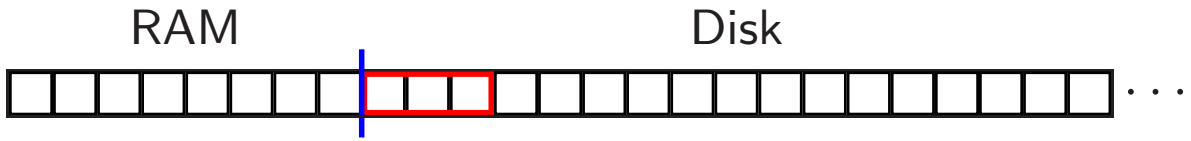
$$O(N/B \log_{M/B}(N/M)) = O(N/B \log_{M/B}(N/B))$$

Defining $n = N/B$ and $m = M/B$ we get

Multiway MergeSort: $O(n \log_m(n))$

# Sorting Lower Bound

Model of memory:




- Comparison based model: elements may be compared in internal memory. May be moved, copied, destroyed. Nothing else.
- Assume $M \geq 2B$.
- May assume I/Os are block-aligned, and that at start, input contiguous in lowest positions on disk.
- Adversary argument: adversary gives order of elements in internal memory (chooses freely among consistent answers).
- Given an execution of a sorting algorithm: $S_t$ = number of permutations consistent with knowledge of order after $t$ I/Os.

# Adversary Strategy

After an I/O, adversary must give new answer, i.e. must give order of elements currently in RAM.

If number of possible (i.e. consistent with current knowledge) orders is $X$, then there exist answer such that

$$S_{t+1} \geq S_t/X.$$

This is because any single answer induces a subset of the $S_t$ currently possible permutations (consisting of the permutations consistent with this answer), and the $X$ such subsets clearly form a partition of the $S_t$ permutations. If no subset has size $S_t/X$, the subsets cannot add up to $S_t$ permutations.

Adversary chooses answer fulfilling the inequality above.

# Possible X's

| Type of I/O | Read untouched block | Read touched block | Write |
|---|---|---|---|
| $X$ | $\binom{M}{B}B!$ | $\binom{M}{B}$ | 1 |

Note: at most $N/B$ untouched blocks read.

From $S_0 = N!$ and $S_{t+1} \geq S_t/X$ we get

$$S_t \geq \frac{N!}{\binom{M}{B}^t (B!)^{N/B}}$$

Sorting algorithm cannot stop before $S_t = 1$. Thus,

$$1 \geq \frac{N!}{\binom{M}{B}^t (B!)^{N/B}}$$

for any correct algorithm making $t$ I/Os.

# Lower Bound Computation

$$1 \geq \frac{N!}{\binom{M}{B}^t (B!)^{N/B}}$$

$$t \log \binom{M}{B} + (N/B) \log(B!) \geq \log(N!)$$

$$3tB \log(M/B) + N \log B \geq N(\log N - 1/\ln 2)$$

$$3t \geq \frac{N(\log N - 1/\ln 2 - \log B)}{B \log(M/B)}$$

$$t = \Omega(N/B \log_{M/B}(N/B))$$

Lemma was used:

a) $\log(x!) \geq x(\log x - 1/\ln 2)$

b) $\log(x!) \leq x \log x$

c) $\log \binom{x}{y} \leq 3y \log(x/y)$ when $x \geq 2y$

# Proof of Lemma

Lemma:

a) $\log(x!) \geq x(\log x - 1/\ln 2)$

b) $\log(x!) \leq x \log x$

c) $\log \binom{x}{y} \leq 3y \log(x/y)$ when $x \geq 2y$

Stirlings formula: $n! = \sqrt{2\pi n} \cdot (n/e)^n \cdot (1 + O(1/12n))$

Proof (using Stirling):

a) $\log(x!) \geq \log(\sqrt{2\pi x}) + x(\log x - 1/\ln 2) + o(1)$

b) $\log(x!) \leq \log(x^x) = x \log x$

c) $\log \binom{x}{y} \leq \log(\frac{x^y}{(y/e)^y}) = y(\log(x/y) + \log(e))$

$\leq 3y \log(x/y)$ when $x \geq 2y$

# The I/O-Complexity of Sorting

Defining

$\mathsf{n} = \mathsf{N}/\mathsf{B}$

$\mathsf{m} = \mathsf{M}/\mathsf{B}$

$N/B \log_{M/B}(N/B) = \mathrm{sort}(N)$

we have proven

**I/O cost of sorting:**

$$\Theta(N/B \log_{M/B}(N/B)) = \Theta(n \log_m(n)) = \Theta(\mathrm{sort}(N))$$