Affine Transformations

Last time: allow translations by "going to 4×4" (and back).

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{bmatrix} & & & \\ & 4\times4 & & \\ & & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \rightarrow \begin{bmatrix} x/t \\ y/t \\ z/t \end{bmatrix}$$

Also called homogeneous coordinates.

But over all 4×4 Matrices M, this captures more than our transformations wanted (eg. also perspective projection [which is good at later stages of pipeline]).

Our wanted transformations are captured by a affine transformations:

<u>Def</u>:
$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} & & \\ & 3\times3 & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$= M \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \vec{D},$$ where M is an invertible matrix.

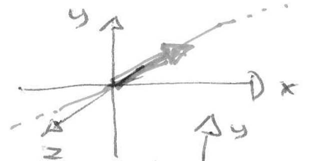
Equivalently, these transformations correspond to 4×4 matrices of the form

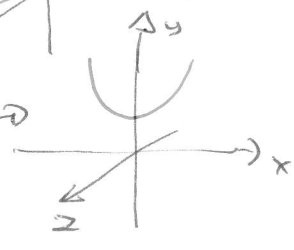
$$3\times 3 \quad \begin{bmatrix} \boxed{\begin{matrix} & & \\ & M & \\ & & \end{matrix}} & \begin{matrix} dx \\ dy \\ dz \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad \text{(with } M \text{ invertible)}$$

Equivalently, they correspond to invertible functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ where each coordinate of the result is given by a polynomial in x, y, z of degree at most one. I.e a function/expression of the form $a \cdot x + b \cdot y + c \cdot z + d$.

Note that polynomials of larger degree do not preserve lines as lines. E.g. for the map $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2 \cdot x \cdot y + 4 \\ z \end{pmatrix}$, the y-coord. is of degree 2.

It maps the line with parametrisation $t \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $[t \in \mathbb{R}]$ into this parabola





Thm Affine transformations map

$$\boxed{\begin{array}{l} \oplus \text{ lines to lines} \\ \oplus \text{ planes to planes} \end{array}}$$

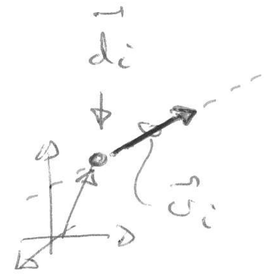
and the images of two lines l_1, l_2 [two planes p_1, p_2]

$$\boxed{\begin{array}{l} \oplus \text{ are parallel} \\ \oplus \text{ intersect} \end{array}}$$

if and only if l_1, l_2 [p_1, p_2] are/do.

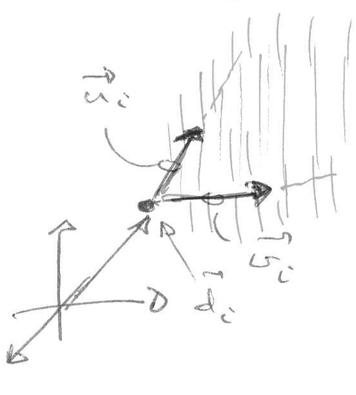
Proof Let l_i have parametrisation

$$t \cdot \vec{v}_i + \vec{d}_i \qquad \begin{array}{l} t \in \mathbb{R} \\ \vec{v}_i \neq \vec{0} \end{array}$$



for $i = 1, 2$ and let p_i have parametrisation

$$s \cdot \vec{v}_i + t \cdot \vec{u}_i + \vec{d}_i \qquad \begin{array}{l} s, t \in \mathbb{R} \\ \vec{v}_i, \vec{u}_i \neq \vec{0} \\ \vec{v}_i \times \vec{u}_i \neq \vec{0} \end{array}$$



Let the affine transf. be $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{D}$

with M invertible.

$$f(l_1) = M\cdot(t\cdot\vec{v}_1 + d_1) + \vec{D}$$

$$= t\cdot\underbrace{(M\cdot\vec{v}_1)}_{\vec{v}_1'} + \underbrace{(M\cdot\vec{d}_1 + \vec{D})}_{\vec{d}_1'}$$

[$\vec{v}_1' \neq \vec{0}$ as M is invertible and $\vec{v}_1 \neq \vec{0}$]

This is a parametrisation $t\cdot\vec{v}_1' + \vec{d}_1'$ of a line, which proves first ⊕.

$$f(p_1) = M\cdot(s\cdot\vec{v}_1 + t\cdot\vec{u}_1 + \vec{d}_1) + \vec{D}$$

$$= s\cdot\underbrace{(M\cdot\vec{v}_1)}_{\vec{v}_1'} + t\cdot\underbrace{(M\cdot\vec{u}_1)}_{\vec{u}_1'} + \underbrace{M\cdot\vec{d}_1 + \vec{D}}_{\vec{d}_1'}$$

[Again, $\vec{v}_1', \vec{u}_1' \neq \vec{0}$]

This is a parametrisation of a plane <u>if</u> $\vec{v}_1' \times \vec{u}_1' \neq \vec{0}$.

From linear algebra we have (long and tedious proof omitted) the following identity for any invertible 3×3 matrix M:

$$(M\cdot\vec{v})\times(M\cdot\vec{w}) = \det(M)\cdot\left(M^{-1}\right)^T\cdot(\vec{v}\times\vec{w})$$

From $\vec{v}_1 \times \vec{u}_1 \neq \vec{0}$, this identity gives the <u>if</u>, and second ⊕ is proved.

For the third +:

[$\Leftarrow$ via M invertible]

l_1 and l_2 parallel $\Leftrightarrow$ $\vec{v}_1 = c \cdot \vec{v}_2$ $\Leftrightarrow$

$M \cdot \vec{v}_1 = c \cdot M \cdot \vec{v}_2$ $\Leftrightarrow$ images of l_1 and l_2 are parallel.

p_1 and p_2 parallel $\Leftrightarrow$ $\underbrace{\vec{v}_1 \times \vec{u}_1}_{\text{normal vector for } p_1} = c \cdot \underbrace{(\vec{v}_2 \times \vec{u}_2)}_{\text{same for } p_2}$

$$\Leftrightarrow \underbrace{(M \cdot \vec{v}_1) \times (M \cdot \vec{u}_1)}_{\text{normal vector for } f(p_1)} = c \cdot \underbrace{((M \cdot \vec{v}_2) \times (M \cdot \vec{u}_2))}_{\text{same for } f(p_2)}$$

using identity at bottom of page 4

$\Leftrightarrow$ images of p_1 and p_2 are parallel.

For the fourth +:

[$\Leftarrow$ because M invertible]

$t_1 \cdot \vec{v}_1 + \vec{d}_1 = t_2 \cdot \vec{v}_2 + \vec{d}_2$ $\Leftrightarrow$

$t_1 \cdot (M \cdot \vec{v}_1) + M \cdot \vec{d}_1 + \vec{D} = t_2 \cdot (M \cdot \vec{v}_2) + M \cdot \vec{d}_2 + \vec{D}$

shows: l_1 and l_2 intersect $\Leftrightarrow$ images intersect.

Same proof applies to the planes

$\square$

<u>Thm.</u> Any 2D affine transformation can be written as

$$g_4 \circ g_3 \circ g_2 \circ g_1$$

where:

g_1, g_3 are rotations about origin

g_2 is a scaling

g_4 is a translation

<u>Thm</u>: Any 3D affine transformation can be written as

$$g_4 \circ g_3 \circ g_2 \circ g_1$$

where

g_1, g_3 are rotations about a radial axis

g_2 is a scaling

g_4 is a translation

So these transformations <u>span</u> the entire set of affine transformations

Proof of first thm : See pages 193 - 5 in text book. Is curriculum.

Proof of second thm : Omitted, as it requires too advanced linear algebra (proof is on p. 235 and 233, but is not exam curriculum).

Def.

Euclidian affine transformations are those preserving distances:

$$|f(\vec{x}_1)\, f(\vec{x}_2)| = |\vec{x}_1\, \vec{x}_2| \quad \text{for all } \vec{x}_1, \vec{x}_2 \in \mathbb{R}^3 \quad [\text{or } \mathbb{R}^2]$$

Thm (not in book)

$$f(\vec{x}) = M \cdot \vec{x} + \vec{D} \quad \text{Euclidian}$$

$$\Updownarrow$$

$$\det(M) = \pm 1$$

Proof: omitted (moderate linear algebra)

Def: $+1$: orientation preserving

-1 : —— " —— reversing

$\boxed{\text{Def}}$ Rigid affine transf. is one which is Euclidean and has det. = +1.

Thm The affine transformations contains ~~the~~ following subclasses, each generated / spanned by the stated transformation types:

	Rigid	Euclidian	All affine
Maps $\vec{o}$ to $\vec{o}$	R	R + M	R + S
Maps $\vec{o}$ anywhere	R + T	R + M + T	R + S + T

R = rotation (radial) [axis through $\vec{o}$]

M = mirror (radial) [plane —"—]

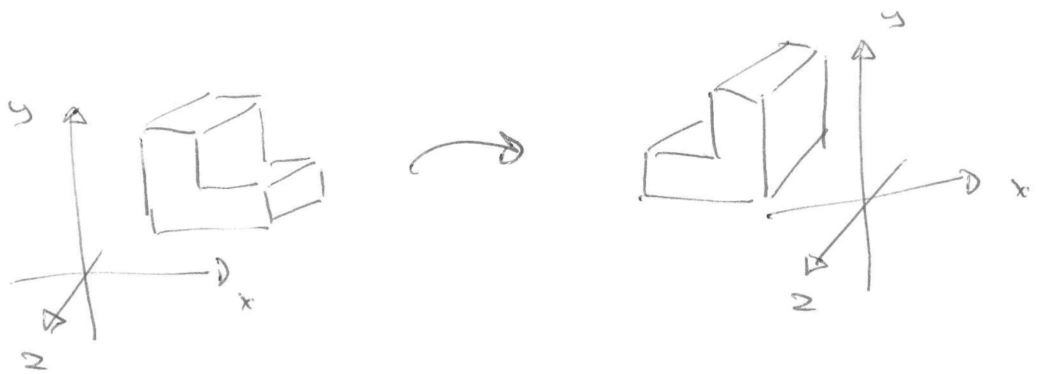
S = scaling (radial) [plane(s) —"—]

T = transformation

Proof: most appears in book, but not curriculum.

Note: mirror [a.k.a. reflection] is a special case of scaling:

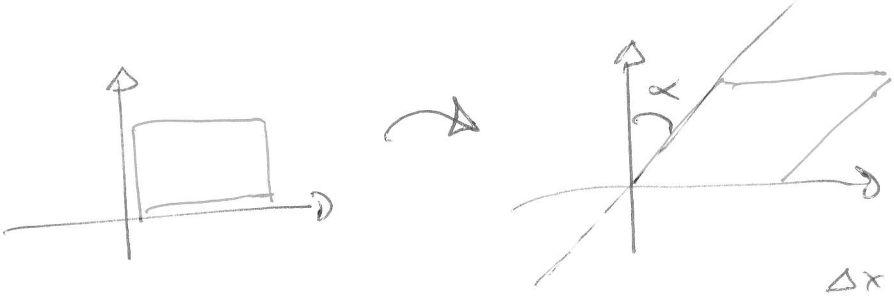
$$M \cdot \vec{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{x}$$

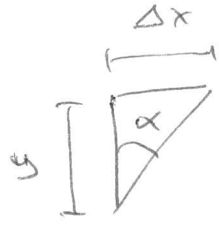


Reflection in y-z plane. For other planes, use "the trick".

Shear

2D:





$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+\Delta x\\ y\end{pmatrix}$$

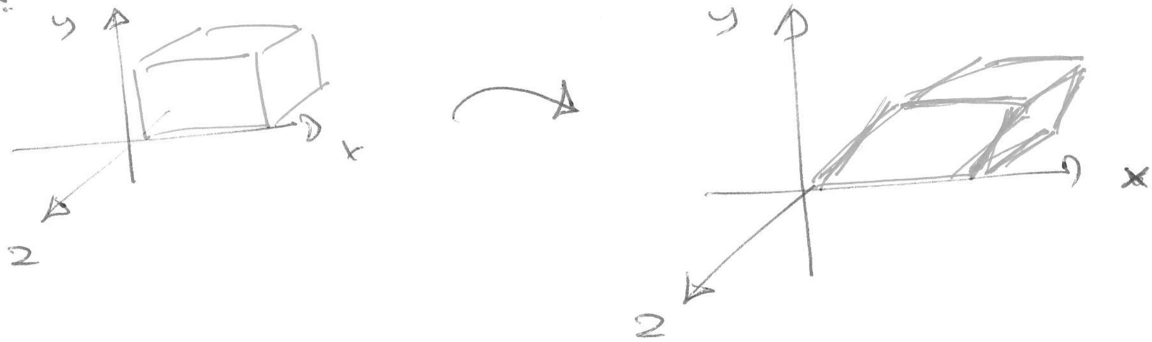
$$= \begin{pmatrix}x+\tan(\alpha)\cdot y\\ y\end{pmatrix}$$

$$= \begin{pmatrix}1 & \tan(\alpha)\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

$$\frac{\Delta x}{y} = \tan(\alpha).$$

Above: line of shear is x-axis. For other axes: use "the trick"

3D:



$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+\tan(\alpha)\cdot y\\ y\\ z\end{pmatrix} = \begin{pmatrix}1 & \tan(\alpha) & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Above: plane of shear is x-z plane and line of shear $\underbrace{\text{is x-axis}}_{\text{in this plane}}$. For other choices, use "the trick".