Affine Transformations

Last time: allow translations by "going to 4x4" (and back).

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{bmatrix} & & & \\ & 4\times 4 & & \\ & & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{bmatrix} x/t \\ y/t \\ z/t \end{bmatrix}$$

Also called homogeneous coordinates.

But over all 4x4 Matrices M, this captures <u>more</u> than our transformations wanted (eg. also perspective projection [which is good at later stages of pipeline]).

Our wanted <u>transformations</u> are captured by <u>affine transformations</u>:

<u>Def.</u>: $$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} & & \\ & 3\times 3 & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$= M \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \vec{D},$$ where M is an invertible matrix.

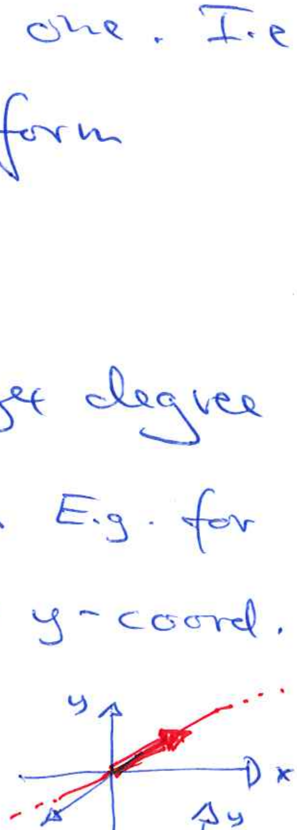
Equivalently, these transformations correspond to 4×4 matrices of the form

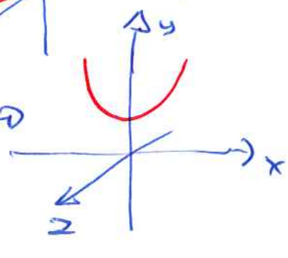
$$\underset{3\times 3}{} \begin{bmatrix} \boxed{\quad M \quad} & \begin{matrix} dx \\ dy \\ dz \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \qquad \left(\text{with } M \text{ invertible}\right)$$

Equivalently, they correspond to invertible functions $f: \mathbb{R}^3 \to \mathbb{R}^3$ where each coordinate of the result is given by a polynomial in x, y, z of degree at most one. I.e a function/expression of the form $a \cdot x + b \cdot y + c \cdot z + d$.

Note that polynomials of larger degree do not preserve Lines as Lines. E.g. for the map $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2 \cdot x \cdot y + 4 \\ z \end{pmatrix}$, the y-coord. is of degree 2.

It maps the line with parametrisation $t \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $[t \in \mathbb{R}]$ into this parabola





Thm Affine transformations map

$$\boxed{\begin{array}{l} \oplus \text{ lines to lines} \\ \oplus \text{ planes to planes} \end{array}}$$

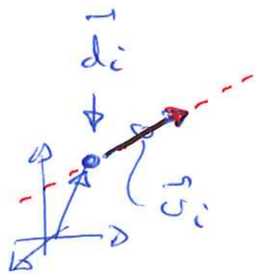
and the images of two lines l_1, l_2 [two planes p_1, p_2]

$$\boxed{\begin{array}{l} \oplus \text{ are parallel} \\ \oplus \text{ intersect} \end{array}}$$

if and only if l_1, l_2 [p_1, p_2] are/do.

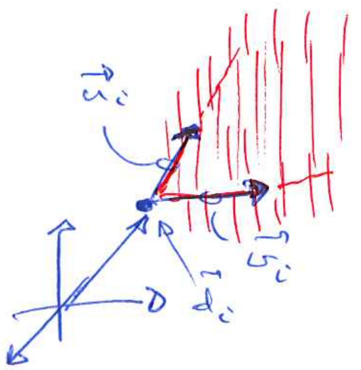
Proof Let l_i have parametrisation

$$t \cdot \vec{v}_i + \vec{d}_i \qquad \begin{array}{l} t \in \mathbb{R} \\ \vec{v}_i \neq \vec{0} \end{array}$$



for $i = 1,2$ and let p_i have parametrisation

$$s \cdot \vec{v}_i + t \cdot \vec{u}_i + \vec{d}_i \qquad \begin{array}{l} s,t \in \mathbb{R} \\ \vec{v}_i, \vec{u}_i \neq \vec{0} \\ \vec{v}_i \times \vec{u}_i \neq \vec{0} \end{array}$$



Let the affine transf. be $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{D}$

with M invertible.

$$f(l_1) = M\cdot(t\cdot\vec{v}_1 + d_1) + \vec{D}$$

$$= t\cdot\underbrace{(M\cdot\vec{v}_1)}_{\vec{v}_1'} + \underbrace{(M\cdot\vec{d}_1 + \vec{D})}_{\vec{d}_1'}$$

[$\vec{v}_1' \neq \vec{0}$ as M is invertible and $\vec{v}_1 \neq \vec{0}$]

This is a parametrisation $t\cdot\vec{v}_1' + \vec{d}_1'$ of a line, which proves first ⊕.

$$f(p_1) = M\cdot(s\cdot\vec{v}_1 + t\cdot\vec{u}_1 + \vec{d}_1) + \vec{D}$$

$$= s\cdot\underbrace{(M\cdot\vec{v}_1)}_{\vec{v}_1'} + t\cdot\underbrace{(M\cdot\vec{u}_1)}_{\vec{u}_1'} + \underbrace{M\cdot\vec{d}_1 + \vec{D}}_{\vec{d}_1'}$$

[Again, $\vec{v}_1', \vec{u}_1' \neq \vec{0}$]

This is a parametrisation of a plane <u>iff</u> $\vec{v}_1' \times \vec{u}_1' \neq \vec{0}$.

From linear algebra we have (long and tedious proof omitted) the following identity for any invertible 3×3 matrix M:

$$(M\cdot\vec{v})\times(M\cdot\vec{w}) = \det(M)\cdot(M^{-1})^T\cdot(\vec{v}\times\vec{w})$$

From $\vec{v}_1 \times \vec{u}_1 \neq \vec{0}$, this identity gives the <u>iff</u>, and second ⊕ is proved.

For the third ⊕ :

l_1 and l_2 parallel $\Leftrightarrow$ $\vec{v}_1 = c \cdot \vec{v}_2$ $\overset{[\Leftarrow \text{ via } M \text{ invertible}]}{\Leftrightarrow}$

$M \cdot \vec{v}_1 = c \cdot M \cdot \vec{v}_2$ $\Leftrightarrow$ images of l_1 and l_2 are parallel.

$$p_1 \text{ and } p_2 \text{ parallel} \Leftrightarrow \underbrace{\vec{v}_1 \times \vec{u}_1}_{\text{normal vector for } p_1} = c \cdot \big(\underbrace{\vec{v}_2 \times \vec{u}_2}_{\text{same for } p_2}\big)$$

$$\Leftrightarrow \underbrace{(M \cdot \vec{v}_1) \times (M \cdot \vec{u}_1)}_{\text{normal vector for } f(p_1)} = c \cdot \big(\underbrace{(M \cdot \vec{v}_2) \times (M \cdot \vec{u}_2)}_{\text{same for } f(p_2)}\big)$$

using identity at bottom of page ④

$\Leftrightarrow$ images of p_1 and p_2 are parallel.

For the fourth ⊕ :

$t_1 \cdot \vec{v}_1 + d_1 = t_2 \cdot \vec{v}_2 + \vec{d}_2$ $\overset{[\Leftarrow \text{ because } M \text{ invertible}]}{\Leftrightarrow}$

$t_1 \cdot (M \cdot \vec{v}_1) + M \cdot \vec{d}_1 + \vec{D} = t_2 \cdot (M \cdot \vec{v}_2) + M \cdot \vec{d}_2 + \vec{D}$

shows: l_1 and l_2 intersect $\Leftrightarrow$ images intersect.

Same proof applies to the planes $\square$

Thm. Any 2D affine transformation can be written as

$$g_4 \circ g_3 \circ g_2 \circ g_1$$

where :

g_1, g_3 are rotations about origin

g_2 is a scaling

g_4 is a translation

Thm: Any 3D affine transformation can be written as

$$g_4 \circ g_3 \circ g_2 \circ g_1$$

where

g_1, g_3 are rotations about a radial axis

g_2 is a scaling

g_4 is a translation

So these transformations <u>span</u> the entire set of affine transformations

Proof of first thm : See pages 193-5 in text book. Is curriculum.

Proof of second thm : Omitted, as it requires too advanced linear algebra (proof is on p. 235 and 233, but is not exam curriculum).

$\boxed{\text{Def.}}$

Euclidian affine transformations are those preserving distances:

$$|f(\vec{x}_1)\, f(\vec{x}_2)| = |\vec{x}_1\, \vec{x}_2| \quad \text{for all } \vec{x}_1, \vec{x}_2 \in \mathbb{R}^3$$

[or $\mathbb{R}^2$]

Thm (not in book)

$f(\vec{x}) = M\cdot\vec{x} + \vec{D}$ Euclidian

$\Updownarrow$

$\det(M) = \pm 1$ Proof: omitted (moderate linear algebra)

$\boxed{\text{Def:}}$ $+1$: orientation preserving
-1 : —"— reversing

$\boxed{\text{Def}}$ <u>Rigid</u> affine transf. is one which is Euclidean and has det. = +1.

<u>Thm</u> The affine transformations contains the following subclasses, each generated/spanned by the stated transformation types:

	Rigid	Euclidian	All affine
Maps $\vec{o}$ to $\vec{o}$	R	R + M	R + S
Maps $\vec{o}$ anywhere	R + T	R + M + T	R + S + T

R = rotation (radial) [axis through $\vec{o}$]

M = mirror (radial) [plane — " —]

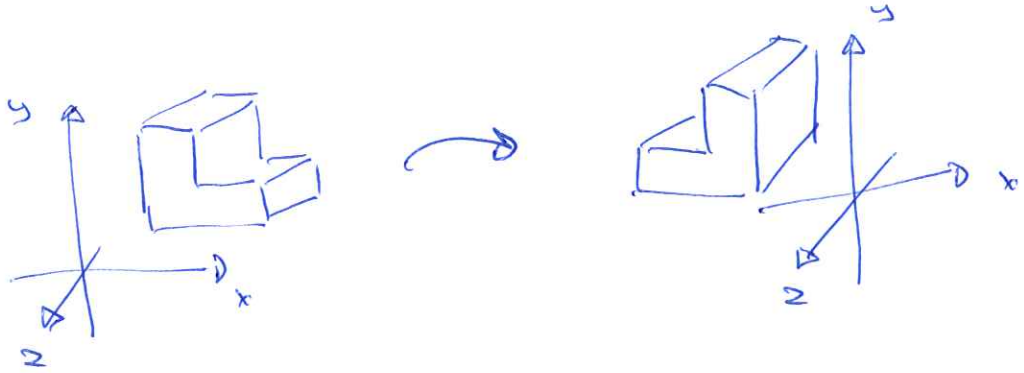
S = scaling (radial) [plane(s) — " —]

T = transformation

Proof: most appears in book, but not curriculum.

Note: mirror [a.k.a. reflection] is a special case of scaling:

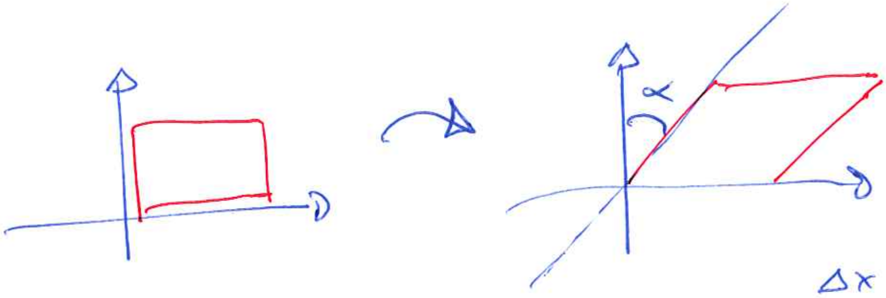
$$M \cdot \vec{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{x}$$

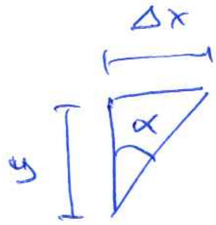


Reflection in y-z plane. For other planes, use "the trick".

Shear

2D :





$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \Delta x \\ y \end{pmatrix}$$

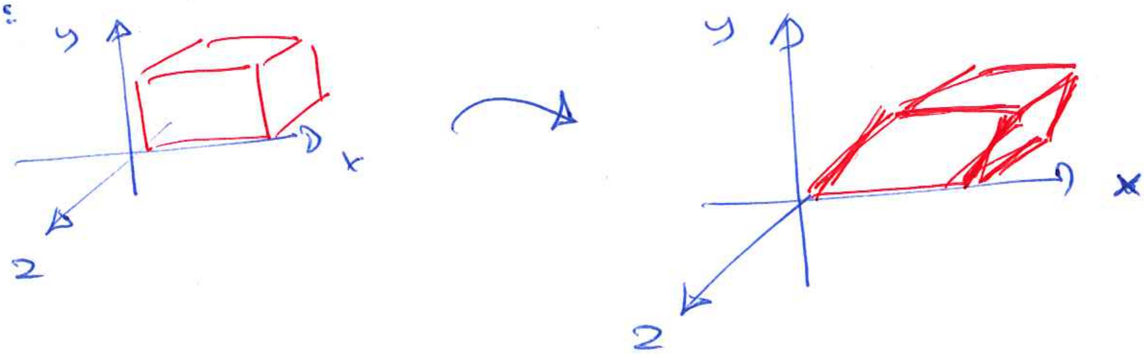
$$\frac{\Delta x}{y} = \tan(\alpha).$$

$$= \begin{pmatrix} x + \tan(\alpha) \cdot y \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \tan(\alpha) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Above : <u>line of shear</u> is x-axis.
For other axes : use "the trick"

3D :



$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \tan(\alpha) \cdot y \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \tan(\alpha) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

<u>Above</u> : <u>plane of shear</u> is x-z plane and <u>line of shear</u> is x-axis. For other choices, use "the trick".
$\underbrace{}_{\text{in this plane}}$