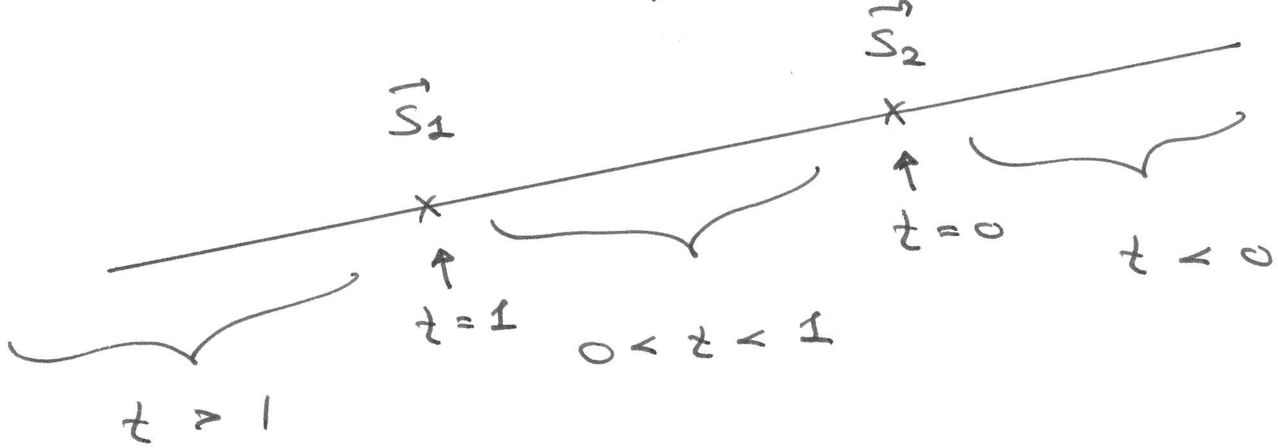
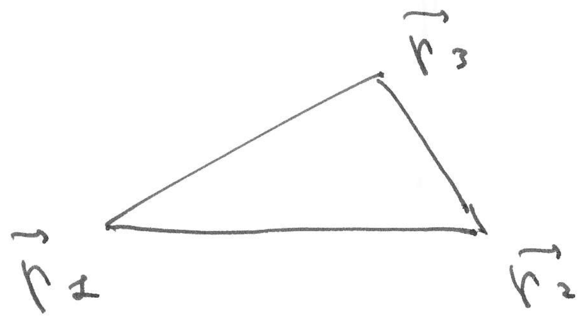
Barycentric Coordinates

Recall: for two points $\vec{s_1}, \vec{s_2} \in \mathbb{R}^3$ (or, more generally, $\mathbb{R}^k$), the parametrization of the line through $\vec{s_1}$ and $\vec{s_2}$ is

$$\vec{l}(t) = t \cdot \vec{s_1} + (1-t) \cdot \vec{s_2}, \quad t \in \mathbb{R}.$$
$$= t \cdot (\vec{s_1} - \vec{s_2}) + \vec{s_2}$$



Now, let $\vec{r_1}, \vec{r_2}, \vec{r_3} \in \mathbb{R}^3$ (or $\mathbb{R}^2$) be the three corners of a triangle T



Assume T is non-degenerate (not all three points lie on same line), so it defines a plane.

Consider an expression of the form

$$\circledast \qquad \vec{x} = c_1 \cdot \vec{p}_1 + c_2 \cdot \vec{p}_2 + c_3 \cdot \vec{p}_3$$

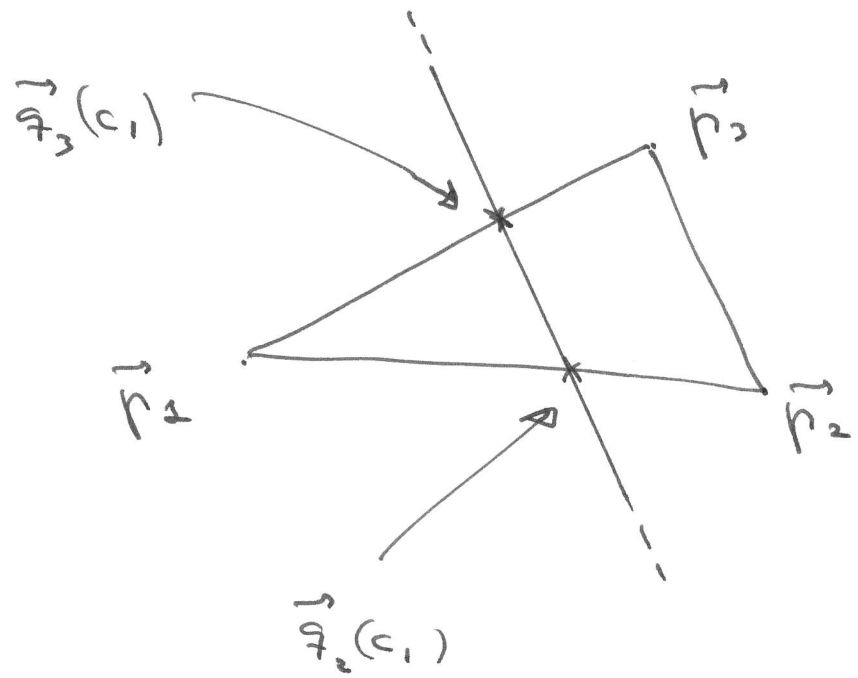
where $c_1, c_2, c_3 \in \mathbb{R}$ and $\sum_{i=1}^{3} c_i = 1$

Call the following a decomposition of $\circledast$ according to $\vec{p}_1$:

1) Define the two points

$$\vec{q}_2(c_1) = c_1 \cdot \vec{p}_1 + (1 - c_1) \cdot \vec{p}_2$$

$$\vec{q}_3(c_1) = c_1 \cdot \vec{p}_1 + (1 - c_1) \cdot \vec{p}_3$$



2) Rewrite ⊛ as follows:

$$\vec{x} = c_1 \cdot \vec{p}_1 + c_2 \cdot \vec{p}_2 + c_3 \cdot \vec{p}_3$$

$$= c_1 \cdot \vec{p}_1 + (c_2 + c_3) \cdot \left(\frac{c_2}{c_2+c_3} \cdot \vec{p}_2 + \frac{c_3}{c_2+c_3} \cdot \vec{p}_3 \right)$$

$$= \frac{c_2}{c_2+c_3} \cdot \vec{q}_2(c_1) + \frac{c_3}{c_2+c_3} \cdot \vec{q}_3(c_1)$$

[use def. of $\vec{q}_2(c_1)$, $\vec{q}_3(c_1)$ and
$c_2 + c_3 = 1 - c_1$ (as $\sum_{i=1}^{3} c_i = 1$)]

$$= t \cdot \vec{q}_2(c_1) + (1-t) \cdot \vec{q}_3(c_1)$$

for $t = \dfrac{c_2}{c_2+c_3}$

I.e., we see that $\vec{x}$ is on the line through $\vec{q}_2(c_1)$ and $\vec{q}_3(c_1)$, a line parallel to the line through $\vec{p}_2$ and $\vec{p}_3$.

Actually, we can see more:

i) For a given $\vec{x}$, the values c_1, c_2, c_3 must be <u>unique</u>. This is because:

a) different values of c_1 would put $\vec{x}$ on different lines parallel to line through $\vec{p_2}$ and $\vec{p_3}$ (not possible for a single point $\vec{x}$)

b) for a given c_1 value (given value of $1 - c_1 = c_2 + c_3$) hence, the value of c_2 determines t, hence determines <u>where</u> on this line $\vec{x}$ is, so $\vec{x}$ can not have two different c_2 values

c) Given c_1 and c_2, $c_3 = 1 - (c_1 + c_2)$ is fixed.

ii) <u>Any</u> point $\vec{x}$ in the plane defined by the triangle T has such an expression

$$\vec{x} = c_1 \cdot \vec{p_1} + c_2 \cdot \vec{p_2} + c_3 \cdot \vec{p_3}$$

(i.e. $c_1, c_2, c_3 \in \mathbb{R}$ exist such that $\vec{x}$ can be expressed in this form).

iii) $\vec{x}$ lies inside T

$$\Updownarrow$$

$$0 \leq c_1 \leq 1 \quad \text{and} \quad 0 \leq t \leq 1$$

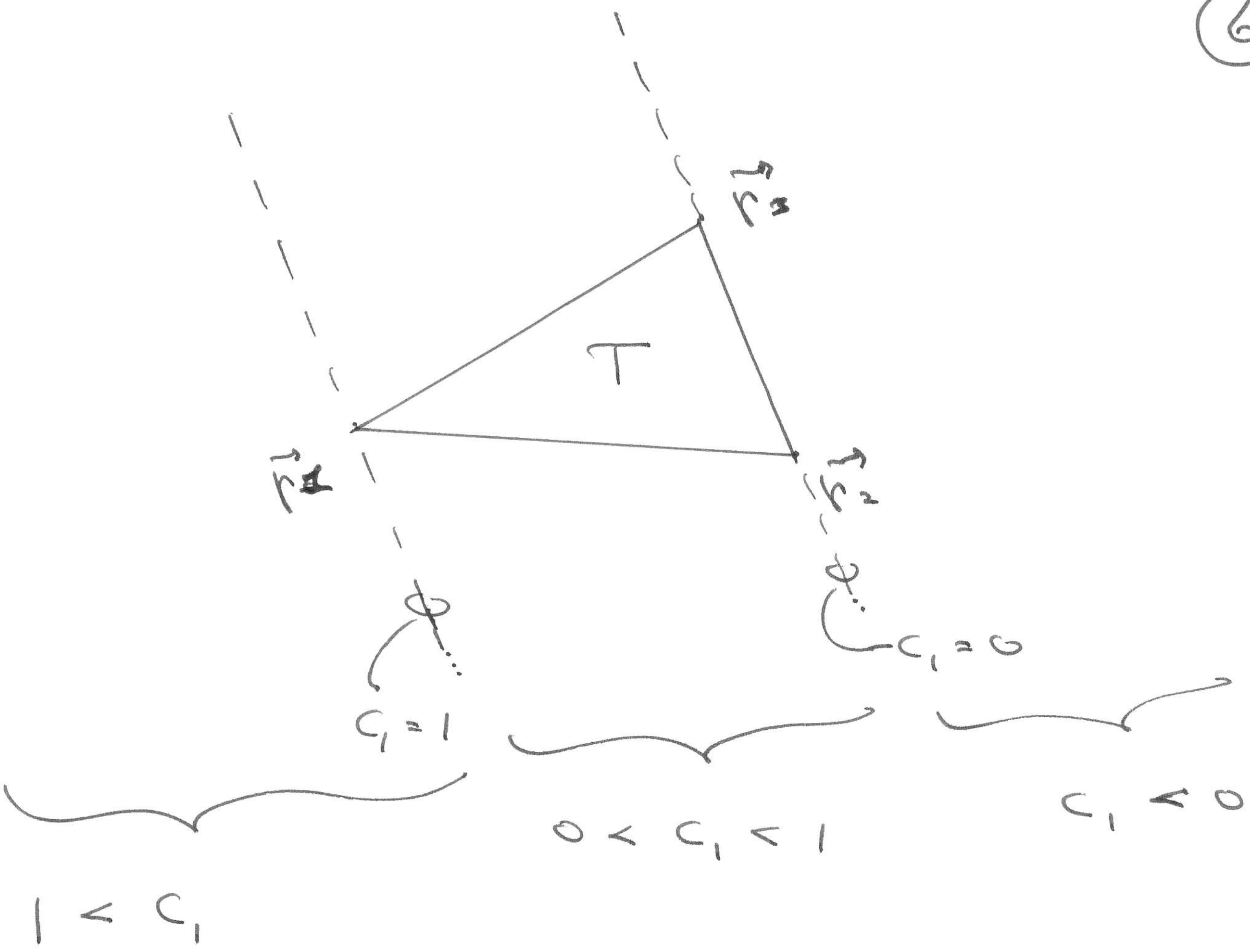
$$\Updownarrow \longleftarrow \left[\begin{array}{l} \text{as } c_1 \leq 1 \Rightarrow c_2 + c_3 = 1 - c_1 \geq 0 \\ \text{and } t = \dfrac{c_2}{c_2 + c_3} \end{array} \right]$$

$$0 \leq c_1, c_2, c_3 \leq 1$$

$$\Updownarrow \longleftarrow \left[\text{as } \sum_{i=1}^{3} c_i = 1 \right]$$

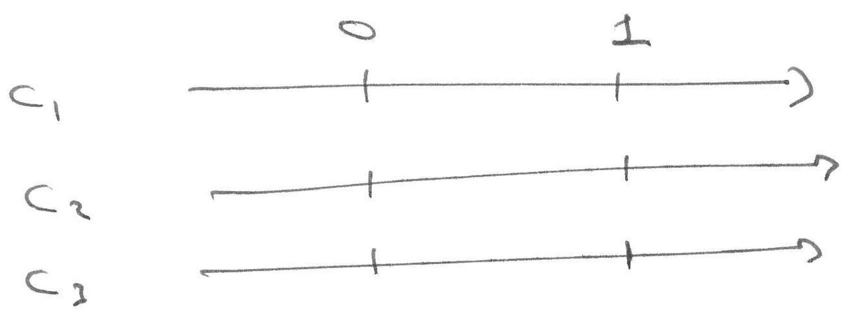
$$\left(0 \leq c_1, c_2, c_3 \right)$$

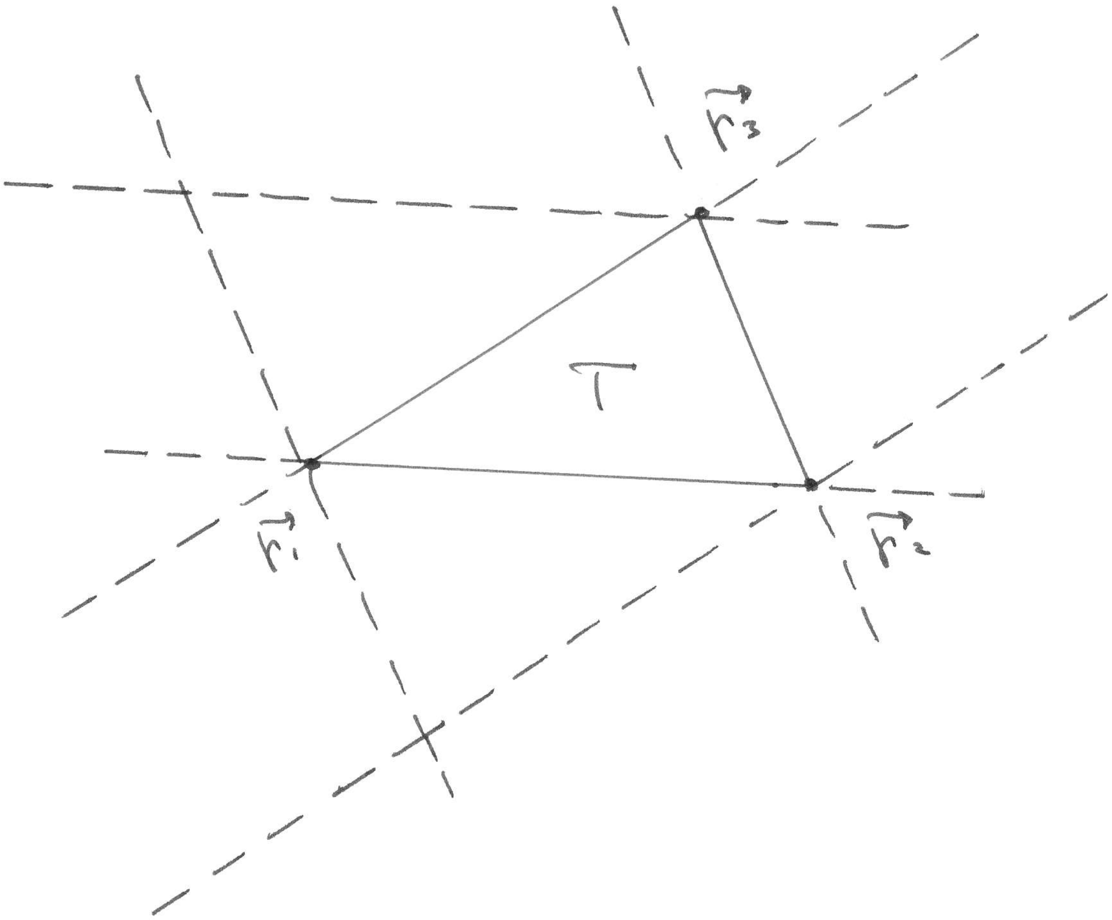
iv) More generally, we know from the above decomposition of $\circledast$ according to $\vec{p}_1$ that the value of c_1 and $\vec{x}$'s position in T's plane is related as follows:



(see def. of $\vec{q}_2(c_1)$ and $\vec{q}_3(c_1)$ and page 1).

By considering similar decompositions of ⊛ [which is symmetric in c_1, c_2, c_3] according to $\vec{r}_2$ and $\vec{r}_3$, we get (naively up to $3 \cdot 3 \cdot 3 = 27$) regions of T's plane characterized by c_1, c_2, c_3's possible relations to 0 and 1.





Of the $3^3 = 27$ possibilities for c_1, c_2, c_3, only 16 can fulfill the $\sum_{i=1}^{3} c_i = 1$ requirement. The figure above shows these 16 possible ones (or rather, their corresponding regions).

The most interesting is the one

$$0 \le c_1, c_2, c_3 \le 1,$$

corresponding to T.

Because of i) [<u>uniqueness</u> of c_1, c_2, c_3] and ii) [<u>existence</u> of c_1, c_2, c_3], we call c_1, c_2, c_3 in

$$(*) \qquad \vec{x} = c_1 \cdot \vec{r_1} + c_2 \vec{r_2} + c_3 \vec{r_3}$$

the barycentric <u>coordinates</u> of $\vec{x}$, for points $\vec{x}$ in T's plane.

For a given $\vec{x}$ (and given T, i.e. given $\vec{r_1}, \vec{r_2}, \vec{r_3}$), they can be found e.g. by solving the coordinate equations of (*) plus the equation $1 = c_1 + c_2 + c_3$

That is three equations in three unknowns when in $\mathbb{R}^2$ [where T's plane is all of $\mathbb{R}$], and four equations in three unknowns in $\mathbb{R}^3$ [where T's plane is not all of $\mathbb{R}^3$, so not all $\vec{x} \in \mathbb{R}$ should have a solution, only those in T's plane].