HARMONIC MORPHISMS, HERMITIAN STRUCTURES AND SYMMETRIC SPACES

In this text we refer to the following papers by capital letters as indicated.

[A] M. Svensson, *On holomorphic harmonic morphisms*, Manuscripta Math. **107** (2002), 1–13.
[B] M. Svensson, *Harmonic morphisms from even-dimensional hyperbolic spaces*, Math. Scand. **92** (2003), 246–260.
[C] M. Svensson, *Holomorphic foliations, harmonic morphisms and the Walczak formula*, J. London Math. Soc. **68** (2003), 781–794.
[D] M. Svensson, *Harmonic morphisms in Hermitian geometry*, J. Reine Angew. Math. (to appear).
[E] S. Gudmundsson and M. Svensson, *Harmonic morphisms from the Grassmannians and their non-compact duals*, preprint, Lund University (2003).
[F] S. Gudmundsson and M. Svensson, *Harmonic morphisms from the compact simple Lie groups and their non-compact duals*, preprint, Lund University (2003).

Numerical references refer to entries in the bibliography.

1. Summary

The purpose of this text is to give an accessible introduction to the theory of harmonic morphisms and to put the methods and results of the papers [A,B,C,D,E,F] into a context which we share with contemporary differential geometers.

Section 2 gives an account of the historical origin of harmonic morphisms and the birth of the theory in modern differential geometry. Section 3 points to an important and closely related theory, the theory of foliations, and its connection with harmonic morphisms.

In Section 4 we move into complex and almost complex geometry and demonstrate how the methods here have been used by geometers to gain a better understanding of the construction and the behaviour of harmonic morphisms. A discussion of the important case of 4-dimensional Einstein manifolds leads us into some of the results of [D].

In Section 5 we continue our discourse on harmonic morphisms and complex structures with an emphasis on Kähler structures. We briefly describe the content of [A] and [C] which deal with holomorphic harmonic morphisms and holomorphic foliations in Kähler geometry, and the content of [D] which deals with some Hodge theory and flag manifolds.

In Section 6 we briefly digress to so-called harmonic morphisms of *warped product type*, and show how methods of [C] can be used to extract information about these.

Finally, in Section 7 we turn our attention to harmonic morphisms from symmetric spaces. We show how in [B] we construct harmonic morphisms from real hyperbolic spaces by retaining some of the methods of complex geometry, and how in [E] we generalize these results to spaces of higher rank and exploit the duality between these spaces and the Grassmannians. Similarly, we use in [F] the duality of Riemannian symmetric spaces of type II and type IV to generate harmonic morphisms from such manifolds.

2. The beginnings of harmonic morphisms

2.1. **Historical background.** The story of harmonic morphisms begins in 1848 with the article [31] by Jacobi[1]. In this article he studied the problem of constructing complex-valued harmonic functions in open subsets of $\mathbb{R}^3$ from holomorphic functions in open subsets of $\mathbb{C}$.

Jacobi's idea was to find conditions for a function $\varphi = \varphi(x)$, defined in some open subset of $\mathbb{R}^3$ and taking values in $\mathbb{C}$, to have the property that *for any local holomorphic function $f = f(z)$ in $\mathbb{C}$, the composition $f \circ \varphi$ is harmonic.* To follow Jacobi, we need only carry out a simple calculation:

$$\begin{aligned}\Delta(f\circ\varphi) =& \frac{\partial^2(f\circ\varphi)}{\partial {x_1}^2} + \frac{\partial^2(f\circ\varphi)}{\partial {x_2}^2} + \frac{\partial^2(f\circ\varphi)}{\partial {x_3}^2} \\ =& \frac{\partial f}{\partial z}\Delta\varphi + \frac{\partial^2 f}{\partial z^2}\Big((\frac{\partial\varphi}{\partial x_1})^2 + (\frac{\partial\varphi}{\partial x_2})^2 + (\frac{\partial\varphi}{\partial x_3})^2 \Big)\end{aligned}$$

As f is an arbitrary holomorphic function, we conclude, as Jacobi did, that φ has the desired property if and only if

(i) the map φ is itself harmonic, i.e.

$$\Delta\varphi = 0,$$

(ii) the map φ satisfies the equation

$$(\frac{\partial\varphi}{\partial x_1})^2 + (\frac{\partial\varphi}{\partial x_2})^2 + (\frac{\partial\varphi}{\partial x_3})^2 = 0. \tag{2.1.1}$$

If we write $\varphi = u + iv$, then

$$(\frac{\partial\varphi}{\partial x_1})^2 + (\frac{\partial\varphi}{\partial x_2})^2 + (\frac{\partial\varphi}{\partial x_3})^2 = |\operatorname{grad} u|^2 - |\operatorname{grad} v|^2 + 2i\langle \operatorname{grad} u, \operatorname{grad} v\rangle.$$

Hence, equation (2.1.1) is equivalent to

$$|\operatorname{grad} u|^2 = |\operatorname{grad} v|^2, \quad \langle \operatorname{grad} u, \operatorname{grad} v\rangle = 0.$$

[1] Jacobi was born in 1804 in Potsdam, Prussia, and studied at the University of Berlin. After a long period in Königsberg, he returned to Berlin in 1844, where he remained until his death in 1851.

These conditions imposed on φ, harmonicity and equation (2.1.1), form an over-determined non-linear system of equations; nevertheless, Jacobi contrived a way to construct solutions. Take a function $F = F(x, z)$ from some open subset of $\mathbb{R}^3 \times \mathbb{C}$ into $\mathbb{C}$ which satisfies

(i) $z \mapsto F(x, z)$ is holomorphic for each fixed x, and
(ii) $x \mapsto F(x, z)$ is harmonic and satisfies (2.1.1) for each fixed z.

It is then easy to see that any smooth local function $z = \varphi(x)$ implicitly defined by the equation

$$F(x, z) = 0$$

has the desired properties. For example, we can choose F in the form

$$F(x, z) = \xi_1(z)x_1 + \xi_2(z)x_2 + \xi_3(z)x_3,$$

where (ξ_1, ξ_2, ξ_3) is a holomorphic function from some open subset of $\mathbb{C}$ into $\mathbb{C}^3$, satisfying

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 0.$$

A range of candidates for such triples are given by

$$(\xi_1, \xi_2, \xi_3) = f(1 - g^2, i(1 + g^2), 2g), \tag{2.1.2}$$

where f and g are two holomorphic functions. Note the similarity between (2.1.2) and the famous Weierstrass representation of minimal surfaces. According to this, for any pair of non-zero holomorphic functions f and g in some open subset $U \subset \mathbb{C}$ and $z_0 \in U$, the mapping

$$X(z) = \text{ Re } \int_{z_0}^{z} f(w)(1 - g(w)^2, i(1 + g(w)^2), 2g(w))\mathrm{d}w$$

parametrizes a minimal surface in $\mathbb{R}^3$. Conversely, any minimal surface in $\mathbb{R}^3$ can locally be represented in this way. In later sections we will return to the concept of minimality and discuss its relevance to the theory of harmonic morphisms.

2.2. **Harmonic morphisms.** Let (M, g) be a Riemannian manifold, i.e. M is a smooth manifold and g is a smooth Riemannian metric on M. As is well known, a local function $f : U \subset M \to \mathbb{R}$ is said to be *harmonic* if it satisfies the Laplace equation in U:

$$\Delta f = \operatorname{div} \operatorname{grad} f = 0.$$

A *harmonic morphism* is a map $\varphi : (M, g) \to (N, h)$ between two Riemannian manifolds with the following property: for any local harmonic function $f : U \subset N \to \mathbb{R}$ on some open subset U of N, the composition $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is again a local harmonic function on M. Thus, a harmonic morphism can be used to construct local harmonic functions on (M, g) from local harmonic functions on (N, h). In the complex plane $\mathbb{C}$, any local harmonic function is the real part of a holomorphic function. Thus, the maps that Jacobi investigated in [31] are exactly the local harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$.

After Jacobi's article, it was to be more than a century before the idea of constructing harmonic functions in this way was considered again. In their article [12] from 1965, potential theorists Constantinescu and Cornea considered so-called *Brelot harmonic spaces*, i.e. topological spaces on which a *harmonic sheaf* has been defined. This is, by definition, a subsheaf of the sheaf of continuous functions, which satisfy two additional axioms. These are both well-known properties for harmonic functions in the Euclidean space. The first axiom is the solubility of the Dirichlet problem, the second axiom a monotone convergence requirement. The elements of this sheaf are called *harmonic functions*. For example, a Riemannian manifold endowed with its usual sheaf of local harmonic functions, is a Brelot harmonic space.

In this context, Constantinescu and Cornea studied maps which preserve the harmonic sheaves, i.e. sheaf morphisms between Brelot harmonic spaces. Somewhat confusingly, they called them *harmonic maps*. Their aim was to generalize known results on Riemann surfaces to these spaces, with the 'harmonic maps' replacing the holomorphic maps between surfaces.

The current terminology, 'harmonic morphisms', was introduced by Fuglede in 1978 in the article [17]. Fuglede studied the geometric properties of the harmonic maps introduced by Constantinescu and Cornea in the specific context of Riemannian manifolds. At about the same time, but independently from Fuglede, Ishihara carried out similar investigations in [30]. They both obtained the same celebrated characterization of harmonic morphisms between Riemannian manifolds.

Theorem 2.2.1. [17, 30] *A smooth map $\varphi : (M, g) \to (N, h)$ between two Riemannian manifolds is a harmonic morphism if and only if φ is both a* harmonic map *and* horizontally (weakly) conformal.

Here, φ is said to be a *harmonic map* if it satisfies the semi-linear elliptic equation

$$\tau(\varphi) = \operatorname{trace} \nabla \mathrm{d}\varphi = 0,$$

where the first equality defines the so-called *tension field* of φ. Here $\mathrm{d}\varphi$ is thought of as a section of the vector bundle $T^*M \otimes \varphi^{-1}(TN)$ with its natural connection ∇ induced from the Levi-Civita connections on M and N. Equivalently, φ is harmonic if and only if, for any relatively compact domain $D \subset M$ and any smooth variation of φ which is compactly supported in D, φ is a critical point of the *energy functional* in D:

$$E^D(\varphi) = \frac{1}{2} \int_D |\mathrm{d}\varphi|^2 \nu.$$

Harmonic maps were first introduced in 1954 by Fuller in an attempt to find energy minimizing maps in given homotopy classes of maps between Riemannian manifolds, see [20]. In their celebrated article [16] from 1964, Eells and Sampson conducted a thorough investigation of harmonic maps, thus laying the ground for what is now one of the most prolific areas in differential geometry.

To explain *horizontal (weak) conformality*, also called *semi-conformality*, introduce at each $x \in M$ the two complementary subspaces of T_xM:

$$\mathcal{V}_x = \ker \mathrm{d}\varphi_x, \quad \mathcal{H}_x = \mathcal{V}_x^\perp \qquad (x \in M).$$

We call $\mathcal{V}_x$ the *vertical* subspace and $\mathcal{H}_x$ the *horizontal* subspace. The map φ is said to be horizontally (weakly) conformal if for every $x \in M$ either $\mathcal{H}_x = 0$, i.e. $\mathrm{d}\varphi_x = 0$, or

$$\mathrm{d}\varphi_x : \mathcal{H}_x \to T_{\varphi(x)}N$$

is surjective and conformal. This means that there is a positive function λ on the set $\{x \in M \mid \mathrm{d}\varphi_x \neq 0\}$ such that

$$h_{\varphi(x)}(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda(x)^2 g_x(X, Y) \qquad (X, Y \in \mathcal{H}_x).$$

The function λ is called the *dilation* of φ; we have $n\lambda^2 = |\mathrm{d}\varphi|^2$, where n is the dimension of N. Thus λ extends continuously to all of M with λ^2 smooth and $\lambda(x) = 0$ if and only if $\mathrm{d}\varphi_x = 0$. It is easy to see that the map φ is horizontally (weakly) conformal with dilation λ if and only if for every $x \in M$, the adjoint $\mathrm{d}\varphi^*$ is conformal with conformal factor λ^2:

$$g_x(\mathrm{d}\varphi_x^*(X), \mathrm{d}\varphi_x^*(Y)) = \lambda(x)^2 h_{\varphi(x)}(X, Y) \qquad (X, Y \in T_{\varphi(x)}N).$$

A point $x \in M$ is said to be a *critical point* if $\mathrm{d}\varphi_x = 0$, or equivalently, $\lambda(x) = 0$. For a non-constant harmonic map, and, in particular, a non-constant harmonic morphism, the set of critical points is nowhere dense; it is in fact a *polar set*, see [17]. A point in M which is not a critical point is said to be a *regular point*.

The characterization of Fuglede and Ishihara tells us that the harmonic morphisms form a very restricted subclass of the harmonic maps.

Returning to Jacobi and looking at a harmonic morphism φ from some open subset of $\mathbb{R}^3$ to $\mathbb{C}$, we see that φ is a harmonic map if and only if φ is a harmonic function, i.e.

$$\Delta\varphi = 0.$$

Moreover, it is easy to see that $\varphi = u + iv$ is horizontally conformal if and only if $\mathrm{d}\varphi \circ \mathrm{d}\varphi^*$ is everywhere a multiple of the identity, i.e. if and only if

$$|\operatorname{grad} u|^2 = |\operatorname{grad} v|^2, \quad \langle \operatorname{grad} u, \operatorname{grad} v \rangle = 0.$$

We conclude that equation (2.1.1) is precisely the condition for horizontal conformality. Hence Fuglede's and Ishihara's characterization of harmonic morphisms generalizes Jacobi's result.

3. Foliations and harmonic morphisms

3.1. **Fibres of horizontally conformal maps.** Geometers have been familiar with the connection between harmonicity and minimal submanifolds for more than a century. Weierstrass himself showed that a conformally immersed surface in $\mathbb{R}^3$ is minimal if and only if the immersion is harmonic. This was later generalized to arbitrary conformally immersed surfaces by Eells and Sampson in [16].

In the same article, the authors proved another (now well-known) result which also relates harmonicity to minimality. Recall that a *Riemannian submersion* is a map $\psi : (M, g) \to (N, h)$ between two Riemannian manifolds with the property that for any $x \in M$, $\mathrm{d}\psi_x$ restricted to the orthogonal complement of its kernel in T_xM, is surjective onto $T_{\psi(x)}N$ and isometric. Thus, a Riemannian submersion is simply a horizontally conformal map with dilation constantly equal to 1.

Theorem 3.1.1. [16] *A Riemannian submersion is a harmonic map if and only if it has minimal fibres.*

We conclude that *a horizontally conformal map with constant dilation is a harmonic map, and so a harmonic morphism, if and only if it has minimal fibres.*

The study of horizontally conformal maps as initiated by the study of harmonic morphisms has made this conclusion even more precise, as we now describe. Assume that $\varphi : (M^m, g) \to (N^n, h)$ is a horizontally conformal submersion. Then φ defines two orthogonal complementary distributions $\mathcal{V}$ and $\mathcal{H}$ on M, called the *vertical* and the *horizontal* distribution, respectively. The vertical distribution is of course *involutive* or *integrable*, i.e. closed under the Lie bracket, and everywhere tangent to the fibres of φ. The distribution $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$. Let λ be the dilation of φ and $B^{\mathcal{V}}$ be the second fundamental form of the vertical distribution. The *fundamental equation* of φ expresses the relationship between the tension field, the dilation and the mean curvature vector of the fibres of φ:

$$\tau(\varphi) = (2 - n)\mathrm{d}\varphi(\operatorname{grad}\log\lambda) - \mathrm{d}\varphi(\operatorname{trace} B^{\mathcal{V}}).$$

This equation is implicit in the paper [6] by Baird and Eells, and immediately leads to the following result. Recall that the regular points are the points where $\mathrm{d}\varphi \neq 0$, or equivalently $\lambda \neq 0$.

Theorem 3.1.2. [6] *Let $\varphi : (M^m, g) \to (N^n, h)$ be a non-constant horizontally (weakly) conformal map.*
(a) *If $n = 2$, then φ is harmonic if and only if the fibres of φ are minimal at regular points.*
(b) *If $n \neq 2$, then any two of the following imply the third:*
 (i) *φ is harmonic,*
 (ii) *$\operatorname{grad}\log\lambda \in \mathcal{V}$ at regular points,*
 (iii) *the fibres of φ are minimal at regular points.*

If $\operatorname{grad}\log\lambda \in \mathcal{V}$ at regular points then φ is said to be *horizontally homothetic*. This condition is surprisingly restrictive; Fuglede showed in [18] that any horizontally homothetic harmonic morphism is submersive, i.e. every point of M is actually regular for such a map.

Because of Theorem 3.1.2, the study of harmonic morphisms is closely related to the study of minimal submanifolds. For instance, Gudmundsson used harmonic morphisms to construct minimal submanifolds of real hyperbolic spaces in [25].

Theorem 3.1.2 also shows that the condition for a map into a surface to be a harmonic morphism is independent of conformal changes of the metric on the codomain. In particular, the concept of *a harmonic morphism into a Riemann surface* is well defined.

3.2. **Foliations which produce harmonic morphisms.** Conversely, we may ask when can a foliation be, at least locally, generated by submersive harmonic morphisms. The study of such foliations was initiated by Wood in [45] and later continued by Pantilie in [37, 38, 39]. Let us take some time here to introduce some important notation and results.

Let (M^m, g) be a Riemannian manifold endowed with a foliation $\mathcal{F}$; let $\mathcal{V}$ denote the involutive distribution associated with the foliation and let $\mathcal{H}$ denote its orthogonal complement. At each point $x \in M$, $\mathcal{V}_x$ is then the tangent space of the leaf of $\mathcal{F}$ through x. We say that $\mathcal{V}$ is a *conformal* distribution, or equivalently, $\mathcal{F}$ is a *conformal foliation*, if Lie transport of vectors in $\mathcal{H}$ in directions tangent to $\mathcal{F}$ is a conformal operation:

$$\mathcal{L}_V g\big|_{\mathcal{H}\times\mathcal{H}} = \eta(V) g\big|_{\mathcal{H}\times\mathcal{H}} \qquad (V \in \Gamma(\mathcal{V})) \tag{3.2.1}$$

for some 1-form η on M. A positive function λ, defined on some open subset $U \subset M$, with the property that

$$\eta(V) = -2\mathrm{d}\log\lambda(V) \qquad (V \in \mathcal{V}\big|_U)$$

is said to be a *local dilation* of $\mathcal{F}$. One may prove that local dilations always exist, see e.g. [9, Chapter 2]. That $\mathcal{F}$ is conformal is equivalent to $\mathcal{H}$ being an *umbilical distribution*, i.e. that

$$B^{\mathcal{H}} = \mathcal{V}(\operatorname{grad}\log\lambda) \otimes g\big|_{\mathcal{H}\times\mathcal{H}}$$

for any local dilation λ. Here we also use the symbol $\mathcal{V}$ to denote orthogonal projection in TM onto $\mathcal{V}$.

It is easy to see that the fibres of a submersive harmonic morphism constitute a conformal foliation, and the dilation of the map serves as a global dilation of the foliation. That a general foliation $\mathcal{F}$ is conformal is thus a necessary condition for the foliation to be locally generated by harmonic morphisms. If $\mathcal{F}$ is minimal and of codimension 2, it is also a sufficient condition, see [45]. The case when the codimension of $\mathcal{F}$ is not equal to 2 is far more complicated. Bryant found the following characterization, the proof of which was later simplified by Pantilie in [37].

Theorem 3.2.1. [11] *Let $\mathcal{F}$ be a conformal foliation of a Riemannian manifold (M, g) of codimension $n \neq 2$. Then the leaves of $\mathcal{F}$ are locally the fibres of harmonic morphisms if and only if*

$$(n-2)\operatorname{trace} B^{\mathcal{H}} - n \operatorname{trace} B^{\mathcal{V}}$$

is locally a gradient vector field.

From a close investigation of this property, Pantilie has found an abundance of conformal foliations, with codimensions not equal to 2, which are locally the fibres of harmonic morphisms (*op.cit.*). These include, for example, foliations locally generated by Killing fields with leaves of dimension 1 or with integrable orthogonal distribution and foliations generated by the action of a unimodular subgroup of the isometry group.

The study of harmonic morphisms has brought about a deeper understanding of minimal and conformal foliations and of how to construct them. Later we shall see how the study of holomorphic harmonic morphisms has also deepened our knowledge of holomorphic conformal foliations.

4. Almost Hermitian structures

4.1. **Harmonic and holomorphic maps.** It was proved already in Eells and Sampson's article [16] that a holomorphic map between two Kähler manifolds is a harmonic map. This result was later generalized by Lichnerowicz, who weakened the conditions substantially.

Theorem 4.1.1. [35] *Any holomorphic map from a cosymplectic manifold to a (1,2)-symplectic one is harmonic.*

Recall that an almost Hermitian manifold (M, g, J) with Kähler form ω, is said to be *cosymplectic* if $\mathrm{d}^*\omega = 0$ or, equivalently, $\operatorname{div} J = 0$. It is said to be *(1,2)-symplectic* if the (1,2)-part of $\mathrm{d}\omega$ vanishes.

Gudmundsson and Wood clarified this result in [29] by showing that, for a holomorphic map φ from an almost Hermitian manifold into a (1,2)-symplectic one, the tension field is given by

$$\tau(\varphi) = -\mathrm{d}\varphi(J \operatorname{div} J),$$

where J denotes the almost Hermitian structure on the codomain. This shows that if φ is holomorphic, it is enough to require that $\operatorname{div} J \in \ker \mathrm{d}\varphi$ to guarantee the harmonicity of φ.

As for harmonic morphisms, Fuglede showed in [17] that any holomorphic map from a Kähler manifold with values in a Riemann surface is a harmonic morphism; indeed, harmonic morphisms were to serve as a generalization of holomorphic maps between Riemann surfaces in Constantinescu and Cornea's early paper. Thus, the close connection between holomorphic maps and harmonic morphisms has been known since the dawn of the theory.

For example, let $\varphi : (M^{2m}, g, J^M) \to (N^2, h, J^N)$ be a holomorphic map from some almost Hermitian manifold into a Riemann surface. Write $\varphi = u + iv$ in some local isothermal coordinate on N^2. Then $\operatorname{grad}\varphi = \operatorname{grad} u + i \operatorname{grad} v$ will belong to the $(0,1)$-tangent space of J. Thus

$$g(\operatorname{grad}\varphi, \operatorname{grad}\varphi) = 0.$$

This condition, just like equation (2.1.1), is easily seen to be equivalent to the horizontal (weak) conformality of φ. Thus, there remains only the harmonicity of φ, which can be translated into conditions on J. As mentioned above, J being cosymplectic is sufficient.

4.2. **The case of dimension 4.** The most important results regarding the connection between harmonic morphisms and almost Hermitian structures have been found when the domain is of (real) dimension 4 and the codomain is a surface.

Let $\varphi : (M^4, g) \to (N^2, h)$ be a harmonic morphism from an orientable 4-dimensional manifold to a Riemann surface. Assuming initially that φ is submersive, we get a splitting of the tangent bundle into two 2-dimensional subbundles:

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

Since $\mathcal{H} \cong \varphi^{-1}(TN)$, both $\mathcal{H}$ and $\mathcal{V}$ inherit natural orientations, and so carry positive (negative) Hermitian structures from rotation $\pi/2$ $(-\pi/2)$ in each fibre. In all, we have, up to sign, two almost Hermitian structures on M with respect to which φ is holomorphic. Wood proved in [46] that if (M, g) is Einstein, then φ will have *superminimal* fibres with respect to one of these structures, i.e. one of these structures will be parallel along $\mathcal{V}$. This will then imply the integrability of this structure. Later, Ville proved in [42] that, if φ is not assumed to be submersive, this Hermitian structure can be smoothly extended to the critical set of φ. In all, we have the following result.

Theorem 4.2.1. [46, 42] *Let $\varphi : (M^4, g) \to (N^2, h)$ be a non-constant harmonic morphism from an orientable Einstein manifold to a Riemann surface. Then φ is holomorphic and has superminimal fibres with respect to a Hermitian structure on (M, g).*

This means that, from a harmonic morphism from an orientable 4-dimensional manifold to a Riemann surface, we get, up to sign, two almost Hermitian structures, one of which is integrable when the manifold is Einstein.

Conversely, let (M^4, g, J) be a Hermitian manifold. If it is a Kähler manifold, then any holomorphic map from M into a Riemann surface is a harmonic morphism. If the manifold is not Kähler, let

$$\Sigma_J = \{x \in M \mid \nabla J = 0\}$$

be the set of *Kähler points* on M. Outside of Σ_J we get a 2-dimensional J-invariant distribution

$$D_J(x) = \{X \in T_xM \mid \nabla_X J = 0\} \qquad (x \in M \setminus \Sigma_J).$$

Wood proved that, under some extra assumptions on (M^4, g), this is indeed a foliation.

Theorem 4.2.2. [46] *Let (M^4, g) be an oriented anti-self-dual Einstein manifold and J a Hermitian structure on it. Then D_J is integrable on $M \setminus \Sigma_J$.*

Recall that a Riemannian manifold (M^4, g) is said to be *anti-self-dual* if the self-dual part of the Weyl tensor vanishes. Apostolov and Gauduchon have since then shown in [2] that this requirement is unnecessarily strong; it is enough to assume that the self-dual part of the Weyl tensor has repeated eigenvalues at each point.

When D_J is integrable, the leaves of the associated foliation are then locally the fibres of some submersive map φ into a surface. We can equip this surface with a Hermitian structure in such a way that φ becomes holomorphic and hence horizontally conformal. Since the fibres of φ are superminimal with respect to J and hence, as is easily seen, minimal, it follows from Theorem 3.1.2 that φ is harmonic and so a harmonic morphism.

Thus, in this special situation, finding local harmonic morphisms is tantamount to finding local Hermitian structures.

4.3. **The complex Euclidean space.** With the previous subsection in mind, it is natural to consider the situation of a holomorphic function

$$\varphi : U \subset \mathbb{C}^2 \to \mathbb{C}.$$

Such a map is of course a harmonic morphism with respect to the flat Kähler metrics on the domain and the codomain. However, to obtain interesting examples, we investigate in [D] the Kähler distribution with respect to a conformally altered metric on the domain. With respect to this changed metric, φ is of course still horizontally (weakly) conformal but not necessarily harmonic.

So let $\langle\ ,\ \rangle$ be the flat Kähler metric on $\mathbb{C}^2$ and J be the standard Kähler structure. Let ∇ be the Levi-Civita connection with respect to the conformally altered metric $e^{2\eta}\langle\ ,\ \rangle$. It turns out that

(i) the Kähler points, i.e. the points where $\nabla J = 0$, are precisely the points where $\operatorname{grad}\eta = 0$,
(ii) away from the Käher points, the Kähler distribution is spanned by $\operatorname{grad}\eta$ and $J\operatorname{grad}\eta$.

With this knowledge, we can extract a sufficient condition for φ to be harmonic.

Theorem 4.3.1. [D] *Assume that $U \subset \mathbb{C}^2$ is open and that η is a real valued function defined in U. If φ is a holomorphic map in U into some Riemann surface N^2 satisfying* $\mathrm{d}\varphi(\operatorname{grad}\eta) = 0$*, then*

$$\varphi : (U, e^{2\eta}\langle\ ,\ \rangle) \to N^2$$

is a harmonic morphism.

Theorem 4.3.1 motivates the following question: given a holomorphic map φ from some almost Hermitian 4-manifold to some Riemann surface, when can we find a real-valued function η on the domain of φ such that $\mathrm{d}\varphi(\operatorname{grad}\eta) = 0$? Under some additional conditions on the almost complex structure and the map φ, we obtain a complete answer. Let

$$I^{\mathcal{H}}(X,Y) = \mathcal{V}[X,Y] \qquad (X, Y \in \Gamma(\mathcal{H}))$$

denote the integrability tensor of $\mathcal{H}$.

Proposition 4.3.2. [D] *Assume that* (M^4, g, J) *is almost Kähler,* N^2 *a Riemann surface and* $\varphi : M \to N$ *a submersive holomorphic function for which* $I^{\mathcal{H}}$ *is everywhere non-vanishing. Then the equation*

$$\mathrm{d}\varphi(\operatorname{grad}\eta) = 0, \quad \operatorname{grad}\eta \neq 0,$$

is locally soluble if and only if the vector field $\mathcal{V}(\operatorname{grad}\lambda^2)$ *is integrable.*

Under the additional assumption that the metric is Einstein, the global converse to Theorem 4.3.1 is also true.

Theorem 4.3.3. [D] *Assume that* $\eta : \mathbb{R}^4 \to \mathbb{R}$ *is a function with the property that the metric* $e^{2\eta}\langle\ ,\ \rangle$ *is Einstein in* $\mathbb{R}^4$*. Assume that*

$$\varphi : (\mathbb{R}^4, e^{2\eta}\langle\ ,\ \rangle) \to N^2$$

is a harmonic morphism into some Riemann surface. Then

$$\mathrm{d}\varphi(\operatorname{grad}\eta) = 0.$$

This result has an interesting consequence when applied to the conformally flat metric on $\mathbb{R}^4$ obtained from stereographic projection from S^4 minus a point. In this case, $\operatorname{grad}\eta(x)$ is parallel to x and so, by Theorem 4.3.3, any harmonic morphism from S^4 minus a point into some Riemann surface is constant along rays emanating from the origin and thus constant altogether.

Theorem 4.3.4. [D] *For a point* $p \in S^4$*, there is no non-constant harmonic morphism from* $S^4 \setminus \{p\}$ *with its standard metric into any Riemann surface.*

4.4. **Twistor spaces and bundles of Hermitian structures.** In the more general case of an orientable 4-dimensional Riemannian manifold, and indeed, for arbitrary even dimensions, local Hermitian structures can still be successfully used to construct local harmonic morphisms.

To explain this assertion, let us digress for a moment to introduce some interesting bundles that appear naturally in this context. Let (M^{2m}, g) be an oriented Riemannian manifold. Its (positive) *twistor space* is the fibre bundle associated to the principal $\mathbf{SO}(2m)$ bundle $\mathbf{SO}(M)$ of (positive) orthonormal frames with fibre $\mathbf{SO}(2m)/\mathbf{U}(m)$:

$$\Sigma^+(M) = \mathbf{SO}(M) \times_{\mathbf{SO}(2m)} \mathbf{SO}(2m)/\mathbf{U}(m).$$

This is the bundle of (positive) almost Hermitian structures on (M, g); for a positive frame $p \in \mathbf{SO}(M)$ over a point $x \in M$ and a coset $g\mathbf{U}(m) \in \mathbf{SO}(2m)/\mathbf{U}(m)$, the pair $(p, g\mathbf{U}(m))$ corresponds to the (positive) Hermitian structure on T_xM which, in the basis p, has matrix gJ_0g^{-1}. Here J_0 denotes the matrix of the canonical complex structure on $\mathbb{R}^{2m}$. Thus, any local section of the bundle is equivalent to a local (positive) almost Hermitian structure; a global section is equivalent to a reduction of the structure group of $\mathbf{SO}(M)$ to $\mathbf{U}(m)$, see e.g. [33, 34].

As $\Sigma^+(M)$ is associated with the principal bundle $\mathbf{SO}(M)$, it inherits a connection from the Levi-Civita connection on M:

$$T\Sigma^+(M) = V \oplus H.$$

Here H denotes the horizontal distribution associated with the connection and V the distribution tangent to the fibres of the bundle projection. From this decomposition, one can equip $\Sigma^+(M)$ itself with a natural almost complex structure which we denote by $\mathcal{J}$. See e.g. [9, Chapter 7] for more details on the construction of $\mathcal{J}$. For $m = 2$, $\mathcal{J}$ is integrable if and only if (M, g) is anti-self-dual, see [5], and for $m > 2$, $\mathcal{J}$ is integrable if and only if (M, g) is conformally flat, see [41]. Moreover, if J is any almost Hermitian structure (M, g), then J corresponds to a unique section σ_J of $\Sigma^+(M)$. It is shown in [15] and [13] that J is integrable if and only if

$$\sigma_J : (M, J) \to (\Sigma^+(M), \mathcal{J})$$

is holomorphic. In addition, it follows from the definition of H, that J is Kähler if and only if $\mathrm{d}\sigma_J(TM) \subset H$.

For example, the twistor bundle of S^4 can be identified with $\mathbb{C}P^3$ with $\mathcal{J}$ corresponding to the usual Kähler structure; the twistor bundle of $\overline{\mathbb{C}P}^2$ (i.e. $\mathbb{C}P^2$ with the opposite orientation) is the flag manifold

$$F_{1,2} = \frac{\mathrm{U}(3)}{\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)},$$

which also carries a natural Kähler structure corresponding to $\mathcal{J}$. For an explanation of these identifications, see e.g. [9, Chapter 7]. See also Subsection 5.4 in this text.

Twistors and *twistor spaces* originally appeared in the work of Penrose on massless fields and gravitation at the end of the 1960s. Later, similar methods were developed to describe instanton and monopole solutions of the Yang-Mills equations, see e.g. [3, 4]. The interpretation of the twistor bundle as the bundle of (positive) almost Hermitian structures on a Riemannian manifold was first made by Atiyah, Hitchin and Singer in [5]. The idea, going back to Penrose, was to 'lift' problems, such as the self-dual Yang-Mills equations on 4-manifolds, to the twistor bundle and thus into the realm of complex differential geometry, see e.g. [3].

In a similar way, twistor theory has helped to simplify the problem of constructing harmonic morphisms on Riemannian manifolds, at least on manifolds for which the twistor bundle is known and easily parametrized. A successful method has been to construct harmonic morphisms of the type $f \circ \sigma$, where σ is some local section of $\Sigma^+(M)$ and f some local holomorphic function on $\Sigma^+(M)$. This problem can be simplified further to finding parametrizations of suitable complex submanifolds of the twistor bundle; when (M, g) is a 4-dimensional Einstein manifold, this construction gives *all* submersive local harmonic morphisms with values in a Riemann surface, up to a possible change of orientation of M and post-composition with conformal diffeomorphisms. See [9, Chapter 7] for proofs and a self-contained investigation of this approach.

5. Holomorphic harmonic morphisms

5.1. **The result of Gudmundsson and Sigurdsson.** Let us now turn to the more general case of holomorphic harmonic morphisms with values in an arbitrary almost Hermitian manifold, not necessarily a Riemann surface. Theorem 3.1.2 does suggest that the dimension of the codomain will have some impact on the theory. We will shortly see that this is indeed the case.

Thus, assume that

$$\varphi : (M^{2m}, g, J^M) \to (N^{2n}, h, J^N)$$

is a harmonic morphism between two almost Hermitian manifolds. Let us also assume that J^M is Kähler: this is the setting which we investigate in [A].

A simple calculation and equation (3.2.1) give us the relations

$$\begin{aligned} \tfrac{1}{2} I^{\mathcal{H}}(X,Y) &= \omega^M(X,Y) J^M(\mathcal{V}(\operatorname{grad}\log\lambda)) \\ B^{\mathcal{H}}(X,Y) &= g(X,Y)\mathcal{V}(\operatorname{grad}\log\lambda) \qquad (X, Y \in \Gamma(\mathcal{H})). \end{aligned} \tag{5.1.1}$$

Here λ is the dilation of φ and ω^M the Kähler form of M. This suggests a close connection between the integrability of the horizontal distribution, the second fundamental form of the horizontal distribution and the vertical part of the gradient of λ.

If $n > 1$, then these quantities can in turn be connected with the curvature of the manifolds involved by a formula for horizontally homothetic maps originally derived by Gudmundsson in [21]. By adapting Gudmundsson's formula to this case we obtain the following result.

Lemma 5.1.1. [A] *Let M be Kähler, N almost Hermitian and $\varphi : M \to N$ a horizontally homothetic holomorphic map with dilation λ, then for a horizontal unit vector X on M:*

$$K^M(X \wedge JX) = \lambda^2 K^N(\varphi_* X \wedge J\varphi_* X) - \lambda^{-4}|\nabla\lambda^2|^2. \tag{5.1.2}$$

In particular, this holds when φ is a holomorphic harmonic morphism and N is of complex dimension at least 2.

The last remark follows from Theorem 3.1.2, as φ is horizontally homothetic, having complex, hence minimal, fibres.

Holomorphic Riemannian submersions, i.e. holomorphic harmonic morphisms for which $\lambda = 1$, have been closely studied by Watson in [44] and Johnson in [32]. In [28] Gudmundsson and Sigurdsson investigated the properties of holomorphic harmonic morphisms from open subsets of $\mathbb{C}^m$ into $\mathbb{C}^n$. Since both these manifolds are flat, assuming that $n > 1$, λ is actually constant by equation (5.1.2), and by equation (5.1.1), the horizontal distribution $\mathcal{H}$ is both integrable and totally geodesic. Furthermore, Gudmundsson and

Sigurdsson showed that φ necessarily is an *affine* map. Recall that a *homothetic* map is a diffeomorphism which preserves the metric up to a constant factor; with this terminology we can state the result of Gudmundsson and Sigurdsson as follows.

Theorem 5.1.2. [28] *Let $\varphi : U \subset \mathbb{C}^m \to \mathbb{C}^n$ be a non-constant holomorphic map. If $n > 1$, then φ is a harmonic morphism if and only if φ is, up to composition with homothetic maps, just the standard orthogonal projection*

$$\mathbb{C}^m = \mathbb{C}^n \times \mathbb{C}^{m-n} \to \mathbb{C}^n.$$

These results show that holomorphic harmonic morphisms between Kähler manifolds have very special properties, at least when the codomain is not a surface.

To introduce the second fundamental form $B^{\mathcal{V}}$ of the fibres into the calculations, we also derive in [A] the useful 'Bochner-type' formula

$$s_{\text{mix}} + |B^{\mathcal{V}}|^2 = \operatorname{div}^{\mathcal{V}} \operatorname{trace} B^{\mathcal{H}}. \tag{5.1.3}$$

Here s_{mix} denotes the *mixed sectional curvature*, which is a sum of sectional curvatures of two-dimensional planes spanned by one horizontal and one vertical vector, and $\operatorname{div}^{\mathcal{V}} W$ denotes the *vertical divergence* of a vertical vector field W, i.e. the divergence of the restriction of W to the fibres.

Equation (5.1.3) is of course most useful when the domain is compact and of non-negative sectional curvature, in which case we can deduce from it that regular fibres are totally geodesic.

Theorem 5.1.3. [A] *If M is a compact Kähler manifold of non-negative sectional curvature and N is an almost Hermitian manifold, then the fibres over regular values of a holomorphic harmonic morphism from M to N are totally geodesic submanifolds of M.*

As an example, any submersive holomorphic harmonic morphism from a complex torus into an almost Hermitian manifold has totally geodesic fibres. Equations (5.1.2) and (5.1.3) also serve to give a new, more intrinsic proof of the result of Gudmundsson and Sigurdsson.

5.2. **The Walczak formula.** As mentioned in earlier sections, the study of harmonic morphisms is intimately related to the study of conformal foliations. In the same vein, the study of holomorphic harmonic morphisms has proved fruitful to the understanding of conformal holomorphic foliations.

In [C] we recognize the Bochner-type formula (5.1.3) as a special case of a formula by Walczak. He proved this formula in the general setting of a Riemannian manifold (M, g), the tangent bundle of which is the direct sum of two orthogonal distributions.

Theorem 5.2.1. [43] *If $\mathcal{V}$ is a distribution on a Riemannian manifold M and $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$, then*

$$\begin{aligned} s_{\text{mix}} + |B^{\mathcal{H}}|^2 + |B^{\mathcal{V}}|^2 &= \operatorname{div}(\operatorname{trace} B^{\mathcal{H}} + \operatorname{trace} B^{\mathcal{V}}) \\ &\quad + |\operatorname{trace} B^{\mathcal{H}}|^2 + |\operatorname{trace} B^{\mathcal{V}}|^2 + \tfrac{1}{4}|I^{\mathcal{H}}|^2 + \tfrac{1}{4}|I^{\mathcal{V}}|^2 \end{aligned} \tag{5.2.1}$$

Here $I^{\mathcal{H}}$ and $I^{\mathcal{V}}$ denote the integrability tensors for $\mathcal{H}$ and $\mathcal{V}$, respectively.

With Walczak's formula in its full generality at our disposal, we show in [C] that many of the results derived in [A] generalize to holomorphic foliations; in this case the formula reduces to equation (5.1.3).

Theorem 5.2.2. [C] *Let (M, g, J) be a Kähler manifold equipped with a holomorphic foliation $\mathcal{F}$ such that the mixed curvature $s_{\text{mix}} \geq 0$. Then $s_{\text{mix}} = 0$ along every compact leaf and every compact leaf is a totally geodesic submanifold of M. In particular, if $s_{\text{mix}} > 0$, then $\mathcal{F}$ has no compact leaves.*

When the foliation is also conformal and has a global dilation (see Subsection 3.2), we find an integrating factor for equation (5.1.3).

Proposition 5.2.3. [C] *Let (M, g, J) be a Kähler manifold equipped with a holomorphic, conformal foliation $\mathcal{F}$ which admits a global dilation λ. Then*

$$\operatorname{div}(\lambda^n \operatorname{trace} B^{\mathcal{H}}) = \lambda^n (s_{\text{mix}} + |B^{\mathcal{V}}|^2), \tag{5.2.2}$$

where $\mathcal{V}$ denotes the involutive distribution associated to $\mathcal{F}$, n the codimension of $\mathcal{V}$ and $\mathcal{H}$ the complementary orthogonal distribution.

By integrating equation (5.2.2), we extend Theorem 5.2.2 to the case when the leaves are not necessarily compact.

Theorem 5.2.4. [C] *Let (M, g, J) be a compact Kähler manifold equipped with a holomorphic, conformal foliation $\mathcal{F}$ which admits a global dilation and $s_{\text{mix}} \geq 0$. Then the leaves of $\mathcal{F}$ are totally geodesic and $s_{\text{mix}} = 0$.*

The foliation $\mathcal{F}$ is said to be *homothetic* if it is conformal and the mean curvature vector field $\operatorname{trace} B^{\mathcal{H}}$ for $\mathcal{H}$ is a gradient vector field. Homothetic foliations are characterized as those which are locally the fibres of horizontally homothetic submersions. In [D] we show how the Walczak formula, when applied to holomorphic homothetic foliations, provides decomposition results.

Theorem 5.2.5. [D] *Let M be a compact Kähler manifold of non-negative sectional curvature and vanishing first Betti number. Let $\mathcal{F}$ be a holomorphic, homothetic foliation of M. Then the orthogonal complementary distribution $\mathcal{H}$ is integrable. Furthermore, both $\mathcal{F}$ and the foliation generated by $\mathcal{H}$ are totally geodesic, so that M is locally a Riemannian product.*

With this result, we extend Theorem 5.1.3 to the following.

Theorem 5.2.6. [D] *Let M and N be compact Kähler manifolds, where M has non-negative sectional curvature and $\dim_{\mathbb{C}} N > 1$. Assume that $\varphi : M \to N$ is a holomorphic harmonic morphism. Then the universal covering $\pi : \tilde{M} \to M$ is biholomorphically isometric to a product of two Kähler manifolds $P \times \tilde{N}$, where $\tilde{N}$ is the universal covering of N, equipped with a constant multiple of the lift of the metric on N. With this identification, $\varphi \circ \pi$ is the projection from $\tilde{M}$ onto $\tilde{N}$ followed by the covering projection onto N.*

5.3. **Holomorphic harmonic morphisms and Hodge numbers.** It is by now a well-known result (see e.g. [14]) that the existence of a harmonic morphism $\varphi : M \to N$, where M and N are both compact, forces the first Betti numbers of M and N to obey the inequality

$$b_1(M) \geq b_1(N).$$

This condition is particularly interesting since it is purely *topological*.

Similarly, the existence of a holomorphic harmonic morphism between two *complex* manifolds imposes conditions on the Hodge numbers, which depend only on the complex structures and not on the chosen Hermitian metrics.

Theorem 5.3.1. [D] *Assume that M and N are connected complex manifolds and that $\varphi : M \to N$ is a holomorphic map which is submersive at some point of M. Then*

$$h^{p,0}(M) \geq h^{p,0}(N)$$

for all p.

By combining this theorem with the well-known *Calabi conjecture*, we obtain many interesting situations when the existence of non-constant holomorphic harmonic morphisms immediately can be ruled out by the properties of the first Chern classes and certain Hodge numbers.

Corollary 5.3.2. [D] *Let M and N be compact Kähler manifolds and assume that $\varphi : M \to N$ is a holomorphic map which is submersive at some point of M.*

(i) *If $c_1(M) > 0$, then $h^{p,0}(N) = 0$ for all $p > 0$.*
(ii) *If $c_1(N) = 0$ and N is simply connected, then $h^{n,0}(M) \neq 0$, where $n = \dim_{\mathbb{C}} N$.*
(iii) *If $c_1(M) = 0$, then $h^{p,0}(N) \leq \binom{m}{p}$, where $m = \dim_{\mathbb{C}} M$ and $0 \leq p \leq \dim_{\mathbb{C}} N$.*

For example, if M^{4m} and N^{4n} are two hyper-Kähler manifolds with holomony equal to $\mathbf{Sp}(m)$ and $\mathbf{Sp}(n)$, respectively, then the only non-constant holomorphic harmonic morphisms between M and N are the holomorphic, homothetic diffeomorphisms. See [D] for a proof of this fact and further applications of Corollary 5.3.2.

5.4. **Flag manifolds.** Most of the results on holomorphic harmonic morphisms discussed so far have dealt with rigidity and non-existence of such maps. For example, a non-constant holomorphic harmonic morphism defined on a Hermitian symmetric space of compact type, with codomain not a surface, is merely the projection onto some of the factors in the de Rham decomposition of the domain (followed by a possible covering projection). This follows from Theorem 5.2.6. Clearly then, if the domain is irreducible, such a map has to be constant. This is, however, part of a more general fact: a compact Kähler manifold with $h^{1,1} = 1$ does not admit *any* non-constant holomorphic mapping into any Kähler manifold of strictly lower dimension, see [40].

Irreducible Hermitian symmetric spaces all have $h^{1,1} = 1$. Indeed, if $M = G/K$ is a Hermitian symmetric space of compact type, where G is semisimple and effective on M, then $h^{1,1}(M)$ is the dimension of the centre of K, see e.g. [10, Remark 8.84]. Thus, to construct non-trivial examples of holomorphic harmonic morphisms from homogeneous manifolds G/K, we would consider the case when K does not have 1-dimensional centre.

With this in mind, we are led to an immediate generalization of Hermitian symmetric spaces of compact type, namely the *flag manifolds*. These can be characterized as the homogeneous spaces G/P, where G is a complex semisimple Lie group and P a parabolic subgroup, see [1].

There is a simple algorithm for constructing flag manifolds. Given a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$, let Π be a basis for the corresponding root system. For any subset $\Pi_0 \subset \Pi$ we may form a parabolic subalgebra $\mathfrak{p}_{\Pi_0}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and denote by P_{Π_0} the Lie subgroup of G with Lie algebra $\mathfrak{p}_{\Pi_0}$. Then G/P_{Π_0} is a flag manifold.

For any *two* subsets $\Pi_0 \subset \Pi_1 \subset \Pi$, we obtain two parabolic subgroups $P_{\Pi_0} \subset P_{\Pi_1}$. The following result gives an abundance of holomorphic harmonic morphisms.

Theorem 5.4.1. [D] *Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Π a basis for the corresponding root system. For any two subsets*

$$\Pi_0 \subset \Pi_1 \subset \Pi,$$

the flag manifolds G/P_{Π_0} and G/P_{Π_1} carry invariant (integrable) cosymplectic structures, such that the natural homogeneous projection

$$\pi : G/P_{\Pi_0} \to G/P_{\Pi_1}$$

is a holomorphic Riemannian submersion with totally geodesic fibres. In particular, π is a harmonic morphism.

These complex structures are in fact Kähler with respect to metrics different from the ones chosen in the above theorem.

6. Warped products

6.1. **Harmonic morphisms of warped product type.** In [C] we also show how the Walczak formula in Theorem 5.2.1 can be utilized to derive results on harmonic morphisms of *warped product type*, i.e. horizontally homothetic harmonic morphisms with totally geodesic fibres and integrable horizontal distribution.

Harmonic morphisms of warped product type are slightly more complicated than *totally geodesic Riemannian submersions*, which, locally, are simply projections of Riemannian products.

Locally, harmonic morphisms of warped product type are instead projections of *warped products* of Riemannian manifolds. The warped product of (M, g) and (N, h) by the function $\eta : M \to \mathbb{R}_+$ is the Riemannian manifold

$$(M \times N, g + \eta^2 h).$$

It is easy to see that the projection of this warped product onto (N, h) is a harmonic morphism of warped product type. As pointed out, any harmonic morphism of warped product type is *locally* of this type. In [C] we show that, under some global conditions of the domain, a harmonic morphism of warped product type is *globally* of this type.

Theorem 6.1.1. [C] *Let $\varphi : (P, k) \to (N, h)$ be a harmonic morphism of warped product type, where (P, k) is complete, connected and simply connected and N is connected. Let λ be the dilation of φ and let $\tilde{N}$ be the universal covering of N. Then (P, k) is isomorphic to a warped product $M \times_{\lambda^{-1}} \tilde{N}$ and φ is the projection onto $\tilde{N}$ followed by the covering projection $\tilde{N} \to N$.*

In the recent article [36], Ou has generalized this result to *p-harmonic morphisms* of *twisted product type*.

By applying the Walczak formula (5.2.1) to this situation, we show how to connect the sectional curvature of (P, k) with the dilation:

$$n\Delta^{\mathcal{V}} \log \lambda = s_{\text{mix}} + |\operatorname{grad} \log \lambda|^2.$$

Here, n is the dimension of N and $\Delta^{\mathcal{V}}$ is the Laplacian along the fibres. In particular, when (P, k) has non-negative sectional curvature, we see that $\log \lambda$ restricted to the fibres is subharmonic. Yau showed in [47] that a non-zero function, defined on a complete, simply connected Riemannian manifold, the logarithm of which is subharmonic, is either constant or of infinite L^p-norm, for any $p > 0$. By recalling that $\lambda^2 = n|\mathrm{d}\varphi|^2$ and applying Yau's result to this situation, we prove the following result.

Theorem 6.1.2. [C] *Let $\varphi : (P, k) \to (N, h)$ be a harmonic morphism of warped product type where (P, k) is complete, connected and simply connected with non-negative sectional curvature and N is connected. If P is compact, then φ is a totally geodesic map. If P is non-compact, then either φ is constant or it has infinite p-energy for all p such that* $\dim N < p < \infty$.

Recall that for $p > 1$, the *p-energy* of a map $\varphi : (P, k) \to (N, h)$ is defined as

$$E_p(\varphi) = \tfrac{1}{p} \int_P |\mathrm{d}\varphi|^p \nu^P,$$

where ν^P is the volume form of (P, k).

Ou has recently clarified this result by showing that under these assumptions, φ is a totally geodesic map even in the non-compact case, see [36].

7. Harmonic morphisms from symmetric spaces

7.1. **Background.** Complex-valued harmonic morphisms from Riemannian (globally) symmetric spaces have been studied in great detail, in particular by Gudmundsson in a series of papers [22, 23, 24, 25, 26]. In these papers, Gudmundsson's main goal has been to produce examples of such maps and thereby to establish his following longstanding conjecture.

Conjecture 7.1.1. *Let (M^m, g) be an irreducible Riemannian symmetric space of dimension $m \geq 2$. For each point $p \in M$ there exists a complex valued harmonic morphism $\varphi : U \to \mathbb{C}$ defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type, then the domain U can be chosen to be the whole of M.*

To prove this conjecture, one may of course restrict oneself to the relatively short list of irreducible Riemannian symmetric spaces, where a case-by-case study seems feasible.

The following approach to constructing harmonic morphisms from symmetric spaces has often proved successful. Let (M, g) be a Riemannian (globally) symmetric space. Choose a principal fibre bundle

$$\pi : P \to M,$$

where the total space P has a Riemannian or semi-Riemannian metric which makes π into a harmonic morphism. (Harmonic morphisms between semi-Riemannian manifolds were introduced by Fuglede in [19], where they were proved to have a similar characterization as in the Riemannian case.) We may now search for local harmonic morphisms on P which are invariant under the action of the structure group of the bundle; such maps will factor through the projection π and induce locally defined harmonic morphisms on M, see [26, Proposition 1]. With any luck, finding invariant harmonic morphisms on P should be easier than finding harmonic morphisms on M directly.

To give a simple example of such a construction, equip the complex projective space $\mathbb{C}P^m$ with its usual Fubini-Study metric. We have a $\mathbb{C}^*$-bundle

$$\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{C}P^m, \quad \pi(z) = [z].$$

The projection π is a harmonic morphism with respect to the usual flat metric on $\mathbb{C}^{m+1} \setminus \{0\}$. Hence we have simplified the question of finding harmonic morphisms on $\mathbb{C}P^m$ to finding harmonic morphisms in $\mathbb{C}^{m+1} \setminus \{0\}$ which are invariant under the action of $\mathbb{C}^*$. This was the approach taken by Gudmundsson in [23] in the search for locally defined *non-holomorphic* harmonic morphisms on $\mathbb{C}P^m$.

This example is of course special, since both spaces carry Kähler structures with respect to which π is holomorphic, and these structures can serve as auxiliary tools in finding even non-holomorphic harmonic morphisms. In the following two subsections we shall see similar and more interesting examples of this construction, where complex structures either play a very different role or are not present at all.

7.2. **The real hyperbolic space.** We turn in [B] to the construction of complex-valued harmonic morphisms from the real hyperbolic space $\mathbb{R}H^m$ or open subsets thereof. Such maps have been studied by Gudmundsson in [25] and [26]. Following the scheme of the previous section, we begin however, by showing a useful way of representing these spaces as the base of a principal fibre bundle.

On $\mathbb{R}^{m+1}$ we have the quadratic form

$$\langle x, y\rangle_L = -x_0y_0 + \sum_{k=1}^{m} x_k y_k \qquad (x, y \in \mathbb{R}^{m+1}).$$

Let $\mathcal{U}_{m+1} = \{x \in \mathbb{R}^{m+1} \mid \langle x, x\rangle_L < 0\}$. The group $\mathbb{R}^*$ acts on $\mathcal{U}_{m+1}$ by multiplication and we can identify the orbit space $\mathcal{U}_{m+1}/\mathbb{R}^*$ with $\mathbb{R}H^m$. If g denotes the standard metric on $\mathbb{R}H^m$ of constant sectional curvature -1, then this identification turns the projection of the $\mathbb{R}^*$-bundle

$$\pi : (\mathcal{U}_{m+1}, \langle\ ,\ \rangle_L) \to (\mathbb{R}H^m, g)$$

into a horizontally homothetic surjective harmonic morphism with totally geodesic fibres. The domain of this map is of course semi-Riemannian.

As pointed out in the previous section, a map

$$\varphi : U \subset \mathbb{R}H^m \to \mathbb{C}$$

is a harmonic morphism if and only if its composition with π is a harmonic morphism from the semi-Riemannian manifold $(\pi^{-1}(U), \langle\ ,\ \rangle_L)$. Thus, to construct local harmonic morphisms from $\mathbb{R}H^m$ we can just as well construct local harmonic morphisms from $\mathcal{U}_{m+1}$ which are invariant under the action of $\mathbb{R}^*$.

To do this, we can apply the following method: choose a function

$$F : V_1 \times V_2 \subset \mathbb{C} \times \mathcal{U}_{m+1} \to \mathbb{C};$$

we require F to be holomorphic in the first variable, a harmonic morphism in the second and, in addition, be homogeneous of some integer degree in the second variable. The idea, which originates from Jacobi's construction, is that if $F(z_0, x_0) = 0$, then any smooth local solution $z = \varphi(x)$ through (z_0, x_0) to the equation

$$F(z, x) = 0$$

is a harmonic morphism. Due to the homogenity of F, φ will be invariant under the action of $\mathbb{R}^*$, and hence induce a local harmonic morphism on $\mathbb{R}H^m$. For example, we can choose F of the form

$$F(z, x) = \sum_{k=0}^{m} \xi_k(z) x_k,$$

where $\xi = (\xi_0, \dots, \xi_m)$ is some holomorphic map into $\mathbb{C}^{m+1}$ satisfying

$$-\xi_0^2 + \sum_{k=1}^{m} \xi_k^2 = 0.$$

Harmonic morphisms constructed in this way will, however, necessarily have totally geodesic fibres. When $m + 1 = 2n$ we can use the complex structure on $\mathcal{U}_{2n} \subset \mathbb{C}^n$ to produce $\mathbb{R}^*$-invariant harmonic morphisms in $\mathcal{U}_{2n}$; these will induce harmonic morphisms on the *odd*-dimensional real hyperbolic space $\mathbb{R}H^{2n-1}$, see [26].

In [B] we demonstrate a method of constructing complex-valued harmonic morphisms, locally and globally defined in *even*-dimensional real hyperbolic spaces, which do not have totally geodesic fibres. Recall that we have transformed the problem into finding $\mathbb{R}^*$-invariant harmonic morphisms on open subsets of the semi-Riemannian manifold $(\mathcal{U}_{m+1}, \langle \ , \ \rangle_L)$. Our construction is a semi-Riemannian version of the work of Baird and Wood in [7] and [8] and it relates to our earlier discussion of twistor spaces.

So, let (M^{2m}, g) be an orientable Riemannian manifold of even dimension. A subbundle E of the complexified tangent bundle $T^{\mathbb{C}}M$ is said to be *isotropic* if

$$g(X, Y) = 0 \qquad (X, Y \in E).$$

Here we have extended the metric tensor g to a complex bilinear form on $T^{\mathbb{C}}M$. In particular, if J is some almost Hermitian structure on (M, g), then its (1,0)-tangent space will give an m-dimensional isotropic subbundle of $T^{\mathbb{C}}M$. Conversely, for any m-dimensional isotropic subbundle of $T^{\mathbb{C}}M$ we can find an almost Hermitian structure on (M, g) which has this space as its (1,0)-tangent space. Thus, by parametrizing m-dimensional isotropic subbundles of $T^{\mathbb{C}}M$, we get a parametrization of the twistor bundle of (M, g).

In [7], Baird and Wood showed how to parametrize the twistor bundle in $\mathbb{R}^{2m}$ by using isotropic subbundles of $T^{\mathbb{C}}\mathbb{R}^{2m} \cong \mathbb{C}^{2m}$. From this parametrization, they found a recipe for constructing complex-valued maps, locally defined on $\mathbb{R}^{2m}$, which are holomorphic with respect to some Hermitian structure. They also showed how such a map can be chosen to have superminimal, hence minimal fibres. Thus, the map so constructed is a harmonic morphism, and it need not have totally geodesic fibres.

For the semi-Riemannian manifold $(\mathbb{R}^{2m}, \langle \ , \ \rangle_L)$, there is no longer a suitable notion of almost Hermitian structures. However, we can still talk about complex isotropic subbundles of the complexified tangent bundle and it is possible to find a parametrization for these.

Following the scheme of [7] and [8], we produce in [B] locally defined complex-valued harmonic morphisms on $(\mathbb{R}^{2m}, \langle \ , \ \rangle_L)$ which need not have totally geodesic fibres. These maps can also be constructed in such a way that they are invariant under both the orthogonal projection $\mathbb{R}^{2m} = \mathbb{R}^{2m-1} \times \mathbb{R} \to \mathbb{R}^{2m-1}$ and under the action of $\mathbb{R}^*$. Thus, if they are locally or globally defined in $\mathcal{U}_{2m}$, they will induce locally or globally defined harmonic morphisms in the even-dimensional hyperbolic spaces $\mathbb{R}H^{2m-2}$, which need not have totally geodesic fibres. As an example, we construct the harmonic morphism

$$\mathcal{U}_7 \ni x \mapsto \frac{(x_2 + x_4 + i(x_5 - x_3))(2x_6 - x_4 + i(2x_3 + x_5))}{(x_0 - x_1)^2},$$

which factors to a globally defined complex-valued harmonic morphism from $\mathbb{R}H^6$.

7.3. **Harmonic morphisms from Grassmannians and their dual spaces.** In our search for harmonic morphisms from symmetric spaces, we generalize in [E] the ideas of the previous section to the non-compact duals of the Grassmannians. Recall the definitions of the Grassmannians and their non-compact duals: the real case

$$\mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q), \quad \mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q),$$

the complex case

$$\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)), \quad \mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)),$$

and the quaternionic case

$$\mathbf{Sp}(p+q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q), \quad \mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q).$$

Let $\mathbb{D}$ denote any of the division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. To each of the above spaces we associate an auxiliary principal fibre bundle, denoted by $U^*_{pq}(\mathbb{D})$ in the compact case and by $U_{pq}(\mathbb{D})$ in the non-compact case.

Let us briefly describe $U_{pq}(\mathbb{D})$. For $X \in \mathbb{D}^{(p+q)\times p}$, write

$$X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix},$$

where $X_0 \in \mathbb{D}^{p\times p}$ and $X_1 \in \mathbb{D}^{q\times p}$. Then

$$U_{pq}(\mathbb{D}) = \{X \in \mathbb{D}^{(p+q)\times p} \mid -X_0^* X_0 + X_1^* X_1 < 0\}.$$

The open submanifold $U_{pq}(\mathbb{D})$ of $\mathbb{D}^{(p+q)\times p}$ is a principal fibre bundle over the non-compact dual of the Grassmannian with group $\mathbf{GL}_p(\mathbb{D})$.

On $U_{pq}(\mathbb{D})$ we have a bilinear form inducing a semi-Riemannian metric

$$\langle X, Y \rangle = \mathfrak{Re}(\operatorname{trace} X^* I_{pq} Y),$$

where

$$I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Here I_p denotes the $p \times p$ identity matrix.

Unfortunately, it is not true in general that the projection of this bundle is a harmonic morphism. However, we can still prove the following result.

Theorem 7.3.1. [E] *Equip $U_{pq}(\mathbb{D})$ with the semi-Euclidean metric $\langle\ ,\ \rangle$. Then the natural projection*

$$\pi : U_{pq}(\mathbb{D}) \to U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$$

has the following property: if $\hat{\varphi}$ is a $\mathbf{GL}_p(\mathbb{D})$-invariant harmonic morphism on $U_{pq}(\mathbb{D})$, then the induced map on $U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$ is also a harmonic morphism.

Thus we have transformed the problem of finding harmonic morphisms from the non-compact duals of the Grassmannians to finding $\mathbf{GL}_p(\mathbb{D})$-invariant harmonic morphisms from the *flat* space $U_{pq}(\mathbb{D})$. We produce several examples of such maps in [E].

As for the Grassmannians themselves, we provide in [E] a useful method of obtaining local harmonic morphisms on a symmetric space of compact or non-compact type from local harmonic morphisms defined on the dual space. A key observation is the following proposition, which follows immediately from the existence of harmonic coordinates.

Proposition 7.3.2. [9] *A harmonic morphism between real analytic manifolds is a real analytic map.*

As is well known, symmetric spaces are all real analytic Riemannian manifolds.

So, assume that M is an irreducible Riemannian symmetric space of, say, non-compact type. We may then represent M as a quotient G/K, where G is a non-compact connected semisimple group of isometries of M and K is a maximal compact subgroup of G. To this quotient corresponds a Cartan decomposition of the Lie algebra of G:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Now assume that $G^{\mathbb{C}}$ is a complex Lie group, the Lie algebra of which is the complexification of $\mathfrak{g}$; assume moreover that $G^{\mathbb{C}}$ contains G as a Lie subgroup. Then we call $G^{\mathbb{C}}$ the *complexification* of G. The Lie group $G^{\mathbb{C}}$ has a compact real form U with Lie algebra

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}.$$

Assume finally that the connected subgroup $U \subset G^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$ is compact. Then $M^* = U/K$ is the compact dual of M.

Let $W \subset M$ be an open subset and $\varphi : W \to \mathbb{C}$ a real analytic map. Then φ extends uniquely to a real analytic map $\varphi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}$ where W^* is an open subset of $G^{\mathbb{C}}/K$. Let $W^* = W^{\mathbb{C}} \cap M^*$ and let $\varphi^* : W^* \to \mathbb{C}$ be the restriction of $\varphi^{\mathbb{C}}$.

Theorem 7.3.3. [E] *The map $\varphi^* : W^* \to \mathbb{C}$ is a harmonic morphism if and only if $\varphi : W \to \mathbb{C}$ is a harmonic morphism.*

This theorem allows us to construct local harmonic morphisms on Grassmannians and thereby establish Conjecture 7.1.1 for these spaces.

7.4. **Harmonic morphisms from compact Lie groups and their dual spaces.** Assume that G is a compact Lie group with a bi-invariant metric and K is a compact subgroup of G, such that (G, K) is a Riemannian symmetric pair. We can then equip G/K with a G-invariant metric such that G/K becomes a Riemannian symmetric space and the projection

$$\pi : G \to G/K$$

becomes a Riemannian submersion with totally geodesic fibres, in particular, a harmonic morphism. Tautologically, the composition of two harmonic morphisms is again a harmonic morphism. Hence, any (locally defined) harmonic morphism on G/K will induce a (locally defined) harmonic morphism on G with respect to the bi-invariant metric. Furthermore, recall that the irreducible Riemannian symmetric spaces of type II are precisely the compact, simple Lie groups provided with a bi-invariant metric.

In [F] we apply this argument to the maps constructed in [E] and thereby prove the local existence of harmonic morphisms from several Riemannian symmetric spaces of type II, more precisely the spaces $\mathbf{SO}(n)$, $\mathbf{SU}(n)$ and $\mathbf{Sp}(n)$. Furthermore, by applying Theorem 7.3.3, we obtain locally defined harmonic morphisms on the dual spaces of type IV:

$$\mathbf{SO}(n, \mathbb{C})/\mathbf{SO}(n), \quad \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n), \quad \mathbf{Sp}(n, \mathbb{C})/\mathbf{Sp}(n).$$

It is easy to see that the maps so constructed from $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$ are globally defined. These maps arise from holomorphic functions on the Hermitian symmetric space $\mathbf{SU}(p, n-p)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(n-p))$, and this observation makes it possible to give a positive answer to the global existence question for $\mathbf{SO}(2n, \mathbb{C})/\mathbf{SO}(2n)$ and $\mathbf{Sp}(n, \mathbb{C})/\mathbf{U}(n)$.

Recall the standard representation as a bounded symmetric domain

$$\mathbf{SO}^*(2n)/\mathbf{U}(n) \cong \{Z \in \mathbb{C}^{n \times n} \mid I_n - ZZ^* > 0,\ Z^t = -Z\}.$$

The matrix entries on the bounded symmetric domain induce harmonic morphisms on $\mathbf{SO}^*(2n)/\mathbf{U}(n)$, which, by duality, induce locally defined harmonic morphisms on $\mathbf{SO}(2n)/\mathbf{U}(n)$. These lift to locally defined harmonic morphisms from $\mathbf{SO}(2n)$, and, once more by duality, induce harmonic morphisms on the dual space $\mathbf{SO}(2n, \mathbb{C})/\mathbf{SO}(2n)$. We prove in [F] that these maps are indeed globally defined, and we do the same for $\mathbf{Sp}(n, \mathbb{R})/\mathbf{U}(n)$, thus adding further examples to the list of irreducible Riemannian symmetric spaces for which Conjecture 7.1.1 has been established.

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