

QUADRATIC FORMS

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1. INTRODUCTION

UPTO this point we have concentrated on applications of vectors and matrices to linear problems, particularly solutions of linear equations. However, the theory we have developed is rather flexible and in particular lends itself well to certain situations that are quadratic in nature.

Consider the following equation in the plane:

$$7x^2 + 8xy + y^2 - 3x + 2y - 1 = 0. \quad (1.1)$$

Here we place one constraint on two variables, so we would expect the solution set to be a curve. What type of curve?

We are familiar with some curves that are described by equations like (1.1). Four particular examples come to mind

Circle: $x^2 + y^2 = r^2$.

Ellipse: $(x^2/a^2) + (y^2/b^2) = 1$

Parabola: $y = ax^2$.

Hyperbola: $(x^2/a^2) - (y^2/b^2) = 1$.

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Below we will see that together with straight lines these are essentially the only curves that can arise. We will also show how we can decide which of these correspond to (1.1).

2. SYMMETRIC MATRICES

LET A be an $n \times n$ symmetric matrix. The quantity $\mathbf{x}^t A \mathbf{x}$ defines a polynomial in the n coordinates of \mathbf{x} . For example, if $A = \begin{bmatrix} 5 & 2 \\ 2 & -3 \end{bmatrix}$ then we have

$$\mathbf{x}^t A \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5x_1^2 + 4x_1x_2 - 3x_2^2.$$

In general this procedure produces a polynomial that is quadratic in the variables x_1, \dots, x_n :

$$\begin{aligned} q_A(x_1, x_2, \dots, x_n) &= \mathbf{x}^t A \mathbf{x} \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j = \sum_{i=1}^n a_{ii} x_i^2 + \sum_{1=i < j \leq n} 2a_{ij} x_i x_j. \end{aligned}$$

Definition 2.1. We call q_A the *quadratic form* associated to A .

Moreover, any quadratic polynomial without linear or constant terms can be written as $\mathbf{x}^t A \mathbf{x}$ for a suitable choice of A : if

$$q(x_1, \dots, x_n) = \sum_{i=1}^n b_i x_i^2 + \sum_{1=i < j \leq n} c_{ij} x_i x_j,$$

then $q = q_A$, where

$$A = \begin{bmatrix} b_1 & \frac{1}{2}c_{12} & \dots & \frac{1}{2}c_{1n} \\ \frac{1}{2}c_{12} & b_2 & \dots & \frac{1}{2}c_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2}c_{1n} & \frac{1}{2}c_{2n} & \dots & b_n \end{bmatrix}$$

Pay particular attention to the factor $1/2$ that appears in front of the coefficients c_{ij} .

The two most common examples one encounters are:

$$n = 2: \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad q_A(x, y) = ax^2 + 2bxy + cy^2,$$

$$n = 3: \quad A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \quad q_A(x, y, z) = ax^2 + by^2 + cz^2 \\ + 2fyz + 2gxz + 2hxy.$$

The naming of the entries in the 3×3 case is standard in mechanics and I find the following mnemonic helpful

All Hairy Gorillas
 Have Big Feet
 Good For Climbing

The other special case to note is when A is a diagonal matrix:

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

$$q_A(x_1, x_2, \dots, x_n) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$$

3. ORTHOGONAL CHANGES OF COORDINATES

SUPPOSE we make a change of coordinates, how does this affect the polynomial q ? If $\mathbf{x} = P\tilde{\mathbf{x}}$ for some invertible $n \times n$ matrix P , then

$$q_A(\mathbf{x}) = \mathbf{x}^t A \mathbf{x} = \tilde{\mathbf{x}}^t P^t A P \tilde{\mathbf{x}} = q_{P^t A P}(\tilde{\mathbf{x}}). \tag{3.1}$$

We thus get a new quadratic form represented by the matrix $P^t A P$. Note that this is *different* from the formula describing how the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ of a linear transformation changes: we have the *transpose* P^t instead of the *inverse* P^{-1} . However, if P is an orthogonal matrix, then $P^t = P^{-1}$. We can thus use our theorem on diagonalisability of symmetric matrices to deduce:

Theorem 3.1. *Let q_A be the quadratic form associated to symmetric matrix A . Then there is an orthogonal matrix P such that*

$$q_{P^t A P}(\tilde{x}_1, \dots, \tilde{x}_n) = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \dots + \lambda_n \tilde{x}_n^2$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A .

The directions defined by the columns of P are called the *principal axes* of q_A .

Example 3.2. Consider the ellipse

$$\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1.$$

In this case the quadratic form is $\frac{1}{4}x^2 + \frac{1}{9}y^2$ and the associated matrix is diagonal, so the principal axes are aligned with the coordinate axes.

Example 3.3. Consider the equation

$$5x^2 - 4xy + 5y^2 = 1. \quad (3.2)$$

The quadratic form $5x^2 - 4xy + 5y^2$ has matrix $A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 5 \end{bmatrix} = \lambda^2 - 10\lambda + 21 = (\lambda - 3)(\lambda - 7),$$

so A has eigenvalues 3 and 7. We now compute the corresponding eigenvectors:

$\lambda = 3$: We solve $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (3I - A) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so $x = y$ and a corresponding unit eigenvector is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = 7$: We now have $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (7I - A) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so $x = -y$ and a corresponding unit eigenvector is $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus taking $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ gives us a coordinate system in which (3.2) becomes

$$3\tilde{x}^2 + 7\tilde{y}^2 = 1,$$

which is an ellipse, with principal axes at 45° to the usual axes (see Figure 1). Note the ellipse crosses these axes at $\tilde{x} = \pm 1/\sqrt{3}$ and $\tilde{y} = \pm 1/\sqrt{7}$, respectively.

Example 3.4. The quadratic part of equation (1.1) is $q(x, y) = 7x^2 + 8xy + y^2$, which is represented by the matrix $A = \begin{bmatrix} 7 & 4 \\ 4 & 1 \end{bmatrix}$. The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 7 & -4 \\ -4 & \lambda - 1 \end{bmatrix} = \lambda^2 - 8\lambda - 9,$$

so A has eigenvalues $\lambda = -1$ and $\lambda = 9$. We now find an orthonormal basis of eigenvectors:

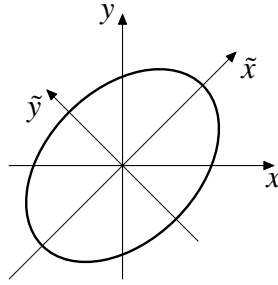


Figure 1: The ellipse $5x^2 - 4xy + 5y^2 = 1$ with its principal axes.

$\lambda = -1$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, which gives $y = -2x$. The corresponding eigenvectors are thus multiples of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, which has norm squared $1^2 + (-2)^2 = 5$. A unit eigenvector is thus given by

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$\lambda = 9$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so $x = 2y$ and a corresponding unit eigenvector is given by

$$\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We build P by using the vectors \mathbf{v}_1 and \mathbf{v}_2 for the columns:

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

The matrix P satisfies

$$P^t A P = \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix}.$$

The new coordinates are related to the old by $\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix}$ since $P^{-1} = P^t$. So $\tilde{x} = \frac{1}{\sqrt{5}}(x - 2y)$ and $\tilde{y} = \frac{1}{\sqrt{5}}(2x + y)$ and the quadratic part of (1.1) becomes

$$-\tilde{x}^2 + 9\tilde{y}^2.$$

4. CLASSIFYING DEGREE TWO CURVES IN THE PLANE

CONSIDER a general curve of degree two in the variables x and y . This is given by an equation

$$ax^2 + 2bxy + cy^2 + dx + ey + g = 0. \quad (4.1)$$

We may rewrite this in terms of matrices as

$$\mathbf{x}^t \mathbf{A} \mathbf{x} + \mathbf{b}^t \mathbf{x} + g = 0, \quad (4.2)$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} d \\ e \end{bmatrix}$.

We can find an orthogonal matrix P such that

$$P^t A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Changing coordinates by $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$ in (4.2), we see that (4.1) becomes

$$\lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 + \tilde{d} \tilde{x} + \tilde{e} \tilde{y} + g = 0,$$

where $\begin{bmatrix} \tilde{d} \\ \tilde{e} \end{bmatrix} = P^t \mathbf{b}$.

If λ_1 and λ_2 are non-zero, then we can remove the linear terms by setting $x' = \tilde{x} + \tilde{d}/(2\lambda_1)$ and $y' = \tilde{y} + \tilde{e}/(2\lambda_2)$ (to complete the square) giving

$$\lambda_1 x'^2 + \lambda_2 y'^2 = g', \quad (4.3)$$

where $g' = \tilde{d}^2/(4\lambda_1) + \tilde{e}^2/(4\lambda_2) - g$. Such a change of coordinates corresponds to moving the origin to the point $(-\tilde{d}/(2\lambda_1), -\tilde{e}/(2\lambda_2))$.

There are now two cases:

λ_1 and λ_2 have the same sign: then (4.3) describes an *ellipse* provided that g' has the same sign as λ_1 and λ_2 . If $g' = 0$, we get just a single point, if g' has the opposite sign to λ_i then (4.3) has no solutions.

λ_1 and λ_2 have the opposite signs: then (4.3) describes a *hyperbola* provided $g' \neq 0$. If $g' = 0$ the hyperbola degenerates to the intersection of two straight lines.

The above cases occur provided both eigenvalues are non-zero. If both eigenvalues vanish, then the original equation (4.1) has no quadratic terms and describes a straight line. If one eigenvalue vanishes, say $\lambda_1 = 0$, then we put $y' = \tilde{y} + \tilde{e}/(2\lambda_2)$ and $x' = \tilde{e}^2/(4\lambda_2^2) - \tilde{d}\tilde{x} - g$, to get $x' = y'^2$, which is

a *parabola*. This requires $\tilde{d} \neq 0$, if that is not the case, we have $y'^2 + g = 0$ which either gives a pair of parallel lines ($g < 0$), a line ($g = 0$) or is empty ($g > 0$).

Thus the only curves other than straight lines that occur are parabolas, hyperbolas or ellipses (including circles).

Example 4.1. Let us continue with (1.1). As the eigenvalues -1 and 9 found in Example 3.4 are of opposite signs this is a hyperbola unless it is degenerate, i.e., unless $g' = 0$. We find

$$\begin{bmatrix} \tilde{d} \\ \tilde{e} \end{bmatrix} = P^t \begin{bmatrix} d \\ e \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -7 \\ -4 \end{bmatrix}.$$

So we have

$$g' = \tilde{d}^2/4 + \tilde{e}^2/(4 \times 9) - g = \frac{1}{4} \frac{1}{5} (49 + \frac{16}{9}) + 1 \neq 0.$$

Therefore (1.1) is a hyperbola.

5. AN EXTREMAL PROBLEM

CHANGING COORDINATES with an orthogonal matrix has the advantage that distances and angles are preserved. This is because

$$P\mathbf{x} \cdot P\mathbf{y} = (P\mathbf{x})^t P\mathbf{y} = \mathbf{x}^t P^t P\mathbf{y} = \mathbf{x}^t \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Thus such coordinate changes are appropriate in the following type of example.

Example 5.1. *Problem:* Find the points on the surface

$$2x^2 + 4y^2 + 2z^2 + 2xz = 4 \tag{5.1}$$

in \mathbb{R}^3 that lie closest the origin and those that lie furthest from the origin.

Solution: The matrix corresponding to the quadratic form $2x^2 + 2y^2 + z^2 + 2xz$ is

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 0 & -1 \\ 0 & \lambda - 4 & 0 \\ -1 & 0 & \lambda - 2 \end{bmatrix}.$$

Expanding this determinant along the middle row we get

$$\begin{aligned}\det(\lambda I - A) &= (\lambda - 4) \det \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = (\lambda - 4)(\lambda^2 - 4\lambda + 3) \\ &= (\lambda - 4)(\lambda - 3)(\lambda - 1),\end{aligned}$$

so the eigenvalues of A are 1, 3 and 4. By the Theorem there is an orthogonal change of coordinates $\mathbf{x} = P\tilde{\mathbf{x}}$ so that

$$P^t A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (5.2)$$

and in this coordinate system equation (5.1) becomes

$$\tilde{x}^2 + 3\tilde{y}^2 + 4\tilde{z}^2 = 4. \quad (5.3)$$

This is an ellipsoid with principal axes aligned with the new coordinate system. The points of (5.3) lying on the new coordinate axes are found by setting two of the variables \tilde{x} , \tilde{y} and \tilde{z} equal to zero and solving (5.3) for the remaining one. Thus the extreme points are among $(\pm 2, 0, 0)$, $(0, \pm 2/\sqrt{3}, 0)$ and $(0, 0, \pm 1)$. Of these $(\pm 2, 0, 0)$ is furthest from $\mathbf{0}$ and $(0, 0, \pm 1)$ is closest.

We now need to find the coordinates of these points in the original coordinate system, i.e., we need $P \begin{bmatrix} \pm 2 \\ 0 \\ 0 \end{bmatrix}$ and $P \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}$. For these we only need to know the first and last columns of P .

The first column is given by a unit eigenvector with eigenvalue 1 (the first entry in (5.2)). To find this eigenvector we solve

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (I - A) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so $z = -x$ and $y = 0$. The unit eigenvector is $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Thus the points on (5.1) furthest from $\mathbf{0}$ are $\pm 2\mathbf{v}_1 = \pm(\sqrt{2}, 0, -\sqrt{2})$.

For the closest points, we need a unit eigenvector with eigenvalue 4 (the last entry in (5.2)). Such a vector is $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the closest points are $\pm\mathbf{v}_3 = (0, \pm 1, 0)$.

6. ARBITRARY CHANGES OF COORDINATE

AS THE discussion of classification of curves shows, often one only needs to know the *signs* of the eigenvalues. In fact, if we allow arbitrary, rather than orthogonal, coordinate changes this is the only information that remains about a quadratic form.

Theorem 6.1. *Let q_A be a quadratic form with matrix A . Then there is an invertible matrix Q such that*

$$q_{Q^t A Q}(\tilde{x}_1, \dots, \tilde{x}_n) = \tilde{x}_1^2 + \dots + \tilde{x}_t^2 - \tilde{x}_{t+1}^2 - \dots - \tilde{x}_r^2,$$

where r is the rank of A , and t is the number of positive eigenvalues of A .

Proof. By Theorem 3.1 there is an orthogonal matrix P diagonalising A . Moreover, we may choose P so that diagonal entries of $P^t A P$ are ordered as we wish. Lets have the positive eigenvalues first, the negative ones next and the zero eigenvalues last. Fix P so that

$$P^t A P = \text{diag}(\mu_1^2, \dots, \mu_t^2, -\mu_{t+1}^2, \dots, -\mu_r^2, 0, \dots, 0),$$

with $\mu_i > 0$ for $i \leq r$. Let D be the diagonal matrix

$$D = \text{diag}\left(\frac{1}{\mu_1}, \dots, \frac{1}{\mu_t}, \frac{1}{\mu_{t+1}}, \dots, \frac{1}{\mu_r}, 1, \dots, 1\right).$$

Then D is an invertible matrix such that $D(P^t A P)D$ has the desired form. As D is symmetric, we take $Q = PD$, so $Q^t A Q = D^t P^t A P D = D P^t A P D$, as required. \square

Example 6.2. In Example 3.4, we have chosen P so that the eigenvalues come out in the wrong order. However, this may be remedied by swapping the columns. So we use

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The eigenvalues are now 9 and -1 , so we take $D = \text{diag}(1/3, 1)$ and Q becomes

$$Q = PD = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}.$$

One can now check that $Q^t A Q = \text{diag}(1, -1)$.

Definition 6.3. The *rank* of q_A is the rank of A . The *signature* of q_A is the number of positive eigenvalues of A minus the number of negative eigenvalues.

For $A \in M(n, n)$, q_A is *non-degenerate* if its rank is n . q_A is *positive definite* if its signature equals n , i.e., if all eigenvalues of A are positive.

In the notation of the Theorem, the rank of q_A is r , the signature of q_A is $s := 2t - r$. Note that $t = (s + r)/2$, so the rank and signature determine the form of q in the Theorem.

Example 6.4. If the rank is 4, q is non-degenerate and the signature -2 , then there is coordinate system in which $q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 - x_3^2 - x_4^2$, which is a Lorentz metric.

7. CRITICAL POINTS OF SMOOTH FUNCTIONS

LET $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. The critical points of f are points satisfying

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \dots, \quad \frac{\partial f}{\partial x_n} = 0.$$

For example, suppose $f(x, y, z) = 4x^2 - 3xy + y^2 - \cos(z) + 1$. Then we have

$$\frac{\partial f}{\partial x} = 8x - 3y, \quad \frac{\partial f}{\partial y} = -3x + 2y, \quad \frac{\partial f}{\partial z} = \sin(z),$$

so there are critical points at $(x, y, z) = (0, 0, n\pi)$ for n an integer.

Suppose $\mathbf{0}$ is a critical point of f and adjust f so that $f(\mathbf{0}) = 0$. The nature of the critical point is determined by the *Hessian* of f which is the symmetric $n \times n$ matrix of all second partial derivatives $\text{Hess}(f) = [\partial^2 f / \partial x_i \partial x_j]$, since by Taylor's Theorem we have that f is approximated near $\mathbf{0}$ by

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{0}) x_i x_j.$$

The Hessian defines a quadratic form, and Theorem 6.1 implies that there are local coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$ such that $f(\tilde{\mathbf{x}})$ near $\mathbf{0}$ is approximated by

$$\tilde{x}_1^2 + \dots + \tilde{x}_t^2 - \tilde{x}_{t+1}^2 - \dots - \tilde{x}_r^2,$$

where r is the rank of $\text{Hess}(f)$ and t is the number of positive eigenvalues.

Suppose that $r = n$. Then the critical point is a

Local Minimum: if $t = n$, i.e., all the eigenvalues are positive,

Local Maximum: if $t = 0$, i.e., all the eigenvalues are negative,
Saddle Point: otherwise.

In our example, we have

$$\text{Hess}(f) = \begin{bmatrix} 8 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Expanding by the last row we get that the characteristic polynomial of this matrix is $(\lambda - 1)((\lambda - 8)(\lambda - 2) - 9) = (\lambda - 1)(\lambda^2 - 10\lambda + 5)$. All the roots of this equation are positive so the critical point is a local minimum.

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