DM515: Introduction to Linear and Integer Programming — Ugeseddel 2

Stuff covered in week 14 MG: pages 1-42.

Lecture April 11 in U20: We will cover sections 4.1-4.2 and most of Chapter 5 on the simplex method.

Exercises April 13 in U140:

- Consider the flow problem in Section 2.2. By a **path flow** (from o to n) we mean a directed path from o to n which caries the same amount of flow on each arc of the path. E.g. the path $oadn$ with flow 1 along it is a path flow of value 1. We will assume that path flows always carry non-negative flow.

  1. Show that the solution shown on page 15 can be written as a sum of path flows as defined above. Here, by "sum" we mean that for each directed edge (called an arc) we add the values of flow for all path flows that use this arc (e.g. for the arc oa we may get the flow 2 by adding the two path flows $oadn$ and $oaben$, each of value 1.
  2. Argue that there is always some optimum solution to the file transfer problem (the orientation and flow values) for which the digraph formed by the edges with positive flow does not contain a directed cycle. Lets call such a solution **cycle-free**. Prove that every cycle-free solution to the file transfer problem (also if we use a much more complicated network between the old and the new computer) can be expressed as a sum of path flows (from o to n).
  3. Describe (but do not write out the full model!) how one could formulate the file transfer problem for the example on page 14 as a linear program where some variables correspond to paths from o to n and the amount of flow on such a path. How do you model that an arc cannot be used in both directions?
  4. What can you say about the efficiency of this approach?

- Consider the flow problem in Section 2.2. Suppose now that we have to pay a fee to use each link for sending one Mbit/s. The price is linear and depends on how much we transfer via the arc, so that for each edge ij we use, we pay $|x_{ij}| * c_{ij}$ and the total price is the sum over all arcs of this price.

  1. The condition above implies that we have to pay the same no matter in which direction we use an arc. We know from the figure on page 15 that we can obtain a file transfer of 4Mbit/s. Show how to model the extended problem of finding such a flow-rate at the lowest cost (here the costs are denoted c_{oa}, c_{ob}, etc.).
  2. Suppose that we are running on a tight budget B and may be willing to accept a smaller rate if we can get a solution of cost at most B per second. We do not take into account that we would then have to transfer for a longer time, the price to use an arc is (unrealistically) assumed to be independent of this. Show how to extend the model above so that we are looking for a maximum transfer rate subject to the condition that we may not violate a given budget B.

- Consider the problem from Section 2.3 and now assume that the demands are not as in the figure but are given values d_i, $i = 1, 2, \ldots, 12$.

  1. Suppose there is also an extra cost on production in each period so that it costs p_i euros to produce one ton of ice cream in month $i = 1, 2, \ldots, 12$. Suppose also that in each month i, the production has to lie between prescribed lower and upper bounds ℓ_i, u_i, $i = 1, 2, \ldots, 12$. Give a formulation of this modified problem as a linear programming problem
  2. Does this modified problem always have a feasible (lovlig) solution?
  3. Can you give a necessary and sufficient condition for the existence of a feasible solution in terms of the bounds on the production?
  4. Show how to extend the model above so that we may start with $s_0 > 0$ and allow that $s_{12} > 0$ also.

- Consider the problem of fitting a line to a set of points from Section 2.4.

  1. Suppose that instead of minimizing the sum $S = \sum_{i=1}^{n} |ax_i + b - y_i|$, we want to find a line which minimizes $M = \max_{i=1}^{n} |ax_i + b - y_i|$. Show how to formulate this problem as a linear programming problem.
  2. Discuss whether (and if so how) one can find a line that both minimizes the sum S above and at the same time also minimizes M (among solutions with sum S).

- Questions 2.4.1 to 2.4.8 in Gutin's notes (see homepage of course).

- The **Steiner Tree problem** for graphs is the following: Given connected graph $G = (V, E)$, a weight function $w : E \rightarrow \mathcal{R}_+$ and a subset $X \subseteq V$. Find a tree T of minimum weight such that T contains all vertices from X. Thus if $X = V$ we have the minimum spanning tree problem.

  1. Formulate an integer programming model for the Steiner Tree problem.
  2. Formulate the LP-relaxation of this model and discuss the quality of the the lower bound we obtain by solving the LP problem to optimality.

- Formulation of an IP-problem

  In the strategic manpower planning problem we consider a strategic assessment of the demand of manpower for an organization. Consider an organization that is open 7 days a week with 1 shift a day.

  - The number of employees needed varies from day to day but is constant on a weekly basis (we will call those numbers b1 for Monday, b2 for Tuesday etc.)
  - All employees must work 5 consecutive days and have two days off.
  - The objective is to minimize the number of employees and find out when they have to work.

  (a) Formulate a mathematical model that solves the strategic manpower planning problem as described above.

  (b) Solve the problem for b1=10, b2=5, b3=10, b4=5, b5=10, b6=5, b7=10.

  (c) Solve the problem for b1=8, b2=8, b3=8, b4=8, b5=8, b6=8, b7=7.

  (d) How can you compute the unused number of man days? Calculate the number of unused days for the two examples above.

  (e) The labour costs vary depending upon the days off. How can we change the objective function to incorporate that?

- Extend the proof of Theorem 3.2.1 to the case when $|X| < |Y|$.

Practical exercises in IMADA's terminal room April 14, 12-14: See the home page for examples that you will work on. Before showing up in the terminalroom you should have read the ZIMPL manual which is available from the homepage of the course (bottom).