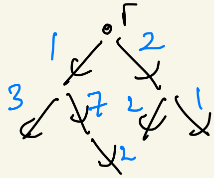
BJG 2nd edition pages 342-345

Minimum cost branchings



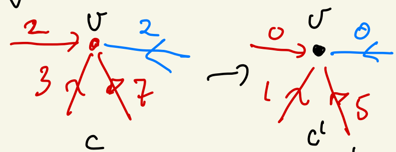
Given $D=(V,A,c)$ $c: A \to \mathbb{R}$ and $r \in V$

Find out-branching B_r^+ s.t. $c(B_r^+) \leq c(\hat{B}_r^+)$

for all out-branchings $\hat{B}_r^+$ from r

<u>lemma 9.2.1</u> Given $D=(V,A,c)$ let $y_v = \min\{c(uv) \mid uv \in A\}$

(y_v is minimum cost of an arc entering v)



let $c'(uv) = c(uv) - y_v$

Then B_r^+ optimal with respect to $c \iff B_r^+$ is optimal wrt c'

P:

$$c(B_r^+) = c'(B_r^+) + \underbrace{\sum_{v \neq r} y_v}_{\text{constant}}$$

□

Given (D, c, r) let $F^* = \{$one min cost arc entering v for each $v \neq r\}$

Then $d^-_{F^*}(v) = 1 \quad \forall v \in V - r$ and

$\boxed{c'(uv) = 0 \quad \text{for every arc } uv \in F^*}$

So if F^* is an out-branching then it is optimal

since $c'(a) \geq \forall a \in A$

Suppose F^* contains a cycle
it must be directed as $d^-_{F^*}(v)=1 \ \forall v \neq r$

Lemma 9.2.3

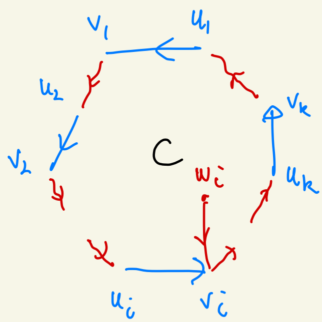
There exists an optimum (min cost) branching rooted at r which contains all but one arc of every cycle in F^*

P: Let B_r^t be a min cost branching from r such that

(□) $|A(B_r^t)\cap F^*|$ is maximized among all out-branchings from r

Let $C \subseteq F^*$ be a cycle and let $A(C)-A(B_r^t)=\{u_1v_1, u_2v_2, \dots, u_kv_k\}$

Suppose $k \geq 2$



all red pieces are part of B_r^t

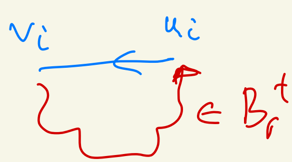
For each $i \in [k]$ let

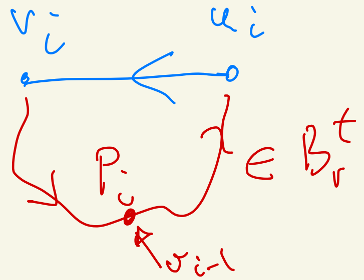
$a_i = w_iv_i$

be the arc entering v_i in B_r^t

$H_i = B_r^t + u_iv_i - w_iv_i$ is not an out-branching by (□)

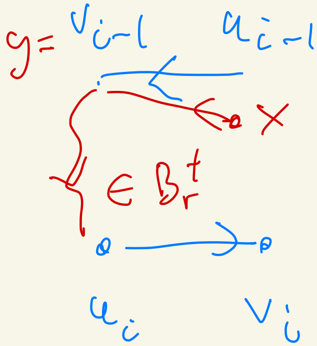
So H_i contains





Let xy be the last arc of P_i which is not on C

Then $y = v_{i-1}$ as $C[v_{i-1}, u_i] \in B_r^t$

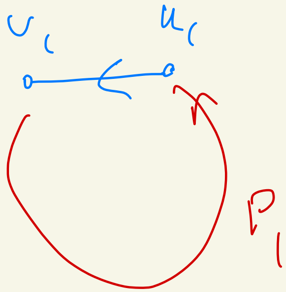


Conclusion B_r^t contains a (v_i, v_{i-1})-path

This holds for each $i \in [k]$

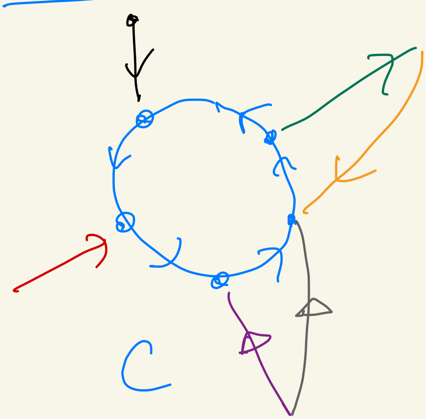
so B_r^t contains a cycle $\rightarrow \Leftarrow$

<u>note:</u> if $k=1$ then

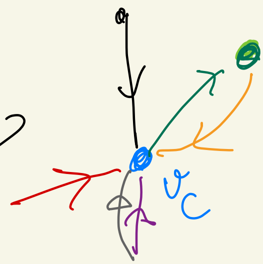


which is not a contradiction and B_1^+ contains all arcs of C except u_1v_1

Contracting a cycle in (D, C):



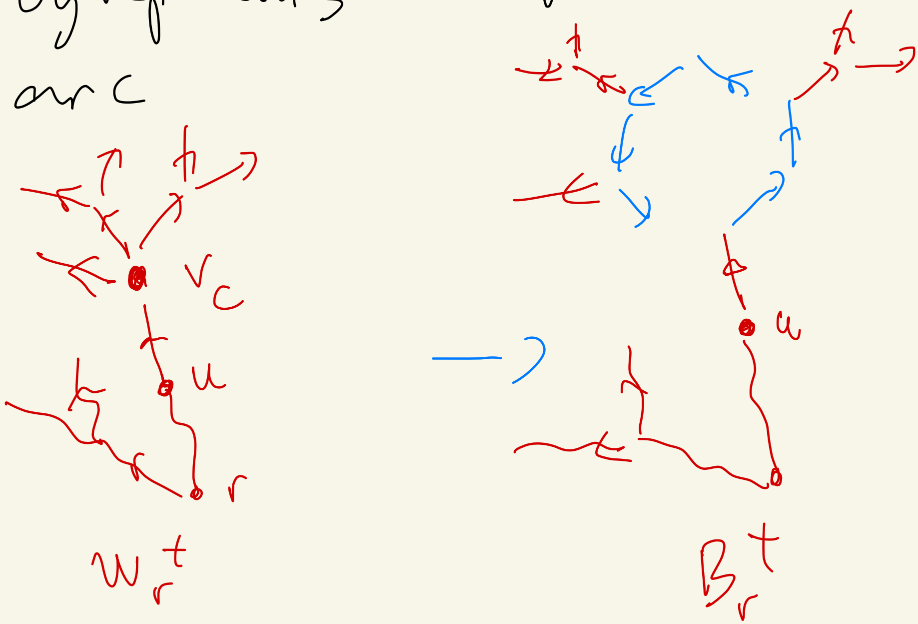
D



D/C

Lemma 9.2.4 Let C be a cycle in F^* and let W_r^t be optimum out-branching from r in D/C wrt c'

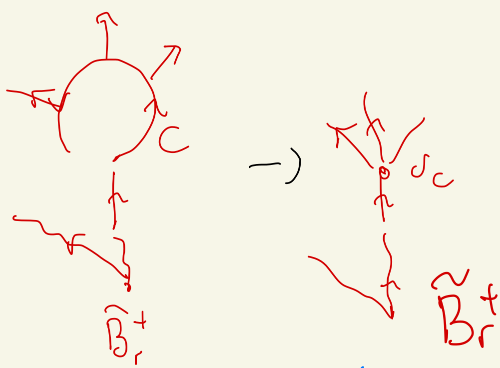
Then we can obtain an optimum out-branching B_r^t in D wrt c' (and c) by replacing v_C by C minus one arc



$$c'(W_r^t) = c'(B_r^t)$$

Proof By lemma 9.2.3 there exist an optimum out-branching $\hat{B}_r^+$ in D which contains all arcs of C except one

If we contract C into s_C, $\hat{B}_r^+$ becomes an out-branching $\tilde{B}_r^+$ in D/C



$c'(pq) = 0$ for every arc pq of C so

$$c'(\hat{B}_r^+) = c'(\tilde{B}_r^+) \geq c'(W_r^+) = c'(B_r^+)$$

$\Rightarrow$ B_r^+ is optimum w.r.t c'

$\Rightarrow$ B_r^+ is optimum w.r.t c

Theorem 9.5 We can find a min cost branching B_r^+ in (D, c, r) in pol time.

P: On input (D, c, r):

1. Check whether r can reach all other vertices and stop if No
2. For $v \in V - r$: $y_v \leftarrow \min\{c(uv) \mid uv \in A\}$
3. For $v \in V - r$: fix one arc a_v entering v with $c(a_v) = y_v$
4. Let $F^* = \{a_v \mid v \in V - r\}$
5. If F^* is a branching (no cycle) return F^*

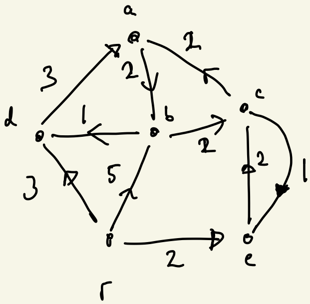
   Else let $C \subseteq F^*$ be a cycle

   a. $D \leftarrow D/C$

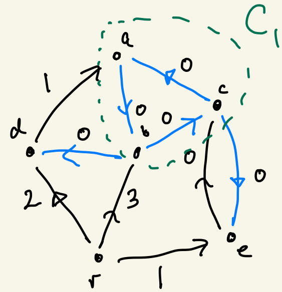
   b. $c' \leftarrow c - y$ $\quad (c'(uv) \leftarrow c(uv) - y_v \ \forall uv \in A)$

   c. Solve recursively on $(D/C, c', r)$

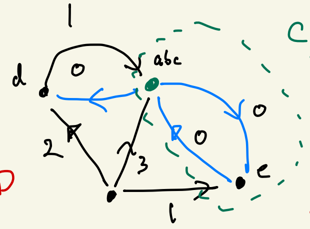
   d. Blow up C again and return the resulting branching



$D, c \quad \sum y_v = 8$

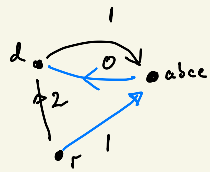


D, c'



$\sum y = 0$

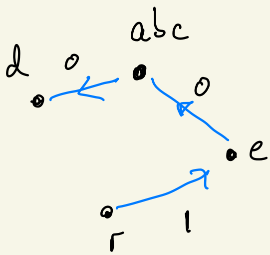
$D/C_1, c' = D/C_1, c^2$



$D/C_1/C_2, c^2$

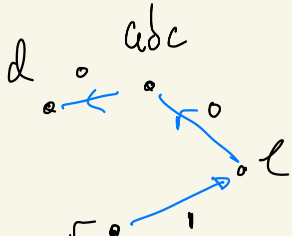
$\sum y = 1$

F^* is a Branching so blow up C_2



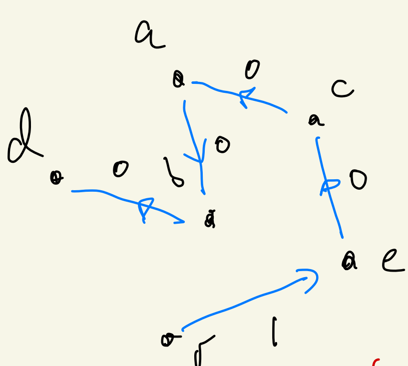
in $D/C', c^2$

$\sim$

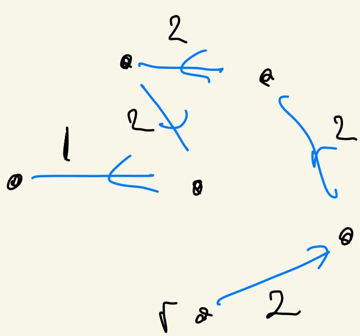


in $D/C', c'$

blow up C'



cost function c'



total cost 9

cost function C

Note (without proof)

cost of the final out-branching

= sum of all y values
during the algorithm.