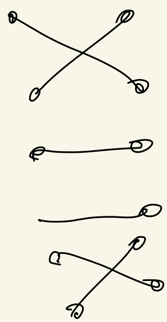
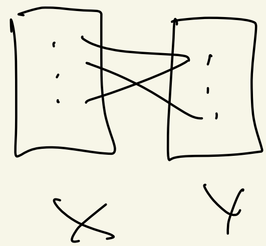
Matchings in bipartite graphs

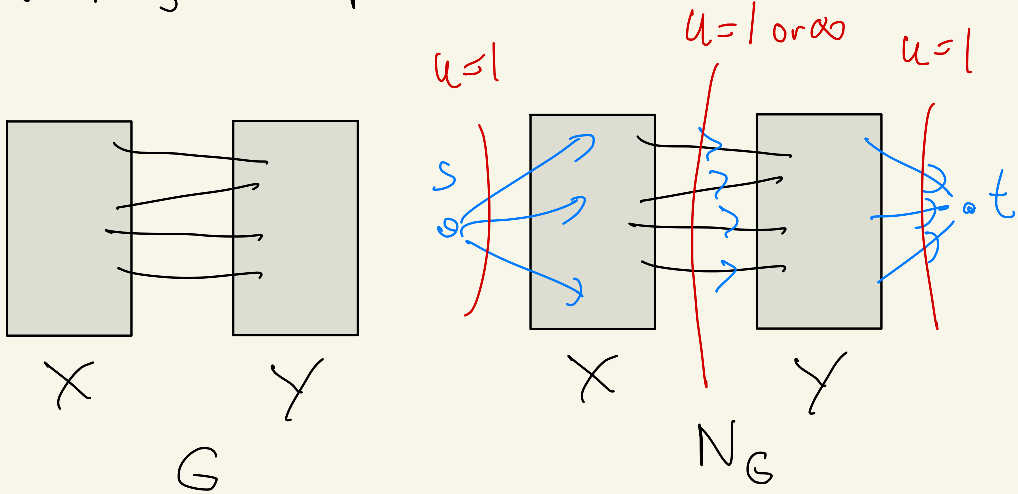
matching



bipartite:



Converting a bipartite graph to a network



Claim $\max\{|M| \mid M \text{ matching in } G\}$

$= \max\{|x| \mid x \text{ } (s,t)\text{-flow in } N_G\}$

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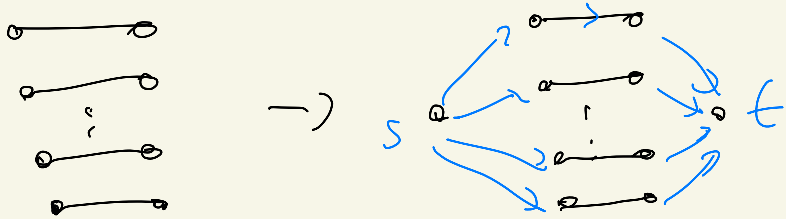
P: $\leq$



$|M| = k$

$x = 1$ on these arcs

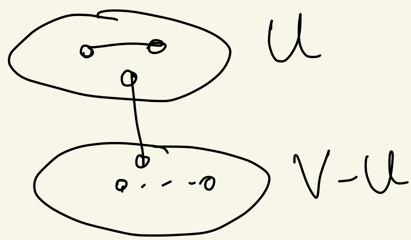
$x = 0$ otherwise

$\geq$ Suppose $|x^*|$ is a maximum integer valued (s,t)-flow. Look at arcs from X to Y with $x_{ij} = 1$. Then form a matching of size $|x^*|$ $\square$

So if we have an algorithm for finding a maximum (s,t)-flow then we can find a maximum matching in a bipartite graph.

In particular, the Ford-Fulkerson algorithm finds a maximum matching in time $O(nm)$

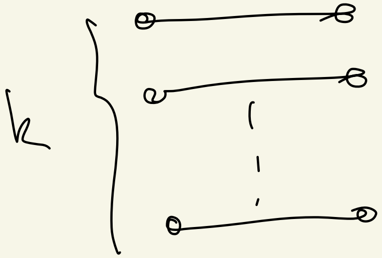
Vertex cover:



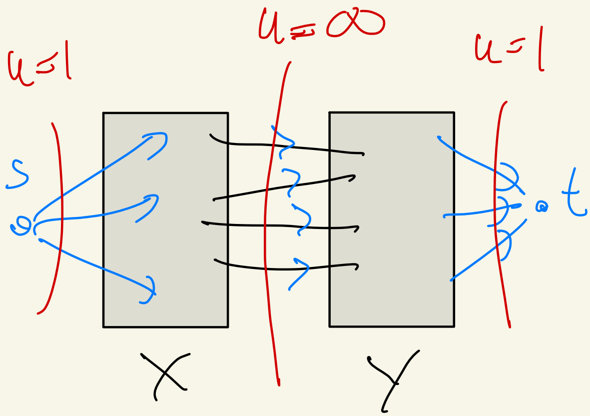
Theorem (König) G bipartite

The max size of a matching in G

= The size of a vertex cover in G

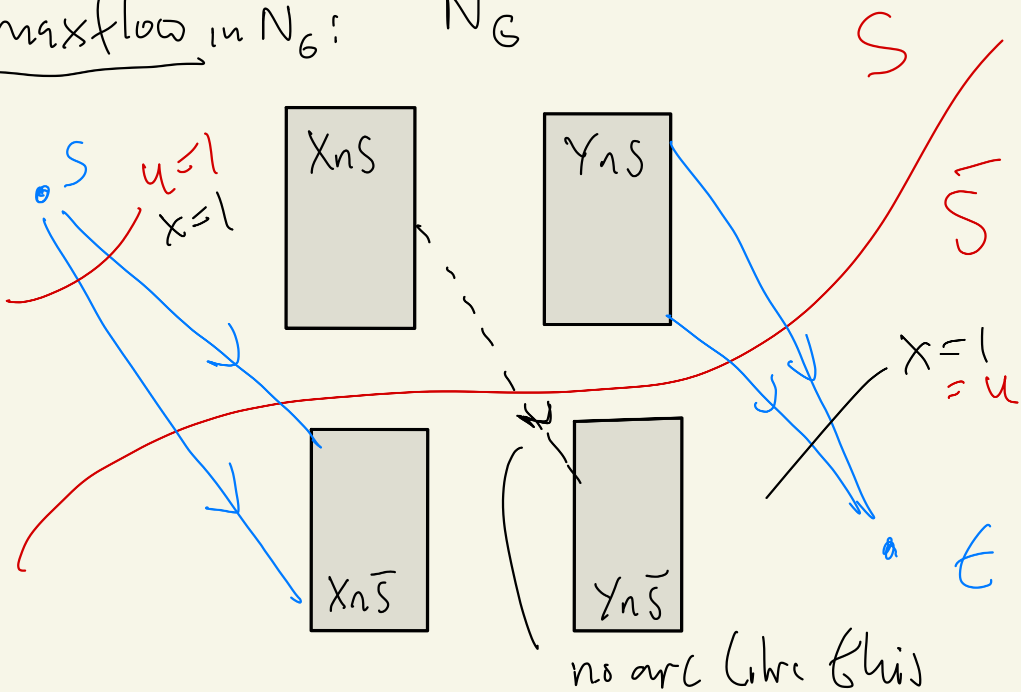


$\Rightarrow$ min VC $\geq k$

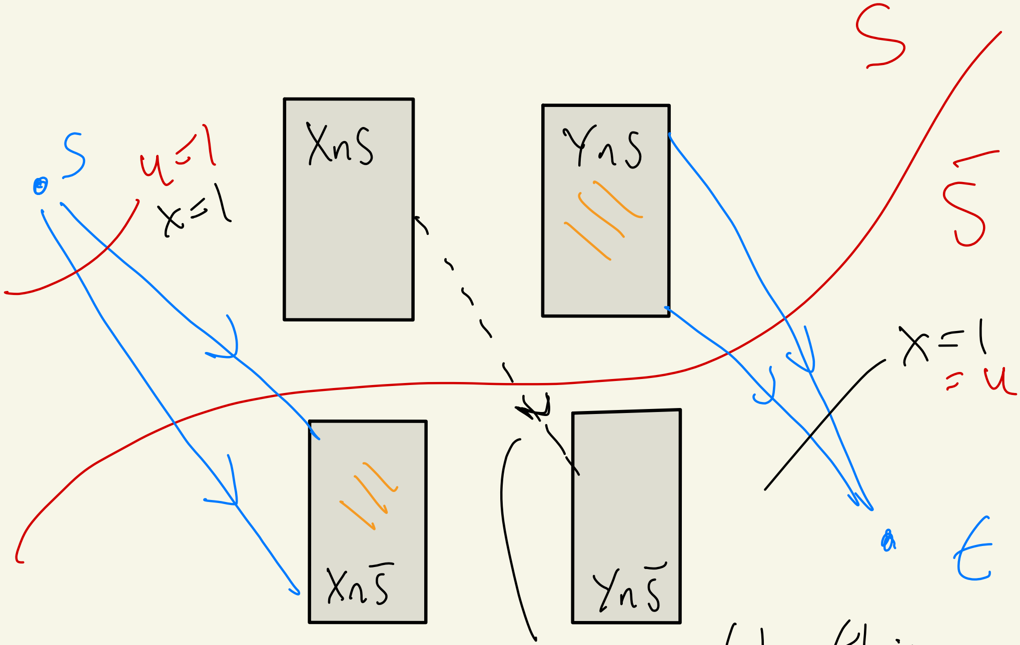


N_G

x maxflow in N_G:



X is a max flow



$$|M^*| = |x| = u(S, \bar{S})$$

$$= |X \cap \bar{S}| + |Y \cap S|$$

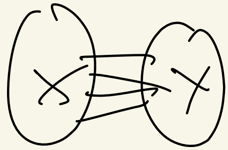
$$= |Z|$$

$$(Z = (X \cap \bar{S}) \cup (Y \cap S))$$

Z is a vertex cover and

$|Z| = |M^*|$ $\square$

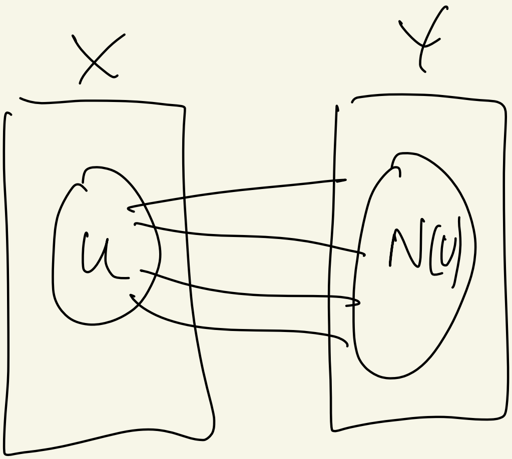
Theorem (Hall)



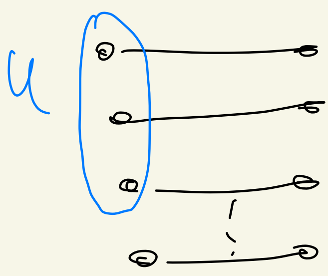
Let $G=(X,Y,E)$ have $|X|=|Y|$

Then G has a perfect matching ($|M^*|=|X|$) if and only if

(*) $|N(U)| \geq |U| \quad \forall U \subseteq X$

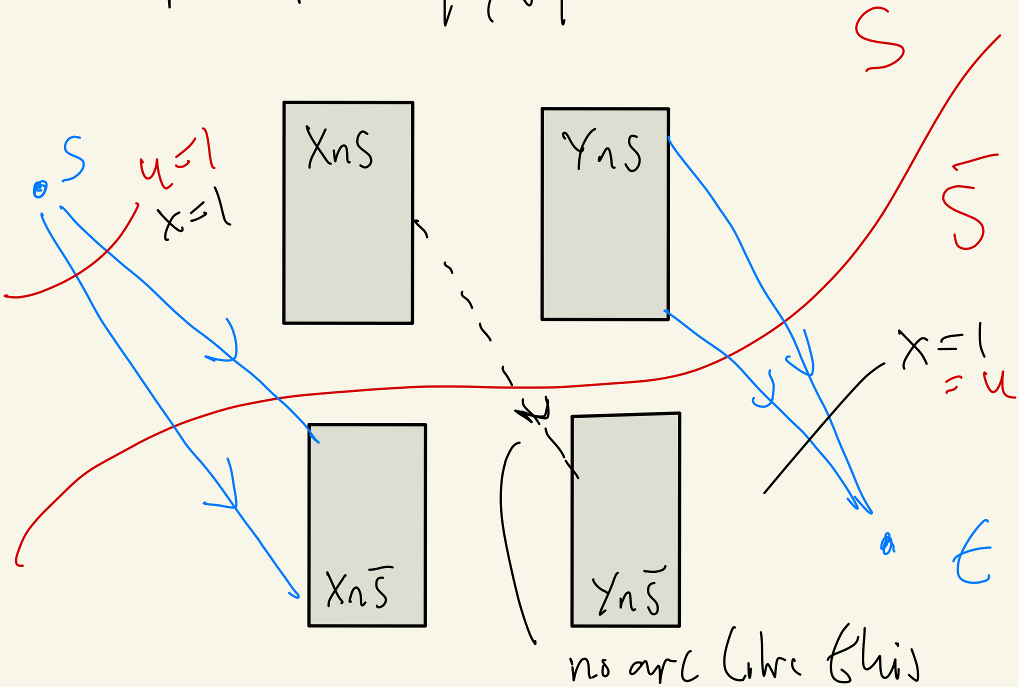


(*) necessary :



Suppose (*) holds but

$$|M^*| < |X|$$



$$|X \cap S| + |\cancel{X \cap \bar{S}}| = |X| > |M^*| = \delta_X(s) = u(S, \bar{S})$$

$$= |\cancel{X \cap \bar{S}}| + |Y \cap S|$$

$\Downarrow$

$$|X \cap S| > |Y \cap S| \geq |N(X \cap S)|$$

$\rightsquigarrow$ (*)

$\square$