Stationary Distributions

$$\bar{p}(t+1) = \bar{p}(t)\mathbf{P}$$

Definition

A *stationary distribution* (also called an *equilibrium distribution*) of a Markov chain is a probability distribution $\bar{\pi}$ such that

$$\bar{\pi} = \bar{\pi}\mathbf{P}.$$

Theorem

Any finite, irreducible, and aperiodic (ergodic) Markov chain has the following properties:

1. *The chain has a unique stationary distribution* $\bar{\pi} = (\pi_0, \pi_1, \ldots, \pi_n)$*;*
2. *For all* j *and* i*, the limit* $\lim_{t\to\infty} P^t_{j,i}$ *exists and it is independent of* j*;*
3. $\pi_i = \lim_{t\to\infty} P^t_{j,i} = \frac{1}{h_{i,i}}$.

- For any distribution vector $\bar{p}$

$$\bar{\pi} = \lim_{t \to \infty} \bar{p} \mathbf{P}^t .$$

- $$\frac{1}{\pi_i} = h_{i,i} = \sum_{t=1}^{\infty} t \cdot r_{i,i}^t$$

Proof

We use:

Lemma

For any irreducible, ergodic Markov chain, and for any state i, *the limit* $\lim_{t\to\infty} P_{i,i}^t$ *exists, and*

$$\lim_{t\to\infty} P_{i,i}^t = \frac{1}{h_{i,i}}.$$

Using the fact that $\lim_{t\to\infty} P^t_{i,i}$ exists, we now show that for any j and i $\lim_{t\to\infty} P^t_{j,i} = \lim_{t\to\infty} P^t_{i,i} = \frac{1}{h_{i,i}}$.

For $j \neq i$ we have $P^t_{j,i} = \sum_{k=1}^{t} r^k_{j,i} P^{t-k}_{i,i}$.

For $t \geq t_1$, $\sum_{k=1}^{t_1} r^k_{j,i} P^{t-k}_{i,i} \leq \sum_{k=1}^{t} r^k_{j,i} P^{t-k}_{i,i} = P^t_{j,i}$.

Since the chain is irreducible $\sum_{t=1}^{\infty} r^t_{j,i} = 1$. For any $\epsilon > 0$ there exists (a finite) $t_1 = t_1(\epsilon)$ such that $\sum_{t=1}^{t_1} r^t_{j,i} \geq 1 - \epsilon$.

$$
\begin{aligned}
\lim_{t\to\infty} P^t_{j,i} &\geq \lim_{t\to\infty} \sum_{k=1}^{t_1} r^k_{j,i} P^{t-k}_{i,i} = \sum_{k=1}^{t_1} r^k_{j,i} \lim_{t\to\infty} P^t_{i,i} \\
&= \lim_{t\to\infty} P^t_{i,i} \sum_{k=1}^{t_1} r^k_{j,i} \geq (1-\epsilon) \lim_{t\to\infty} P^t_{i,i}.
\end{aligned}
$$

Similarly,

$$P_{j,i}^{t} = \sum_{k=1}^{t} r_{j,i}^{k} P_{i,i}^{t-k} \leq \sum_{k=1}^{t_1} r_{j,i}^{k} P_{i,i}^{t-k} + \epsilon,$$

$$\begin{aligned}
\lim_{t\to\infty} P_{j,i}^{t} &\leq \lim_{t\to\infty} (\sum_{k=1}^{t_1} r_{j,i}^{k} P_{i,i}^{t-k} + \epsilon) \\
&= \sum_{k=1}^{t_1} r_{j,i}^{k} \lim_{t\to\infty} P_{i,i}^{t-k} + \epsilon \\
&\leq \lim_{t\to\infty} P_{i,i}^{t} + \epsilon.
\end{aligned}$$

For any pair i and j

$$\lim_{t\to\infty} P_{j,i}^{t} = \lim_{t\to\infty} P_{i,i}^{t} = \frac{1}{h_{i,i}}.$$

Let

$$\pi_i = \lim_{t\to\infty} P^t_{j,i} = \frac{1}{h_{i,i}}.$$

We show that $\bar{\pi} = (\pi_0, \pi_1, \dots)$ forms a stationary distribution.
For every $t \geq 0$, $P^t_{i,i} \geq 0$, and thus $\pi_i \geq 0$. For any $t \geq 0$, $\sum_{i=0}^{n} P^t_{j,i} = 1$, and thus

$$1 = \lim_{t\to\infty} \sum_{i=0}^{n} P^t_{j,i} = \sum_{i=0}^{n} \lim_{t\to\infty} P^t_{j,i} = \sum_{i=0}^{n} \pi_i,$$

and π is a proper distribution. Now,

$$P^{t+1}_{j,i} = \sum_{k=0}^{n} P^t_{j,k} P_{k,i}.$$

Letting $t \to \infty$ we have

$$\pi_i = \sum_{k=0}^{n} \pi_k P_{k,i},$$

proving that $\bar{\pi}$ is a stationary distribution.

Suppose that there was another stationary distribution $\bar{\phi}$.

$$\phi_i = \sum_{k=0}^{n} \phi_k P^t_{k,i},$$

and taking the limit as $t \to \infty$ we have

$$\phi_i = \sum_{k=0}^{n} \phi_k \pi_i = \pi_i \sum_{k=0}^{n} \phi_k .$$

Since $\sum_{k=0}^{n} \phi_k = 1$, we have $\phi_i = \pi_i$ for all i, or $\bar{\phi} = \bar{\pi}$.

Computing the Stationary Distribution

1. Solve the system of linear equations $\bar{\pi}\mathbf{P} = \bar{\pi}$.
2. Solving equilibrium equations.

Theorem

Let S be a set of states of a finite, irreducible, aperiodic Markov chain. In the stationary distribution, the probability that the chain leaves the set S equals the probability that it enters S.

Proof.

(Only for single states.) For any state i:

$$\sum_{j=0}^{n} \pi_j P_{j,i} = \pi_i = \pi_i \sum_{j=0}^{n} P_{i,j}$$

$$\sum_{j \neq i} \pi_j P_{j,i} = \sum_{j \neq i} \pi_i P_{i,j}.$$

Theorem

Consider a finite, irreducible, and ergodic Markov chain on n states with transition matrix $\mathbf{P}$. If there are non-negative numbers $\bar{\pi} = (\pi_0, \ldots, \pi_n)$ such that $\sum_{i=0}^{n} \pi_i = 1$, and for any pair of states i, j,

$$\pi_i P_{i,j} = \pi_j P_{j,i},$$

then $\bar{\pi}$ is the stationary distribution corresponding to $\mathbf{P}$.

Proof.

$$\sum_{i=0}^{n} \pi_i P_{i,j} = \sum_{i=0}^{n} \pi_j P_{j,i} = \pi_j.$$

Thus $\bar{\pi}$ satisfies $\bar{\pi} = \bar{\pi}\mathbf{P}$, and $\sum_{i=0}^{n} \pi_i = 1$, and $\bar{\pi}$ must be the unique stationary distribution of the Markov chain. $\square$

Theorem

Any irreducible aperiodic Markov chain belongs to one of the following two categories:

1. *The chain is ergodic. For any pairs of states i and j, the limit $\lim_{t\to\infty} P^t_{j,i}$ exists and is independent of j. The chain has a unique stationary distribution $\pi_i = \lim_{t\to\infty} P^t_{j,i} > 0$.*
   or
2. *No state is positive recurrent. For all i and j, $\lim_{t\to\infty} P^t_{j,i} = 0$, and the chain has no stationary distribution.*

Example: A Simple Queue

Discrete time queue.

At each time step, exactly one of the following occurs:

- If the queue has fewer than n customers, then with probability λ a new customer joins the queue.
- If the queue is not empty, then with probability μ the head of the line is served and leaves the queue.
- With the remaining probability the queue is unchanged.

$X_t =$ the number of customers in the queue at time t.

$$
\begin{aligned}
P_{i,i+1} &= \lambda \text{ if } i < n \\
P_{i,i-1} &= \mu \text{ if } i > 0 \\
P_{i,i} &= \begin{cases} 1-\lambda & \text{if } i = 0 \\ 1-\lambda-\mu & \text{if } 1 \le i \le n-1 \\ 1-\mu & \text{if } i = n. \end{cases}
\end{aligned}
$$

The Markov chain is irreducible, finite, and aperiodic, so it has a unique stationary distribution $\bar{\pi}$.

We use $\bar{\pi} = \bar{\pi}\mathbf{P}$ to write

$$\begin{aligned} \pi_0 &= (1-\lambda)\pi_0 + \mu\pi_1, \\ \pi_i &= \lambda\pi_{i-1} + (1-\lambda-\mu)\pi_i + \mu\pi_{i+1},\ 1 \le i \le n-1, \\ \pi_n &= \lambda\pi_{n-1} + (1-\mu)\pi_n. \end{aligned}$$

$$\pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i$$

Adding the requirement $\sum_{i=0}^{n} \pi_i = 1$, we have

$$\sum_{i=0}^{n} \pi_i = \sum_{i=0}^{n} \pi_0 \left(\frac{\lambda}{\mu}\right)^i = 1,$$

$$\pi_0 = \frac{1}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i}.$$

For all $0 \leq i \leq n$,

$$\pi_i = \frac{\left(\frac{\lambda}{\mu}\right)^i}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i}. \tag{1}$$

Use cut sets to compute the stationary probability:
For any i, the transitions $i \to i+1$ and $i+1 \to i$ are a cut-set.

$$\lambda \pi_i = \mu \pi_{i+1}.$$

By induction

$$\pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i.$$

Removing the limit on n, the Markov chain is no longer finite.
The Markov chain has a countably infinite state space. It has a stationary distribution if and only if the following set of linear equations has a solution with all $\pi_i > 0$:

$$
\begin{aligned}
\pi_0 &= (1-\lambda)\pi_0 + \mu\pi_1 \\
\pi_i &= \lambda\pi_{i-1} + (1-\lambda-\mu)\pi_i + \mu\pi_{i+1}, \, , i \geq 1.
\end{aligned}
$$

$$
\pi_i = \frac{\left(\frac{\lambda}{\mu}\right)^i}{\sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^i} = \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right)
$$

is a solution of the above system of equations.
All of the π_i are greater than 0 if and only if $\lambda < \mu$.
If $\lambda > \mu$, no stationary distribution, each state in the Markov chain is transient.
If $\lambda = \mu$ there is no stationary distribution, and the queue length will become arbitrarily long, but now the states are null recurrent.

Random Walks on Undirected Graph

Let $G = (V, E)$ be a finite, undirected, and connected graph.

Definition

A *random walk* on G is a Markov chain defined by the movement of a particle between vertices of G. In this process, the place of the particle at a given time step is the state of the system. If the particle is at vertex i, and i has $d(i)$ outgoing edges, then the probability that the particle follows the edge (i, j) and moves to a neighbor j is $1/d(i)$.

Lemma

A random walk on an undirected graph G is aperiodic if and only if G is not bipartite.

Proof.

If the graph is bipartite then the random walk is periodic, with a period $d = 2$.
If the graph is not bipartite, then it has an odd cycle, and by traversing that cycle we have an odd length path from any vertex to itself. Since we also have a path of even length 2 from any vertex to itself, the walk cannot be periodic. $\square$

Theorem

A random walk on G converges to a stationary distribution π, where

$$\pi_v = \frac{d(v)}{2|E|}.$$

Proof.

Since $\sum_{v \in V} d(v) = 2|E|$,

$$\sum_{v \in V} \pi_v = \sum_{v \in V} \frac{d(v)}{2|E|} = 1,$$

and π_v is a proper distribution over $v \in V$.
Let $N(v)$ be the set of neighbors of v. The relation $\bar{\pi} = \bar{\pi}\mathbf{P}$ gives

$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2|E|} \frac{1}{d(u)} = \frac{d(v)}{2|E|}$$

$h_{v,u}$ denotes the expected number of steps to reach u from v.

Corollary

For any vertex u in G,

$$h_{u,u} = \frac{2|E|}{d(u)}.$$

Lemma

If $(u, v) \in E$, *then* $h_{v,u} < 2|E|$.

Proof.

Let $N(u)$ be the set of neighbors of vertex u in G. We compute $h_{u,u}$ in two different ways.

$$\frac{2|E|}{d(u)} = h_{u,u} = \frac{1}{d(u)} \sum_{w \in N(u)} (1 + h_{w,u}).$$

Hence

$$2|E| = \sum_{w \in N(u)} (1 + h_{w,u}),$$

and we conclude that $h_{v,u} < 2|E|$. $\square$

Definition

The *cover time* of a graph G is the maximum over all vertices of the expected time to visit all nodes of the graph starting the random walk from that vertex.

Lemma

The cover time of $G = (V, E)$ *is bounded above by* $4|V| \cdot |E|$.

Proof.

Choose a spanning tree on G, and an Eulerian cycle on the spanning tree.
Let $v_0, v_1, \ldots, v_{2|V|-2} = v_0$ be the sequence of vertices in the cycle.

$$\sum_{i=0}^{2|V|-3} h_{v_i, v_{i+1}} + h_{v_{2|V|-2}, v_1} < (2|V| - 2)2|E| < 4|V| \cdot |E|,$$

□

Application: An $s - t$ Connectivity Algorithm

Given an undirected graph $G = (V, E)$, and two vertices s and t in G.
Let $n = |V|$ and $m = |E|$.
We want to determine if there is a path connecting s and t.
Easily done in $O(m)$ time and $\Omega(n)$ space.
$s - t$ **Connectivity Algorithm**

- Start a random walk from s.
- If the walk reaches t within $4n^3$ steps, return that there is a path. Otherwise, return that there is no path.

Theorem

The algorithm returns the correct answer with probability $1/2$, and it only errs by saying that there is no path from s to t when there is such a path.

Proof.

If there is no path, the algorithm returns the correct answer.
If there is a path, the expected time to reach t from s, is bounded by $4nm < 2n^3$.
By Markov's inequality, the probability that a walk takes more than $4n^3$ steps to reach s from t is at most $1/2$. $\square$

The algorithm must keep track of its current position, which takes $O(\log n)$ bits, and the number of steps taken in the random walk, which also takes only $O(\log n)$ bits (since we count up to only $4n^3$).