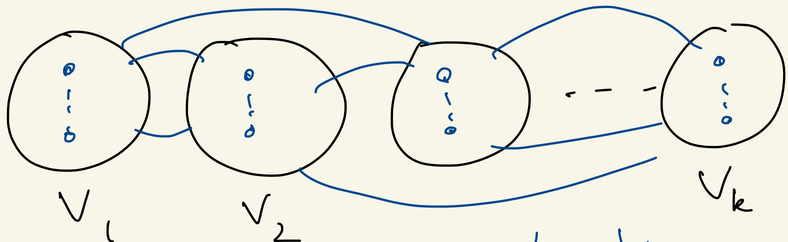
(Algorithmic) Properties of Chordal graphs

Denote by $\chi(G)$ the chromatic number of $G=(V,E)$
that is the minimum $k \geq 1$ s.t we can partition
V as $V = V_1 \cup V_2 \cup \dots \cup V_k$ and each V_i is
an independent set



Denote by $\omega(G)$ the size of a largest clique
(= complete subgraph) in G.

Clearly $\chi(G) \geq \omega(G)$

Definition: G is perfect
$\Updownarrow$
$\chi(G') = \omega(G')$ $\forall G'$ induced subgraph of G

Complete graphs are perfect and also chordal

We will show that every chordal graph
is perfect.

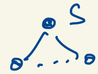
Theorem 4.9 Let G be a non-complete graph and let S be a vertex separator of G s.t $G-S$ has connected components $A_1, A_2 \ldots A_t$

Then $\chi(G) = \max_{i \in [t]} \chi(G[S \cup A_i])$

and $\omega(G) = \max_{i \in [t]} \omega(G[S \cup A_i])$

P: This follows from the fact that there are no edges between G_{A_i} and G_{A_j} for $i \neq j$ so every clique is a clique of some $G[S \cup A_i]$ and we can colour G by $\max_{i \in [t]} \chi(G[S \cup A_i])$ colours by colouring S optimally and then extend this colouring into each A_i □.

Corollary 4.10 Let G be connected and not complete. Let S be a separating set inducing a clique and $A_1, A_2 \ldots A_t$ the connected comp. of $G-S$. If $G[S \cup A_i]$ is perfect $\forall i \in [t]$, then G is perfect

p: induction on $|V(G)|$ base case  ✓

It suffices to show that $\chi(G) = \omega(G)$

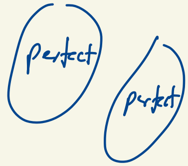
By Thm 4.9 and the assumption that $G[S \cup A_i]$ is perfect for $i \in [t]$

we get $\chi(G) = \max_{i \in [t]} \chi(G[S \cup A_i]) = \max_{i \in [t]} \omega(G[S \cup A_i]) = \omega(G)$

Theorem 4.11 Chordal graphs are perfect

P: Induction on $n = |V(G)|$

- If G is not connected we are done by induction



- If $G = K_n$ we are done

- So G has a non trivial separator S
  $G - S = A_1, A_2 \cdots A_t$

- $G[S]$ is a clique (by theorem 4.1)

- By induction $G[S \cup A_i]$ is perfect for $i \in [t]$

- Now Corollary 4.10 $\Rightarrow$ G is perfect $\square$.

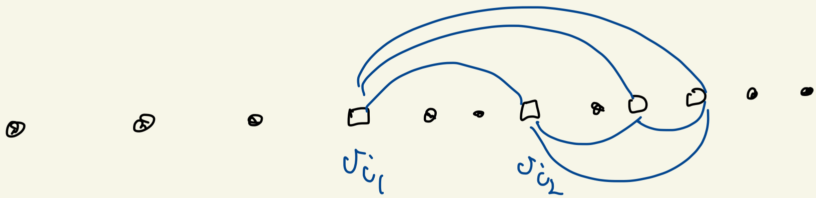
Proposition 4.12

Every chordal graph G on n vertices has at most n maximal cliques and equality holds only if $E(G) = \emptyset$

P: Let $\sigma=[v_1, v_2 \cdots v_n]$ by a p.e.s for G and consider a maximal clique A

Then $A=\{v_{i_1}, v_{i_2}, \dots, v_{i_{|A|}}\}$ where

$$\sigma^{-1}(v_{i_1})<\sigma^{-1}(v_{i_2})<\dots<\sigma^{-1}(v_{i_{|A|}})$$

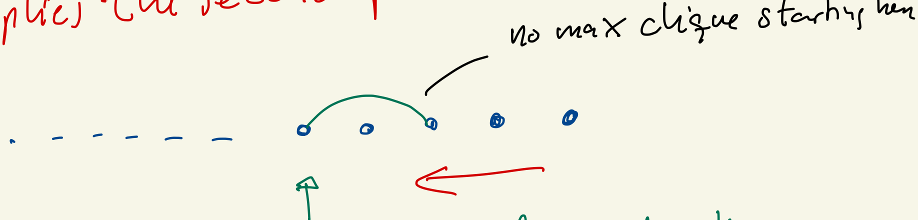


Now $A=\{v_{i_1}\}\cup X_{v_{i_1}}$ where $X_{v_{i_1}}=\{v_j \mid v_{i_1}v_j\in E \wedge j>i_1\}$

So there is at most one maximal clique starting at any vertex v_p.

Note that if some v_k with $k<i_1$ is adjacent to all of A then there is ~~no~~ maximal clique starting at v_{i_1}!

This implies the second part of the proposition

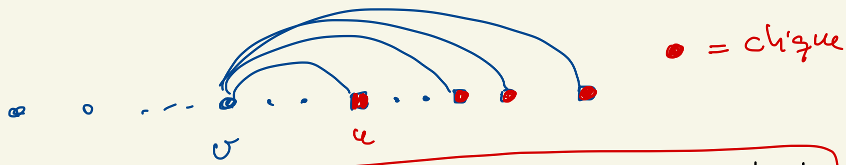


It is easy to list all the cliques of the form $v_{i_1} \cup X_{v_{i_1}}$, but how do we find the maximal ones?

Algorithm 4.3

Scan the vertices in the order according to $\sigma = [v_1 v_2 \cdots v_n]$

when considering vertex v we update $S(u)$:



$\boxed{S(u) = \text{size of largest clique currently known which starts at vertex } u}$

Note that there may have been a vertex v' before v in σ whose first vertex in $X_{v'}$ is also u and which gives a larger clique. In that case $S(u)$ does not change when scanning v.

When v is scanned we also test whether we should update $S(v)$: if $S(v) \leq |X_v|$ then $S(v) \leftarrow |X_v| + 1$

We can also update χ at the same time:

if current value for χ is less than $1 + |X_v|$

then $\chi \leftarrow 1 + |X_v|$

Construction a colouring with $\chi(G)$ colours:

- Start at v_n $\quad (\sigma = [v_1 v_2 \dots v_n])$
- assign colour 1 to v_n and so to v_{n-1}
- when considering v_i assign it the smallest colouring not used on a vertex in X_{σ_i}

This gives a colouring with $k = \omega(G)$ colours:

v_i is assigned colour $r+1$

$\Updownarrow$

$v_i \cup X_{\sigma_i}$ is an $(r+1)$-clique

Independence number of chordal graphs

We show how to produce an independent set I with $I = \alpha(G)$ and a covering of V by $\alpha(G)$ cliques (certifying) that I is a maximum independent set.

Let $\sigma = [v_1 v_2 \dots v_n]$ be a p.e.s. and define $y_1, y_2, \dots, y_t$ inductively:

$y_1 = \sigma(1) = v_1$

for $i > 1$ y_i is the first vertex after y_{i-1} s.t. $y_i \in V - (X_{y_1} \cup X_{y_2} \cup \dots \cup X_{y_{i-1}})$

stop when $V - (X_{y_1} \cup X_{y_2} \cup \dots \cup X_{y_t}) = \emptyset$

output $I = \{y_1, y_2, \dots y_t\}$

Theorem 4.18 The set $I = \{y_1, y_2, \ldots, y_t\}$ is a maximum independent set and $Y_1, Y_2, \ldots, Y_t$ is a minimum clique cover of G when $Y_i = y_i \cup X_{y_i}$

P: • $\{y_1, y_2 \ldots y_t\}$ is independent by construction

• Y_i is a clique for $i = 1, 2, \ldots, t$

Then two things imply that $\alpha(G) = t$ and $Y_1, Y_2, \ldots, Y_t$ is a minimum clique cover of $V(G)$. $\square$