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Arc-disjoint spanning sub(di)graphs in digraphs☆

Jørgen Bang-Jensen [a,*], Anders Yeo [b]

[a] Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark
[b] Department of Computer Science, Royal Holloway, University of London, Egham Surrey TW20 0EX, United Kingdom

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ABSTRACT

We prove that a number of natural problems concerning the existence of arc-disjoint directed and "undirected" (spanning) subdigraphs in a digraph are $\mathcal{NP}$-complete. Among these are the following of which the first settles an open problem due to Thomassé (see e.g. Bang-Jensen and Gutin (2009) [1, Problem 9.9.7] and Bang-Jensen and Kriesell (2009) [5,4]) and the second settles an open problem posed in Bang-Jensen and Kriesell (2009) [5].

- Given a directed graph D and a vertex s of D; does D contain an out-branching B_s^+ rooted at s such that the digraph remains connected (in the underlying sense) after removing all arcs of B_s^+?
- Given a strongly connected directed graph D; does D contain a spanning strong subdigraph D' such that the digraph remains connected (in the underlying sense) after removing all arcs of D'?

1. Introduction

Notation not given below is consistent with [1]. Paths and cycles are always directed unless otherwise specified. For a digraph D we denote by V(D) and A(D), respectively, the set of vertices and the set of arcs of D. An (s, t)-path in a digraph D is a directed path from the vertex s to the vertex t. A digraph $D = (V, A)$ is **strongly connected** (or just **strong**) if there exists an (x, y)-path and a (y, x)-path in D for every choice of distinct vertices x, y of D, and D is k-**arc-strong** if $D - X$ is strong for every subset $X \subseteq A$ of size at most $k - 1$. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and replacing multiple edges by one edge. A digraph D is **connected** if $UG(D)$ is a connected graph. If $D = (V, A)$ is a digraph and $X \subseteq V$ then we use the notation $D\langle X\rangle$ to denote the subdigraph of D **induced** by the vertices in X. We shall often use the shorthand notation $i \in [m]$ for $i \in \{1, 2, \dots, m\}$.

An **out-branching** B_s^+ in a digraph $D = (V, A)$ is a connected spanning subdigraph of D in which each vertex $x \neq s$ has precisely one arc entering it and s has no arcs entering it. The vertex s is the **root** of B_s^+. The structure of digraphs with arc-disjoint out-branchings from the same root is well understood due to the following important result by Edmonds.

Theorem 1.1 (*Edmonds [9]*). *A digraph $D = (V, A)$ with a special vertex s has k-arc-disjoint out-branchings rooted at s if and only if there are k-arc-disjoint (s, v)-paths in D for every $v \in V - s$.*

Using flows in networks, it is easy to check whether a given digraph D with special vertex s has k arc-disjoint (s, v)-paths for every $v \in V - s$ (see e.g. [1, Section 5.5]) and thus checking whether D has k arc-disjoint out-branchings

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* Corresponding author. Tel.: +45 65502335; fax: +45 65502325.
E-mail addresses: jbj@imada.sdu.dk (J. Bang-Jensen), anders@cs.rhul.ac.uk (A. Yeo).

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from s can be done efficiently. Furthermore, the proof of Theorem 1.1 by Lovász [11] implies that there is a polynomial algorithm for constructing a set of k arc-disjoint branchings when they exist (for details see [1, Section 9.3]). Similarly packing edge-disjoint spanning trees in undirected graphs is also well understood, namely there is a (more complicated) necessary and sufficient condition for the existence of k edge-disjoint spanning trees in a graph G.

Theorem 1.2 (*Tutte [13]*). *A graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if, for every partition $\mathcal{F} = \{X_1, X_2, \ldots, X_t\}$ of V into non-empty sets, the number $\epsilon_{\mathcal{F}}$ of edges intersecting two of these sets is at least $k(t-1)$.*

Furthermore, it is a celebrated result due to Edmonds that using any algorithm for matroid partition, in polynomial time, one can check whether the condition above is satisfied and find k-edge-disjoint trees if it is. For details see e.g. [12].

Motivated by the fact that both the existence of arc-disjoint out-branchings from the same root in a digraph and the existence of edge-disjoint spanning trees in a graph can be decided in polynomial time and that both problems have good (polynomially verifiable) characterizations, Thomassé posed the following problem around 2005, a positive solution to which would be a first step for providing a link between Theorems 1.1 and 1.2. The problem is well known in the community and has been published on the Egres open problem list for several years.[1]

Problem 1.3 (*Thomassé*). Find a good characterization of directed graphs D whose underlying undirected graph $UG(D)$ has two edge-disjoint spanning trees such that one of these is an out-branching rooted at a given vertex in D.

Clearly the existence of such spanning trees is equivalent to the existence of an out-branching rooted at the given vertex s such that removing the arcs of this branching leaves a connected digraph. In the case where we replace "out-branching" by "a path with specified end vertices s, t" and "connected" by "existence of a path in the underlying graph between s and t" the problem is $\mathcal{NP}$-complete as was shown recently by the first author and Kriesell.

Theorem 1.4 (*[4]*). *It is $\mathcal{NP}$-complete to decide for a given digraph and specified vertices s, t of D whether D contains a directed (s, t)-path P such that $UG(D - A(P))$ contains a path from s to t.*

The proof of Theorem 1.4 does not generalize to the case of directed spanning trees. Furthermore, the fact that in Problem 1.3 we want spanning subdigraphs and that one of these does not have to respect the orientation of the arcs could indicate that there might be a nice characterization or at least a polynomial algorithm for testing the existence of a non-separating out-branching. However, we are going to prove the following which implies that such a characterization does not exist unless $\mathcal{P} = \mathcal{NP}$. Our proof technique does not apply to the problem of Theorem 1.4 because we strongly use the fact that at least one of the two arc-disjoint digraphs we are looking for is a spanning subdigraph.

Theorem 1.5. *It is $\mathcal{NP}$-complete to decide for a given digraph $D = (V, A)$ and a vertex $s \in V$ whether D contains an out-branching B_s^+ such that $UG(D - A(B_s^+))$ is connected.*

We shall also prove that a number of related problems are NP-complete. In particular we prove the following. A digraph is k**-regular** if every vertex has precisely k-arcs out of it and k-arcs into it.

Theorem 1.6. *It is $\mathcal{NP}$-complete to decide whether a 2-regular digraph D contains a spanning strong subdigraph D' such that $UG(D - A(D'))$ is connected.*

This result may be considered slightly surprising, given that if a positive solution exists, then the number of arcs in D' and $D - A(D')$ is either n and n, or $n+1$ and $n-1$, where n is the number of vertices of the given digraph, that is, either D' is a hamiltonian cycle or it has just one more arc than a hamiltonian cycle.

2. Main proofs

We shall use reductions from 3-CNF satisfiability (3-SAT) and Not-All-Equal 3-SAT (NAE-3-SAT). Recall that a boolean formula is in 3-conjunctive normal form, or 3-CNF, if it is expressed as an AND of clauses, each of which is an OR of exactly 3 distinct literals. In this paper, by 3-SAT we mean the problem of deciding whether a boolean 3-CNF formula $\mathcal{F}$ is satisfiable (that is whether there exist a truth assignment t to the variables of $\mathcal{F}$ such each clause of $\mathcal{F}$ has at least one true literal). By NAE-3-SAT we mean the problem of deciding whether a boolean 3-CNF formula $\mathcal{F}$ has a truth assignment such that for each clause there is at least one literal which is true and at least one literal which is false. Note that this is equivalent to saying that both $\mathcal{F}$ and its negation (obtained by negating all literals in the clauses of $\mathcal{F}$) can be satisfied by the same truth assignment t. It is well known that both 3-SAT and NAE-3-SAT are NP-complete problems (see e.g. [10, p. 259]).

Proof of Theorem 1.5. The reduction used here uses the same type of variable gadget as the one used in the proof of Theorem 1 of [8].[2] We shall show how to reduce 3-SAT to the problem of Theorem 1.5. Let $H(r)$ be the digraph (the clause gadget) on 7 vertices $\{a_{r,1}, a_{r,2}, a_{r,3}, b_{r,1}, b_{r,2}, b_{r,3}, c_r\}$ and arcs $a_{r,i}b_{r,i}$, $b_{r,i}a_{r,i}$, $c_r a_{r,i}$, $c_r b_{r,i}$, $i = 1, 2, 3$ (see Fig. 1). Let $W[u, v, p, q]$ be the digraph (the variable gadget) with vertices $\{u, v, y_1, y_2, \ldots y_p, z_1, z_2, \ldots z_q\}$ and the arcs of the two (u, v)-paths $uy_1y_2 \ldots y_pv$, $uz_1z_2 \ldots z_qv$. Note that we allow $\min\{p, q\} = 0$ but $p + q \geq 1$ must hold.

[1] It currently appears on the Egres open problem page: URL http://lemon.cs.elte.hu/egres/open/Category:Trees_and_branchings.

[2] Which again uses ideas from another proof.

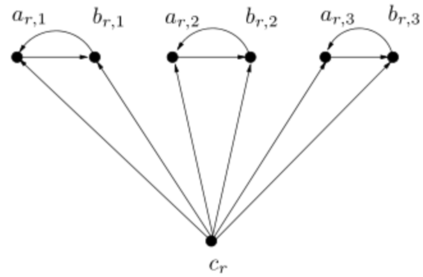

Fig. 1. The clause gadget $H(r)$.

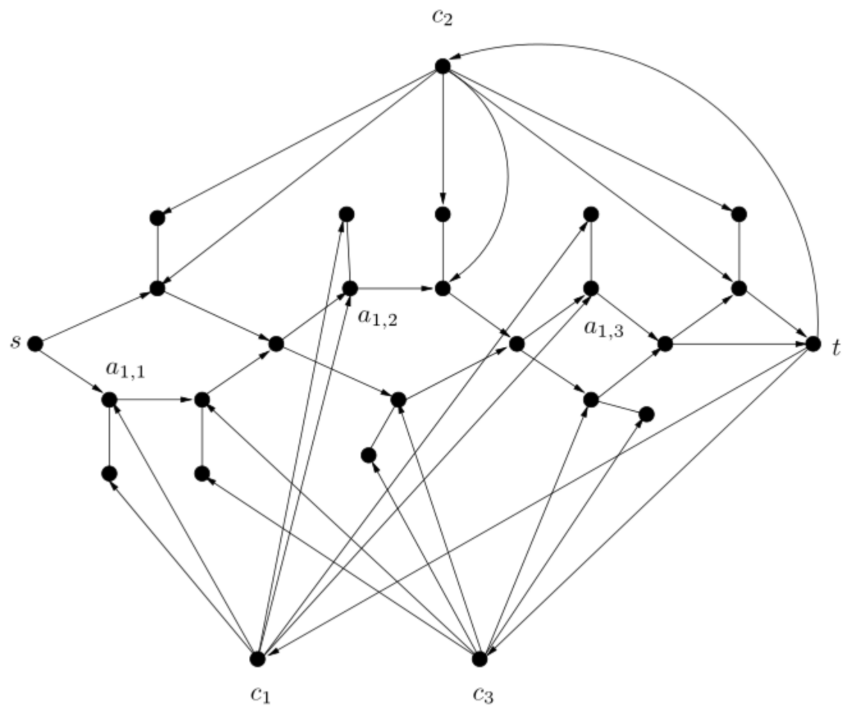

Fig. 2. A schematic picture of $D_{\mathcal{F}}$ where $\mathcal{F}$ has variables x_1, x_2, x_3, x_4 and clauses $C_1 = (\bar{x}_1 \vee x_2 \vee x_3)$, $C_2 = (x_1 \vee x_2 \vee x_4)$, $C_3 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$. For convenience only some vertices are labelled and the 2-cycles of the type $a_{j,i}b_{j,i}a_{j,i}$ are shown as one undirected edge.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. We may assume that each variable x occurs at least once either in the negated form or non-negated in $\mathcal{F}$. For each variable x the ordering of the clauses $C_1, C_2, \ldots, C_m$ induces an ordering of the occurrences of the literal x and the literal $\bar{x}$ in these. With each variable x_i we associate a copy of $W[u_i, v_i, p_i, q_i]$ where the literal x_i occurs p_i times and the literal $\bar{x}_i$ occurs q_i times in the clauses of $\mathcal{F}$. Identify end vertices of these digraphs by setting $v_i = u_{i+1}$ for $i = 1, 2, \ldots, n-1$. Let $s = u_1$ and $t = v_n$. Next, for each clause C_j we take a copy $H_j = H(j)$ of the clause gadget and identify the vertices $a_{j,1}, a_{j,2}, a_{j,3}$ of H_j with vertices in the chain we build above as follows: assume C_j contains literals involving the variables x_i, x_k, x_l. If C_j contains the literal x_i and this is the r'th copy of the literal x_i (in the order of the clauses that use literal x_i), then we identify $a_{j,1}$ with $y_{i,r}$ and if C_j contains the literal $\bar{x}_i$ and this is the k'th occurrence of literal $\bar{x}_i$, then we identify $a_{j,1}$ with $z_{i,k}$. We make similar identifications for $a_{j,2}, a_{j,3}$. Finally we add all the arcs tc_j for $j \in [m]$. This concludes the description of the digraph $D_{\mathcal{F}}$ with special vertices s, t. Let D' be the subdigraph induced by the union of all the vertices from $W[u_i, v_i, p_i, q_i]$, $i \in [n]$. Recall that by the identifications above D' contains all the vertices $a_{j,r}$, $j \in [m]$, $r \in [3]$. See Fig. 2 for an example.

Claim 1. *D' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$ if and only if $\mathcal{F}$ is satisfiable.*

Proof of Claim 1. Suppose P is an (s, t)-path which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$. By construction, for each variable x_i, P traverses either the subpath $Q_i = u_i y_{i,1} y_{i,2} \ldots y_{i,p_i} v_i$ or the subpath $P_i = u_i z_{i,1} z_{i,2} \ldots z_{i,q_i} v_i$. Now define a truth assignment by setting x_i false when P traverses Q_i and true if P traverses P_i for $i \in [n]$. This is a satisfying truth assignment for $\mathcal{F}$ since for any clause C_j at least one literal is avoided by P and hence becomes true by the assignment (the literals traversed become false and those not traversed become true). Conversely, given a truth assignment for $\mathcal{F}$ we can form P by routing it through all the false literals in the chain of variable gadgets. $\square$

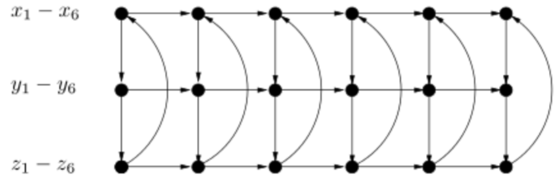

Fig. 3. The gadget $H(x, y, z)$. The vertices are ordered from the left to the right and labelled as indicated in the left part of the Figure [1, Figure 6.1].

Claim 2. *$D_{\mathcal{F}}$ has an out-branching B_s^+ such that $D_{\mathcal{F}} - A(B_s^+)$ is connected if and only if D' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$.*

Proof of Claim 2. Suppose first that there exists B_s^+ such that $D - A(B_s^+)$ is connected. It follows from the structure of $D_{\mathcal{F}}$ that the (s, t)-path P in B_s^+ lies entirely inside D' and since tc_j is the only arc entering c_j, all arcs of the form $tc_j, j \in [m]$ are in B_s^+. Now it follows that P cannot contain all of $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for some clause C_j because that would disconnect the vertices of H_j from the remaining vertices in $D - A(B_s^+)$. Conversely, suppose that D' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$. Then we form an out-branching B_s^+ by adding the following arcs to P: all arcs of the form $tc_j, j \in [m]$ and for each clause $C_j, j \in [m]$ and $r \in [3]$ if P contains the vertex $a_{j,r}$ we add the arc $a_{j,r}b_{j,r}$ and otherwise we add the arcs $c_jb_{j,r}, b_{j,r}a_{j,r}$. This clearly gives an out-branching B_s^+ of $D_{\mathcal{F}}$. It remains to show that $D^* = D_{\mathcal{F}} - A(B_s^+)$ is connected. First observe that $D^*\langle V(D')\rangle$ contains either all arcs of the subpath $u_iy_{i,1}y_{i,2}\ldots y_{i,p_i}v_i$ or all arcs of the subpath $u_iz_{i,1}z_{i,2}\ldots z_{i,q_i}v_i$ for each $i \in [n]$ and hence it contains an (s, t)-path which passes through all the vertices $u_1, u_2, \ldots, u_n, t$. By the description of P above, for each clause $C_j, j \in [m]$ and $r \in [3]$, if P contains the vertex $a_{j,r}$ then D^* contains the arcs $c_jb_{j,r}, c_ja_{j,r}$ and if P does not contain the vertex $a_{j,r}$ then D^* contains the arcs $c_ja_{j,r}, a_{j,r}b_{j,r}$. Now it is easy to see that D^* is connected and spanning. □

Theorem 1.5 now follows by combining Claims 1 and 2. □

In the proof of Theorem 1.6 we shall use the following result due to the second author (the result is mentioned in [1, Section 13.10] and in [7]). Since a proof has never appeared in print before and the proof of this result plays an important role in the proof below, we include a proof here for completeness (the proof is a refinement of the proof of Theorem 6.1.3 in [1]).

We recall from [1] that a ***k*-path factor** of a digraph H is a collection of k vertex disjoint paths that cover all vertices of $V(H)$.

Theorem 2.1. *It is $\mathcal{NP}$-complete to decide whether a 2-regular digraph D contains a pair of arc-disjoint hamiltonian cycles.*

Proof. We will reduce the Not-All-Equal 3-SAT (NAE-3-SAT) problem to the problem of deciding whether a 2-regular digraph has two arc-disjoint hamiltonian cycles. Consider the following digraph $H(x, y, z)$

$$\begin{aligned} V(H(x,y,z)) &= \{x_i, y_i, z_i : i = 1, 2, 3, 4, 5, 6\}, \\ A(H(x,y,z)) &= \{x_iy_i, y_iz_i, z_ix_i : i = 1, 2, 3, 4, 5, 6\} \cup \\ &\quad \{x_jx_{j+1}, y_jy_{j+1}, z_jz_{j+1} : j = 1, 2, 3, 4, 5\} \end{aligned}$$

(see Fig. 3). It is easy to verify that the digraph $H(x, y, z)$ has the following properties:

(i) There is a unique hamiltonian path P of $H(x, y, z)$ starting at x_1 (y_1, z_1, respectively) and this terminates at x_6 (y_6, z_6, respectively). Furthermore, when P denotes this hamiltonian path from x_1 to x_6 then $H(x, y, z) - A(P)$ has a unique 2-path factor $R \cup S$ and R is a (y_1, y_6)-path and S is a (z_1, z_6)-path. Similarly, when P is a hamiltonian path from y_1 to y_6 or from z_1 to z_6.

(ii) Let $P \cup Q$ be a 2-path factor of $H(x, y, z)$ such that the path P starts at x_1 and the path Q starts at y_1 and both paths end in the set $\{x_6, y_6, z_6\}$. Then P terminates at x_6 and Q at y_6. Furthermore, $H(x, y, z) - A(P) - A(Q)$ is a hamiltonian path starting at z_1 and terminating at z_6. Similarly for the pairs x_1, z_1 and y_1, z_1.

(iii) Let $P \cup Q \cup R$ be a 3-path factor of $H(x, y, z)$ such that the paths P, Q and R start at x_1, y_1 and z_1, respectively and all three paths end in the set $\{x_6, y_6, z_6\}$. Then P, Q and R terminate at x_6, y_6 and z_6, respectively. Furthermore, after removing the arcs of $P \cup Q \cup R$ we obtain 6 vertex disjoint 3-cycles with no arcs between them.

That (iii) holds is obvious. To see that property (i) holds it suffices to check that the unique hamiltonian path starting in x_1 in $H(x, y, z)$ is $x_1y_1z_1z_2x_2y_2y_3z_3x_3x_4y_4z_4z_5x_5y_5y_6z_6x_6$ and that after deleting these arcs the unique 2-path factor of the remaining digraph consists of the paths $y_1y_2z_2z_3z_4x_4x_5x_6y_6$ and $z_1x_1x_2x_3y_3y_4y_5z_5z_6$. We leave to the reader to verify the (ii) holds (again the paths are unique and easy to construct).

We are going to use $H(x, y, z)$ as a building block in a bigger digraph below and since we will only connect the vertices $x_1, x_6, y_1, y_6, z_1, z_6$ to other parts of the digraph, we will use the names x, x', y, y', z, z' for these below and denote the subdigraph by $H(x, x', y, y', z, z')$.

Consider an instance $\mathcal{I}$ of NAE-3-SAT with variables $v_1, \ldots, v_k$ and clauses $C_1, \ldots, C_p$. Since we require that every clause contains both true and false literals in any satisfying truth assignment, we may assume that every variable and its

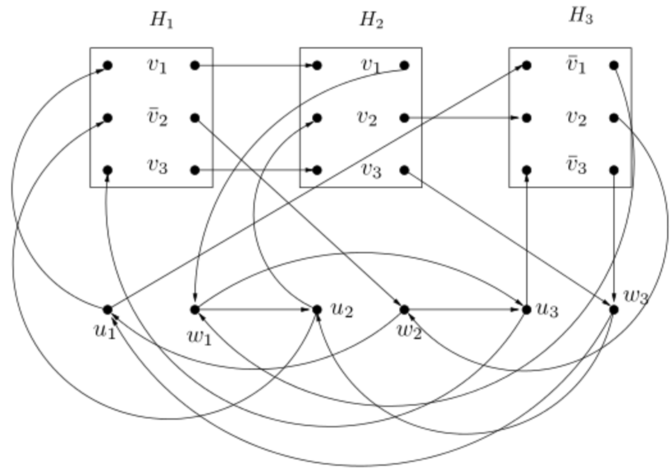

Fig. 4. An illustration of the digraph $D_{\mathcal{I}}$ for the formula $\mathcal{I} = (v_1 \vee \bar{v}_2 \vee v_3)(v_1 \vee v_2 \vee v_3)(\bar{v}_1 \vee v_2 \vee \bar{v}_3)$. For convenience only the six important vertices of each H_i is shown and in the middle column of each H_i we show the 3 literals in the order they appear in C_i. Thus the top literal corresponds to the two top vertices etc.

negation appear in $\mathcal{I}$ as literals (otherwise we can add negated copies of some of the clauses). Construct a digraph $D_{\mathcal{I}}$ as follows: start from a disjoint union $U = H_1 \cup H_2 \cup \cdots \cup H_p$, where $H_j = H(a_j, a'_j, b_j, b'_j, c_j, c'_j)$ and a_j, b_j, c_j are the literals in $C_j, j \in [p]$.

For every variable v the ordering of $C_1, \ldots, C_p$ induces an ordering $C_{v,1}, \ldots, C_{v,p_v}$ of the clauses containing the literal v and an ordering $C_{\bar{v},1}, \ldots, C_{\bar{v},q_{\bar{v}}}$ of the clauses containing the literal $\bar{v}$. Based on this ordering we join the vertices of different pairs among $H_1, H_2, \ldots, H_p$ as follows (where we denote the clause gadget corresponding to the clause $C_{v,r}$ by $H_{v,r}$): for each variable v and $r \leq p_v - 1$ we add an arc $\alpha \to \beta$ from $H_{v,r}$ to $H_{v,r+1}$ where α equals one of the vertices $a'_{v,r}, b'_{v,r}, c'_{v,r}$ depending on whether the first, second or third literal in $C_{v,r}$ is equal to v and β equals one of the vertices $a_{v,r+1}, b_{v,r+1}, c_{v,r+1}$ depending on whether the first, second or third literal in $C_{v,r+1}$ is equal to v. Similarly, for each variable v and $r \leq q_{\bar{v}} - 1$ we add an arc $\alpha \to \beta$ from $H_{\bar{v},r}$ to $H_{\bar{v},r+1}$ where α equals one of the vertices $a'_{\bar{v},r}, b'_{\bar{v},r}, c'_{\bar{v},r}$ depending on whether the first, second or third literal in $C_{v,r}$ is equal to $\bar{v}$ and β equals one of the vertices $a_{\bar{v},r+1}, b_{\bar{v},r+1}, c_{\bar{v},r+1}$ depending on whether the first, second or third literal in $C_{\bar{v},r+1}$ is equal to $\bar{v}$. See Fig. 4.

Next we add $2k$ new vertices $u_1, w_1, u_2, w_2, \ldots, u_k, w_k$ where the vertices u_i, w_i correspond to the variable v_i, for $i \in [k]$. Each vertex u_i dominates one vertex in each of $H_{v_i,1}$ and $H_{\bar{v}_i,1}$, namely one of the vertices $a_{v_i,1}, b_{v_i,1}, c_{v_i,1}$ depending on whether v_i is the first, second or third literal in $C_{v_i,1}$ and one of the vertices $a_{\bar{v}_i,1}, b_{\bar{v}_i,1}, c_{\bar{v}_i,1}$ depending on whether $\bar{v}_i$ is the first, second or third literal in $C_{\bar{v}_i,1}$. Each vertex w_i is dominated by one vertex from each of $H_{v_i,p_{v_i}}$ and $H_{\bar{v}_i,q_{\bar{v}_i}}$, namely one of the vertices $a'_{v_i,p_{v_i}}, b'_{v_i,p_{v_i}}, c'_{v_i,p_{v_i}}$ depending on whether v_i is the first, second or third literal in $C_{v_i,p_{v_i}}$ and one of the vertices $a_{\bar{v}_i,q_{\bar{v}_i}}, b_{\bar{v}_i,q_{\bar{v}_i}}, c_{\bar{v}_i,q_{\bar{v}_i}}$ depending on whether $\bar{v}_i$ is the first, second or third literal in $C_{\bar{v}_i,q_{\bar{v}_i}}$. Finally, we add the arcs w_iu_{i-1}, w_iu_{i+1} for every $i \in [k]$, where $u_0 = u_k,\ u_{k+1} = u_1$.

It is easy to verify that D is 2-regular.

Suppose $\mathcal{I}$ is a 'yes' instance of NAE-3-SAT and consider a satisfying truth assignment t. Note that the complementary truth assignment $\bar{t}$ (where we set a variable true if and only if it is false in t) is also a satisfying truth assignment for $\mathcal{I}$. We will show how to construct arc-disjoint hamiltonian cycles C, C' of $D_{\mathcal{I}}$ based on the values of the variables in t. For each variable v_j such that v_j is true in t we let C contain the arc w_ju_{j+1}, the arc from u_j to $H_{v_j,1}$, the arc from $H_{v_j,p_{v_j}}$ to w_j and all the arcs from $H_{v_j,r}$ to $H_{v_j,r+1}, r = 1, 2, \ldots, p_{v_j} - 1$ that were described above corresponding to the occurrences of v_j in the clauses $C_{v_j,1}, \ldots, C_{v_j,p_{v_j}}$. For each variable v_f such that v_f is false in t we let C contain the arc w_fu_{f+1}, the arc from u_f to $H_{\bar{v}_f,1}$, the arc from $H_{\bar{v}_f,q_{\bar{v}_f}}$ to w_f and all the arcs from $H_{\bar{v}_f,r}$ to $H_{\bar{v}_f,r+1}, r = 1, 2, \ldots, q_{\bar{v}_f} - 1$ that were described above corresponding to the occurrences of $\bar{v}_f$ in the clauses $C_{\bar{v}_f,1}, \ldots, C_{\bar{v}_f,q_{\bar{v}_f}}$. Similarly (except we use arcs from w_j to u_{j-1} for all $j \in [k]$, where indices are taken modulo k) we define C' from the truth assignment $\bar{t}$.

Since every clause is satisfied by t, the cycle C uses vertices from each digraph in the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_p$. By the properties (i) and (ii) of $H(x, y, z)$ above, if s $(1 \leq s \leq 2)$ literals are satisfied in a clause C_j by t, all vertices of the corresponding digraph H_j can be used in C due to the existence of an appropriate s-path factor in H_j. Thus, C is indeed hamiltonian. Similarly C' is a hamiltonian cycle and it is arc-disjoint from C by the way we constructed it.

Suppose now that $D_{\mathcal{I}}$ has a pair of arc-disjoint hamiltonian cycles C, C'. It follows from (iii) above that none of the cycles C, C' passes through any H_j more than two times. Hence if we set v_i true if C uses the arc from u_i to $H_{v_i,1}$ and false otherwise, then we obtain a truth assignment t such that both t and $\bar{t}$ satisfy all clauses of $\mathcal{I}$ (a literal will be set to true if and only if C uses the arcs of $D_{\mathcal{I}}$ that correspond to this literal). □

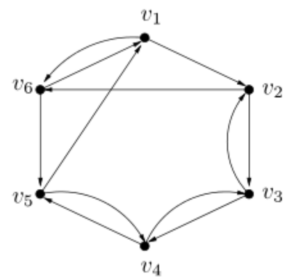

Fig. 5. A 2-regular digraph Q with no hamiltonian cycle.

Lemma 2.2. *The digraph Q in Fig. 5 has the following properties.*

(i) *Q has no hamiltonian cycle.*
(ii) *Q contains the strong spanning subdigraph induced by the arcs $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_2v_6, v_6v_1\}$ which is arc-disjoint from the connected spanning subgraph of $UG(D)$ formed by the arcs $\{v_1v_6, v_6v_5, v_5v_4, v_4v_3, v_3v_2\}$.*
(iii) *There is no strong spanning subdigraph Q' of Q such that $Q - A(Q')$ is connected and both of the arcs v_4v_3, v_4v_5 are in Q'.*

Proof. We leave it to the reader to make the easy check that Q has no hamiltonian cycle. That (ii) holds is easy to verify so it only remains to prove (iii). Let Q' be any strong spanning subdigraph of Q which contains both of the arcs v_4v_3, v_4v_5 and assume that $Q - A(Q')$ is connected. Then precisely one of the arcs v_3v_4, v_5v_4 is in Q'. Note that, by (i), Q' must contain 7 arcs and $Q - A(Q')$ must be a spanning tree of Q. Therefore all vertices in Q' except one have out-degree one and all vertices in Q' except one have in-degree one. Suppose first that Q' contains v_3v_4. Then $v_3v_2 \notin A(Q')$ and $v_1v_2 \in A(Q')$, which implies that $v_1v_6 \notin A(Q')$ and $v_2v_6 \in A(Q')$. Therefore $v_2v_3 \notin A(Q')$ and $Q - A(Q')$ contains both v_2v_3 and v_3v_2 a contradiction to $Q - A(Q')$ being a spanning tree of Q.

So Q' contains v_5v_4 but not v_3v_4. Analogously to above we note that $v_5v_1 \notin A(Q')$ and thus $v_6v_1 \in A(Q')$, which implies that $v_6v_5 \notin A(Q')$. As $v_3v_4 \notin A(Q')$ we note that $v_3v_2 \in A(Q')$. As $Q - A(Q')$ is connected Q' cannot contain the arc v_2v_3 but then $\{v_1, v_2, v_6\}$ has out-degree 0 in Q', a contradiction. □

Proof of Theorem 1.6. Consider an instance $\mathcal{I}$ of Not-All-Equal 3-SAT (NAE-3-SAT) with variables $v_1, \ldots, v_k$ and clauses $C_1, \ldots, C_p$ such that every variable and its negation appear in $\mathcal{I}$ as literals and let $D_{\mathcal{I}}$ be the digraph constructed as in the proof of Theorem 2.1.

We form a new digraph W from $D_{\mathcal{I}}$ and a copy of Q as follows: fix a vertex $x \in V(D_{\mathcal{I}})$ and delete the two arcs xy, xz to its two out-neighbours in $D_{\mathcal{I}}$. Delete the two arcs v_4v_3, v_4v_5 from Q and add the arcs xv_3, xv_5, v_4y, v_4z. Clearly W is a 2-regular digraph. We claim that W has a spanning strong subdigraph D' such that $W - A(D')$ is connected if and only if $\mathcal{I}$ is satisfiable and by the proof above we know that this happens if and only if $D_{\mathcal{I}}$ has two arc-disjoint hamiltonian cycles. Suppose first that H_1, H_2 are arc-disjoint hamiltonian cycles of $D_{\mathcal{I}}$. Without loss of generality H_1 contains the arc xy. Thus we obtain a strong spanning subdigraph D' of W such that $W - A(D')$ is connected by replacing the arc xy by the arcs $\{v_1v_2, v_2v_3, v_3v_4, v_4y, xv_5, v_5v_1, v_2v_6, v_6v_1\}$ and the arc xz (which is in H_2) by the arcs xv_3, v_4z.

Conversely, suppose that W has a spanning strong subdigraph D' such that $W - A(D')$ is connected. First observe that when we replace back the arcs $\{xy, xz, v_4v_3, v_4v_5\}$ for the arcs $\{xv_3, xv_5, v_4y, v_4z\}$ D' splits up into disjoint strong spanning subdigraphs R of Q and S of $D_{\mathcal{I}}$, respectively. This implies that D' uses exactly one of the arcs xv_3, xv_5. If both of these arcs are in D' then the fact that $W - A(D')$ is connected implies that $Q - A(R)$ must be connected, contradicting Lemma 2.2(iii). It follows from the fact that Q has no hamiltonian cycle that R uses 7 arcs and since W has $2(n+6)$ arcs and $n+6$ vertices we get that S is a hamiltonian cycle of $D_{\mathcal{I}}$. Therefore D' cannot contain both of the arcs v_4y, v_4z. Furthermore, it follows from (iii) in the proof of Theorem 2.1 that S cannot traverse any H_j more than twice since if it did then $W - A(D')$ could not be connected as $D_{\mathcal{I}} - A(D')$ would contain many vertex disjoint 3-cycles and we have only two arcs coming from $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ to connect these. Thus we can use S to define a satisfying truth assignment for $\mathcal{I}$ just as we did in the proof above. □

3. Further results and related open problems

In the digraph $D_{\mathcal{F}}$ used in the proof of Theorem 1.5 the vertex s is a source (has in-degree 0) and hence is the only vertex from which an out-branching can start in $D_{\mathcal{F}}$. Thus the proof of Theorem 1.5 shows that the following holds.

Theorem 3.1. *It is NP-complete to decide for a given digraph D whether D contains some vertex s and an out-branching B_s^+ such that $D - A(B_s^+)$ is connected.* □

It is easy to check that the proof of Theorem 1.5 still works if we add the arc ts in which case the digraph $D_{\mathcal{F}}$ becomes strongly connected. Hence we have shown the following.

Theorem 3.2. *It is NP-complete to decide for a given strongly connected digraph D and a specified vertex s of D whether D contains an out-branching B_s^+ such that $D - A(B_s^+)$ is connected.* □

Clearly, by Theorem 1.1, every 2-arc-strong digraph has an out-branching B_s^+ such that $D - A(B_s^+)$ is connected for every choice of the root s. On the other hand, there exist strong digraphs D with $UG(D)$ arbitrarily highly edge-connected and a vertex s which can reach all other vertices by a directed path and yet D has no out-branching B_s^+ s.t. $D - A(B_s^+)$ is connected. To see this take a directed path $P = u_1u_2u_3 \ldots u_k$ and add all arcs u_iu_j where $j < i$ and both i and j are even or both odd. This has no good out-branching from s because every B_s^+ will use all arcs on P.

Theorem 3.3. *The following problems are all NP-complete*

(i) *Given a digraph D and $s, t \in V(D)$; does D have an (s, t)-path P such that $D - A(P)$ is connected?*
(ii) *Given a digraph D and $s, t \in V(D)$; does $UG(D)$ have an (s, t)-path P such that $D - A(P)$ contains an out-branching rooted in s?*
(iii) *Given a strong digraph D; does D contain a cycle C such that $D - A(C)$ is connected?*
(iv) *Given a strong digraph D; does D contain a cycle C such that $D - A(C)$ is strong?*
(v) *Given a strong digraph D; does $UG(D)$ contain a cycle W such that $D - A(W)$ is strongly connected?*

Proof. Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$, let $D^*_{\mathcal{F}}$ be the digraph that we build as in the proof of Theorem 1.5, except that instead of using $H(j)$ as the clause gadget for C_j we use a directed 6-cycle $a_{j,1}d_{j,1}a_{j,2}d_{j,2}a_{j,3}d_{j,3}a_{j,1}$ as clause gadget and where the vertices $a_{j,1}, a_{j,2}, a_{j,3}$ are identified with vertices of the variable gadgets as we did in the proof of Theorem 1.5. Define D' as we did in the proof of Theorem 1.5. To prove that problem (i) is NP-complete, it suffices to note that if P is an (s, t)-path in $D^*_{\mathcal{F}}$ such that $D^*_{\mathcal{F}} - A(P)$ is connected, then P does not use any arc from any of the clause gadgets. Now it is easy to see that $D^*_{\mathcal{F}}$ contains such a path if and only if D' contains a path which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$ and we are done by Claim 1. If Q is path between s and t in $UG(D^*_{\mathcal{F}})$ such that $D^*_{\mathcal{F}} - A(Q)$ contains an out-branching, then Q does not use any arc from any of the clause gadgets. Again this and Claim 1 easily implies that (ii) is NP-complete. To prove that (iii) is NP-complete we consider the digraph D^* that we obtain from $D^*_{\mathcal{F}}$ by adding the arc ts. Then the argument above for (i) shows that D^* has a cycle C such that $D - A(C)$ is connected if and only if $\mathcal{F}$ is satisfiable so (iii) is NP-complete. To prove that (iv) and (v) are NP-complete we consider D^{**} which we obtain from D^* by adding a new vertex t' and the arcs tt', $t's$. By the choice of clause gadget, a cycle C such that $D^{**} - A(C)$ is strong is a cycle formed by an (s, t)-path P in D' and the arc ts with the property that P avoids at least one vertex from each of the sets $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ and now we can apply Claim 1 to see that (iv) is NP-complete. Finally, observe that $UG(D^{**})$ has a cycle W such that $D - A(W)$ is strongly connected if and only if W is formed by an (s, t)-path P in D' and the arc ts with the property that P avoids at least one vertex from each of the sets $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ and again we can use Claim 1. □

It was shown in [3] that there is a polynomial algorithm to check whether the underlying digraph $UG(D)$ of a given strong digraph D contains two vertex disjoint cycles C, C' such that C is also a cycle in D. On the other hand it was shown in [6] that the same problem becomes NP-complete if we do not require that D is strong.

In [7] the authors posed the following conjecture and proved it for semicomplete digraphs where $N = 3$ is necessary and sufficient (a digraph is semicomplete if it has no non-adjacent vertices). Recently [2] the conjecture was also confirmed for locally semicomplete digraphs and again $N = 3$ is best possible for this much larger class of digraphs. A digraph is locally semicomplete if and only if the in-neighbourhood and the out-neighbourhood of every vertex induces a semicomplete digraph.

Conjecture 3.4 ([7]). *There exist a natural number N such that every N-arc-strong D contains two arc-disjoint spanning strong subdigraphs.*

A (much) weaker version of this is the following.

Conjecture 3.5 ([5]). *There exist a natural number K such that every K-arc-strong digraph D has a strong spanning subdigraph D' such that $UG(D - A(D'))$ is connected.*

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