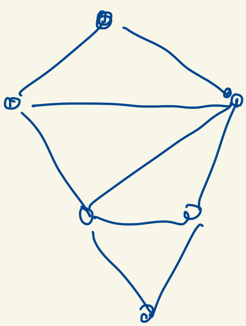
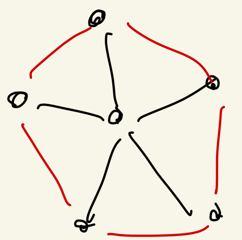
Chordal Graphs (Golumbic ch. 4)

Definition G is chordal
$\Updownarrow$
G has no induced cycle of length ≥ 4



Chordal

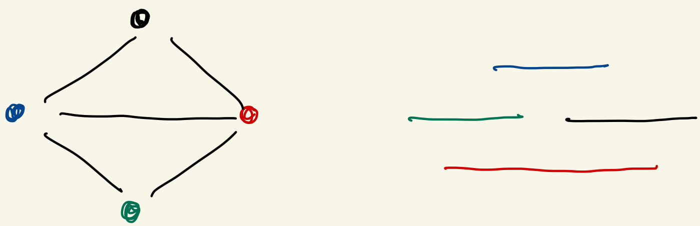


not chordal due to red cycle

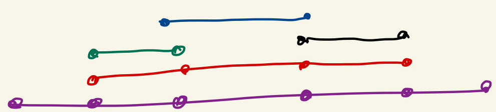
Chordality is a hereditary property:
If $G=(V,E)$ is chordal and H is an induced subgraph of G, then H is chordal

Important subclass of chordal graphs: interval graphs

Def. $G=(V,E)$ is an interval graph is there exists a set of intervals $I = \{ I_v \mid v \in V \}$ where each I_v is an interval along the real line and $uv \in E \Leftrightarrow I_u \cap I_v \neq \emptyset$



Observe that every finite interval graph is the intersection graph of sub paths of a path

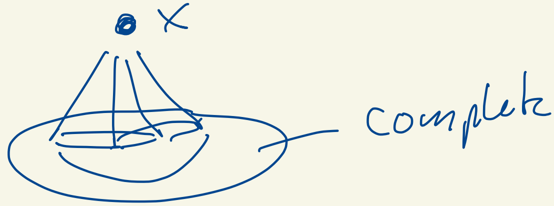


Interval graphs are chordal (exercise)

Interval graphs are important in applications such as scheduling.

Simplicial vertex

$x \in V(G)$ is simplicial $\Leftrightarrow$ $N[x]$ is a clique in G.



Theorem (Dirac 1961)

Every chordal graph has a simplicial vertex

This leads to a polynomial algorithm for recognizing chordal graphs:

While $G \neq \emptyset$ and G has a simplical vertex x
    $G \leftarrow G - x$

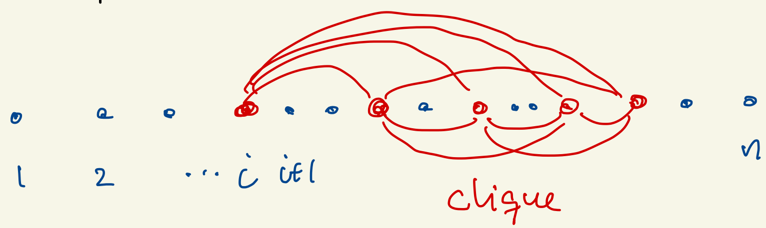
If $G \neq \emptyset$
    report G is not chordal (we prove this later)

This leads to the notion of a

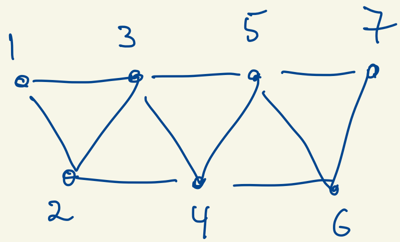
perfect elimination scheme (p.e.s)

Definition Let $G=(V,E)$

An ordering $\sigma = [v_1, v_2, \ldots, v_n]$ of V is a perfect elimination scheme for G if each v_i is a simplicial vertex in $G[v_i, v_{i+1}, \ldots, v_n]$



Example



Definition A subset $S \subset V$ in a graph $G=(V,E)$ is a vertex separator for $a,b \in V$ with $ab \notin E$ if $S \subseteq V - \{a,b\}$ and $G-S$ has no (a,b)-path

S is a minimal (a,b)-separator if $G-S$ has no (a,b)-path but $G-S'$ has such a path for every $S' \subset S$

Theorem 4.1 Let $G=(V,E)$. Then

(i) G is chordal

$\Updownarrow$

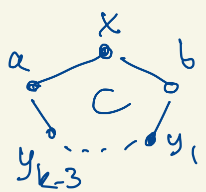
(ii) G has a p.e.s. and every simplicial vertex can start a p.e.s.

$\Updownarrow$

(iii) Every minimal vertex separator induces a clique

Proof

(iii) $\Rightarrow$ (i): Suppose C is an induced k-cycle for some $k \geq 4$



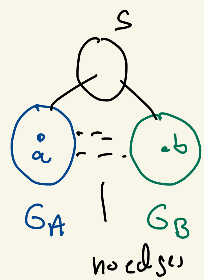
Every minimal (a,b)-separator contains x and some y_i, $i \in [k-3]$

Then (iii) $\Rightarrow$ $xy_i \in E$ so C is not induced

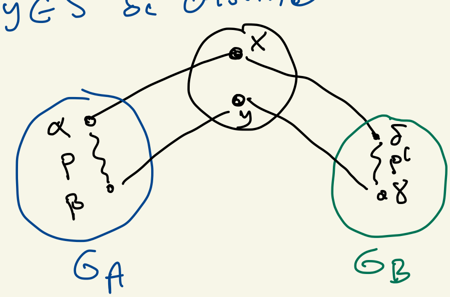
(i) $\Rightarrow$ (iii)

Let S be a minimal (a,b)-separator

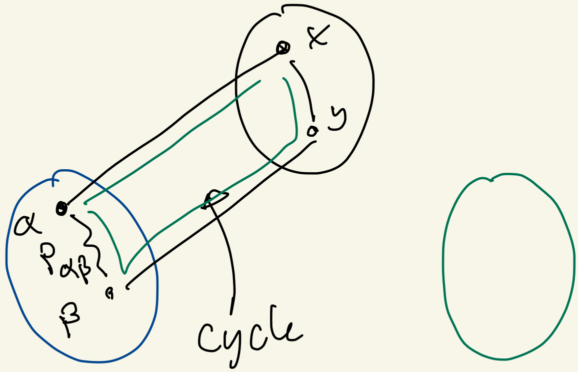
S minimal $\Rightarrow$ $\forall s \in S$ $\exists$ edge from s to G_A and $\exists$ edge from s to G_B



Let $x, y \in S$ be distinct then



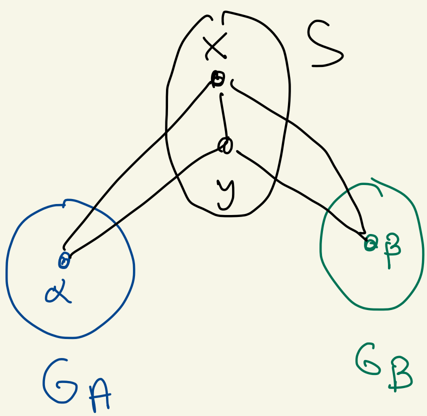
We must have $xy \in E$ as then is no edge from G_A to G_B



assume α and β are chosen such that
α is the only neighbour of x on $P_{\alpha\beta}$
β - - -   - - - y - - -

Then $\alpha = \beta$ as G is chordal so
in fact we have shown:

$\forall x, y \in S \ \exists \alpha \in G_A$ and $\beta \in G_B$ such that
$\alpha x, \alpha y, x\beta, y\beta \in E$

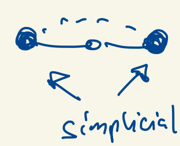


With (i) $\Leftarrow$ (iii) proved, we can now show

Lemma Every chordal graph G has a simplicial vertex. If G is not complete it has a pair of non adjacent simplicial vertices.

Proof We may assume G is not complete as every vertex of K_n is simplicial

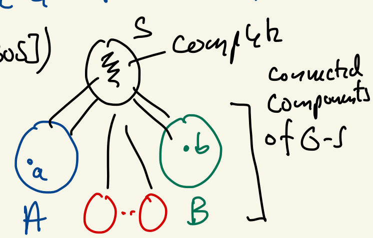
Base case is clear (assume G is connected)



Assume Lemma holds for all connected chordal graphs G' with fewer vertices than G

Let $a, b \in V$ s.t. $ab \notin E$ and let S be a minimal (a,b)-sep.

Note that $x \in A$ (B) is simplicial in $G[A \cup S]$ $(G[B \cup S])$ iff x is simplicial in G (x has no other neighbours)



By induction, either $G[A \cup S]$ has 2 non-adj simplicial vertices p, q or $G[A \cup S]$ is complete and one of p, q, say p, is in A (S is complete)

Similarly, either $G[B \cup S]$ is complete or it has non-adj simplicial p', q' vertices where $q' \in B$

- If p, q and p', q' exist p, q' is the desired pair
- if p, q exist and $G[B \cup S]$ is complete then p, b is ok
- if $G[A \cup S]$ is complete and p', q' exist, then a, q' is the desired pair $\square$

Back to the proof of Thm 4.1:

(i) $\Rightarrow$ (ii) Let $x \in V$ be an arbitrary simplicial vertex of G. Then $G' = G - x$ is chordal.

By induction G' has a simplicial ordering $\sigma' = [v_1', v_2', \dots, v_{n-1}']$ and now $\sigma = [x, v_1', v_2', \dots, v_{n-1}']$ is a simplicial ordering of G starting with x.

(ii) $\Rightarrow$ (i):

Suppose G satisfies (ii) but C is an induced k-cycle in G for some $k \geq 4$.

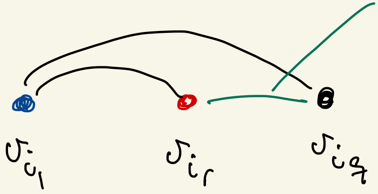
Let $\sigma = [v_1, v_2, \dots, v_n]$ be a simplicial ordering of G.

This induces a simplicial ordering

$$[v_{i_1}, v_{i_2}, \dots, v_{i_k}] \text{ of } G[V(C)]$$

v_{i_1} has two neighbours on C so, as v_{i_1} is simplicial in $G[V(C)]$ these neighbours are adjacent, contradicting that C is induced.

must be there as v_{i_1} is simplicial in $G[V(C)]$

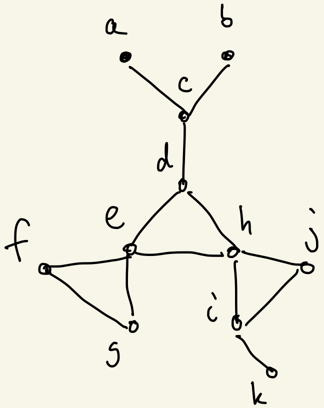


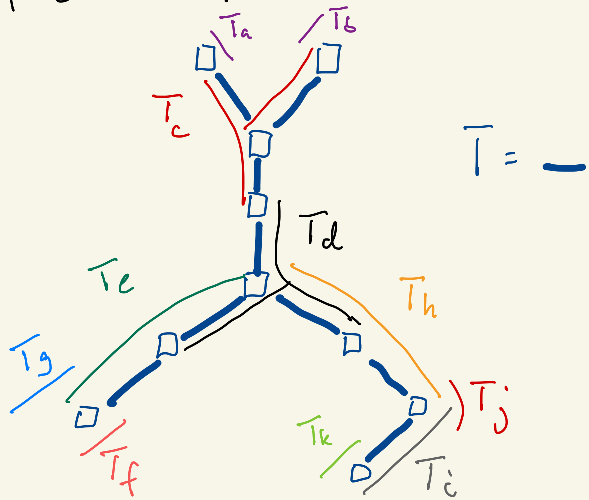
$\square$

5. Chordal graphs as intersection graphs

Motivation: Interval graphs ⊆ Chordal Graphs
So can we also represent chordal graphs as intersection graphs of something?

Goal: show that every chordal graph is the intersection graph of subtrees of some tree





Definition

A family $\{T_i\}_{i\in I}$ of subsets of a set T

has the Helly property if for each $J \subseteq I$

s.t. $T_i \cap T_j \neq \emptyset \;\forall i,j \in J \Rightarrow \bigcap_{j\in J} T_j \neq \emptyset$

Proposition 4.7 Let $\{T_i\}_{i\in I}$ be a family of subtrees of a tree T.

Then $\{T_i\}_{i\in I}$ has the Helly property.

proof skipped (not pensum)

Theorem 4.8 For a graph $G=(V,E)$ the following are equivalent

(i) G is chordal

(ii) G is the intersection graph of subtrees of some tree

(iii) $\exists$ a tree $T=(\mathcal{K},\mathcal{E})$ whose vertex set $\mathcal{K}$ is the set of maximal cliques of G and T has the property that $T[\mathcal{K}_v]$ is connected $\forall v \in V$

where $\mathcal{K}_v$ is the set of maximal cliques containing v

Proof

(iii) $\Rightarrow$ (ii): Let $T=(\mathcal{K},\mathcal{E})$ be as in (iii)

Consider an edge $vw \in E$ and let $A \in \mathcal{K}$ be a maximal clique containing v,w

So $T[\mathcal{K}_v] \cap T[\mathcal{K}_w] = \emptyset$ (A is in both)

and G is the intersection graph of the family

$\{T[\mathcal{K}_v] \mid v \in V\}$ of subtrees of T

Note that if $T[\mathcal{K}_v] \cap T[\mathcal{K}_w] \neq \emptyset$ then v and w are in the same maximal clique in G so $vw \in E$

(ii) $\Rightarrow$ (i) assume (ii) holds and let $\{T_v \mid v \in V\}$ and T represent G

Suppose $G \supseteq C_k = v_0 v_1 \cdots v_{k-1} v_0$ induced cycle $k \geq 4$

Let $T_i = T_{v_i}$

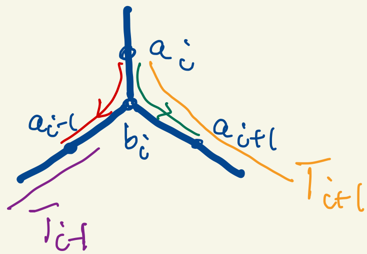
As C_k has no chord $T_i \cap T_j \neq \emptyset \Leftrightarrow |i-j| \leq 1$ modulo k $(*)$

Now choose vertices $a_0, a_1, \ldots a_{k-1}$ of T such that

$a_i \in T_i \cap T_{i+1}$ $\quad i = 0, 1, 2, \ldots, k-1$

Let b_i be the last common vertex of the paths

$P_{a_i a_{i+1}}$ and $P_{a_i a_{i-1}}$ in T



recall that

$T_{i-1} \cap T_{i+1} = \emptyset$

by $(*)$

Then $b_i \in T_i \cap T_{i+1}$ as $P_{a_i a_{i+1}} \in T_{i+1}$ and $P_{a_i a_{i-1}} \in T_i$

Let P_{i+1} be the (b_i, b_{i+1})-path in T

Then $P_i \subseteq T_i$ so $(*) \Rightarrow P_i \cap P_j = \emptyset$ if $|i-j| > 1$ mod k

Moreover $P_i \cap P_{i+1} = \{b_i\}$ $\quad i = 0, 1, 2 \ldots, k-1$ (by definition of b_i)

But now we see that $\bigcup_{i=0}^{k-1} P_i$ is a simple cycle in T,

contradicting that T is a tree.

(i) $\Rightarrow$ (iii) Induction on $|V(G)|$

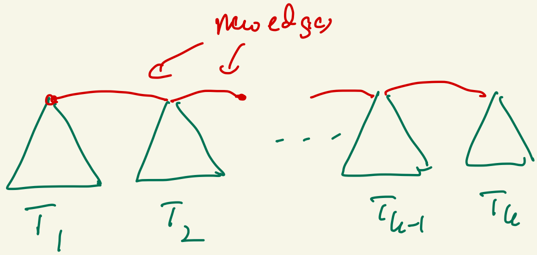
base case $G = \underset{v}{\bullet}$ trivial $\mathcal{K} = \{v\}$, $\mathcal{E} = \emptyset$ $T = \square\{v\}$

assume claim holds for all chordal G' with $|V(G')| < |V(G)|$

Case 1 G is complete. Take $\mathcal{K} = \{V\}$, $\mathcal{E} = \emptyset$ $T = \square\{V\}$

Case 2 G is not complete

If G is not connected with $G_1, G_2, \ldots, G_k$ its components then by induction there are trees $T_1, T_2, \ldots T_k$ with T_i satisfying (iii) wrt G_i and then we can obtain T as follows



So we may assume G is connected and not complete

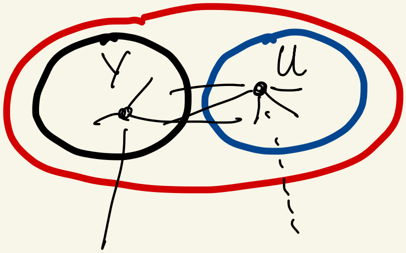
$$\boxed{\text{Let } a \in V \text{ be simplicial and set } A = N[a]}$$

Then A is a maximal clique in G

Define the sets $\mathcal{U}$, Y as follows

$\mathcal{U} = \{u \in A \mid N(u) \subseteq A\}$

$Y = \{v \in A \mid N(v) \cap V - A \neq \emptyset\}$



A $\quad (A = U \cup Y)$

$V - A \neq \emptyset$ as G is not complete

$\mathcal{U} \neq \emptyset$ as $a \in \mathcal{U}$

$Y \neq \emptyset$ as G is connected

By induction (iii) holds for $G' = G[V - \mathcal{U}]$

So let $T' = (K', \mathcal{E}')$ be a tree repr. G'

where K' are the maximal cliques of G' and

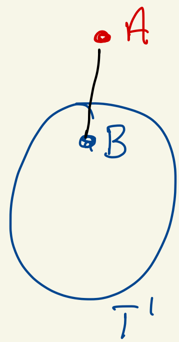
$\forall v \in V - \mathcal{U}$: $K'_v = \{X \in K' \mid v \in X\}$ induces a subtree of T'

<u>Remark</u>: We shall obtain a tree $T = (K, \mathcal{E})$ for G

where

$$K = \begin{cases} (K' - \{Y\}) + \{A\} & \text{if } Y \text{ is a maximal clique in } G' \\ K' + \{A\} & \text{otherwise} \end{cases}$$

Let $B \in K'$ be a maximal clique of G' s.t $Y \subseteq B$



Case 1 $B = Y$:

Then just rename B by A and let $T = T'$

Case 2 $B \neq Y$ then let $T = T' + AB$

In both cases let $K_u = A \quad \forall u \in U$

$K_v = K'_v \quad \forall v \in V - A$

For $y \in Y$ we define K_y as follows:

In case 1 $(B = Y)$ let $K_y = K'_y - \{B\} + \{A\}$

In case 2 let $K_y = K'_y + \{A\}$

In both cases K_y induces a subtree of T as K'_y contains B

This completes the proof of Theorem 4.8