Random orientation method & Color-coding

Based on Alon et al STOC paper 1994 see home page

Recall from probability that if an experiment has probality p of success then we expect to repeat it $1/p$ times before to first success occors.

Random acyclic subdigraph method:

Given digraph $D=(V,A)$ with $V=\{v_1, v_2, \dots, v_n\}$ and a permutation $\pi : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$

Let $H_\pi = (V, A_\pi)$ be the subdigraph consisting of the arcs that go forward wrt π: $A_\pi = \{ v_{\pi(i)} v_{\pi(j)} \mid \pi(i) < \pi(j) \}$

$$\xrightarrow{A_\pi}$$

$$v_{\pi(1)} \quad v_{\pi(2)} \quad \cdot \quad \cdot \quad \dots \quad v_{\pi(n)}$$

$$\boxed{\text{Every walk in } H \text{ is a path since } H_\pi \text{ is acyclic}}$$

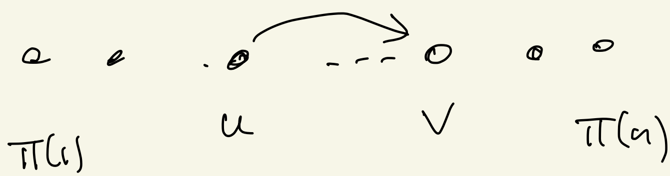
From now on π is a <u>random</u> permutation of $\{1,2,\dots,n\}$

Note: if $G=(V,E)$ is undirected and π is a permutation of V, then π induces a random acyclic orientation of G by orienting an edge $uv \in E$ as $u \to v$ precisely if $\pi(u) < \pi(v)$ (when π is thought of as a permutation of the indices of $V = \{v_1, v_2 \dots v_n\}$)

So when we deal with an undirected graph the derived digraph (the random acyclic orientation) D' still has G as its underlying graph

If G is already a digraph, then we only keep those arcs of D which agree with π so

$uv \in A$ is in A_π if and only if



Simple observation: If $P = v_{i_1} v_{i_2} \cdots v_{i_{k+1}}$ is a simple path in D, then P is a path in H_π with probability $\frac{1}{(k+1)!}$

$(P \in H \iff \pi(i_1) < \pi(i_2) < \dots < \pi(i_{k+1}))$

Theorem 2.1 Let G be a (di)graph which contains a simple (directed) path of length k. Such a path can be found in $O((k+1)!\, m)$ time when m is the number of edges/arcs of G.

Proof Let π be a random permutation of V and form H_π as above (include only forward arcs)

Find a longest path Q in H_π (easy as H_π is acyclic)

If Q has length at least k we output a subpath of length k.

If not we try with a new random permutation

Expected # repetitions is $\frac{1}{\frac{1}{(k+1)!}} = (k+1)!$

So expected running time $O((k+1)!\, m)$

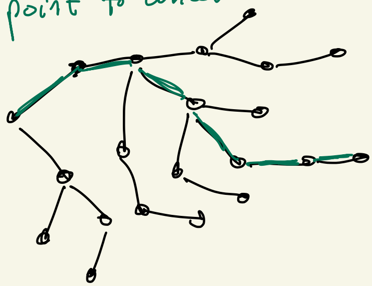
as longest path in H can be found in time $O(m)$ $\square$

Improvement if $G=(V,E)$ is undirected:

First we depth first search from an arbitrary vertex v

If we find a vertex w at depth k then we output the (v,w)-path P of the DFS tree

If no vertex of depth k is ever found then G has at most $k|V|$ edges since all back edges to the DFS tree point to ancestors:



Every back edge of green path go to a vertex on that path

So $|E| = |V|-1 + \#\text{back edges}$
$\leq |V|-1 + |V|\cdot(k-1)$
$< k|V|$

So either we found the desired path via DFS or we have $m \leq kn$

Now run the random permutation method on G. As $m \in O(kn)$ the expected time to find a good path is

$$O((k+1)!\,kn) = O((k+2)!\,n)$$

Theorem 2.2 Let $G=(V,E)$ be a (di)graph that contains a simple cycle of length k. Then such a cycle can be found in expected time $O(k!\log k\, V^{\omega})$ time, when ω is the constant such that multiplying two $n\times n$ matrices can be done in $O(n^{\omega})$ time.

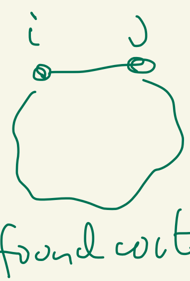
Proof Case 1 $G=(V,E)$ a graph $V=\{1,2,\dots,n\}$

Let D be a random acyclic orientation of G and consider the M^{k-1} when M is the adjacency matrix of D. So $M^{k-1}_{ij}=1 \Leftrightarrow D$ has an (i,j)-path of length $k-1$. And we can remember such a path Q_{ij}

If $\exists\, i\neq j$ s.t $M^{k-1}_{ij}=1$ and $ij\in E$ then the edge ij and any (i,j)-path of length $k-1$ (showing that $M^{k-1}_{ij}=1$) shows that G has a k-cycle.

The probability that an (i,j)-path Q of length $k-1$ in G will occur as a directed (i,j)-path in D is $\frac{2}{k!}$ (2 good permutations $\overrightarrow{Q}$ and $\overleftarrow{Q}$)

So if



k-cycle in G then this cycle is found with probability $\frac{2}{k!}$

It takes time $O(\log k |V|^{\omega})$ to calculate M^{k-1}

$M, M^2, M^4 \ldots M^p, M^{2p}$ $\quad p < k-1 < 2p$

write $k-1$ in binary eg 101101

and use those powers of M to find M^{k-1}

e.g $M^6 = M^4 \times M^2$ as $6 = 1100$

Case 3 D is a digraph on n vertices

Let π be a random permutation of $\{1, 2 \ldots n\}$

Form H_π from D and let M_H be adjacency matrix of H_π

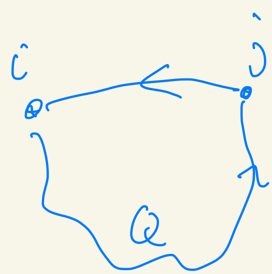
If M_H^{k-1} has a 1 in position (i, j) we check

if $j \to i$ is an arc of D.

If yes for some (i, j) we return yes (and a k-cycle)

Else repeat with new π.

k-cycle in D



Q has probability $\frac{1}{k!}$ of being a path in H_π

Expected # of repetitions is $\frac{1}{\frac{1}{k!}} = k!$

Random colorings

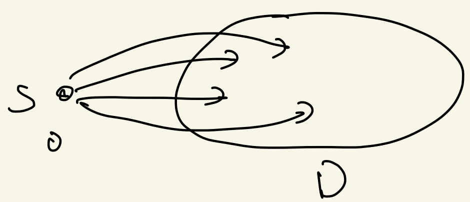
Given a (di)graph $G=(V,E)$ and a coloring $c: V \to k$. Then a path P in G is colorful if each vertex on P recived a different color via c.

Observations assuming $c: V \to k$ is random coloring

- If P is a colorful path, then P is a simple path
- Each simple path P of length $k-1$ has a chance of $\frac{k!}{k^k} > e^{-k}$ of becoming colorful

Lemma 3.1 Let $G=(V,E)$ be a (di)graph and $c: V \to [k]$ a random k-col of V. If G has a colorful path of length $k-1$ then such a path can be found in expected time $2^{O(k)} m$

Proof Add a new vertex s of color 0 with (arc) edges to all of V



The new (di)graph D' has a path of length k from s

$\Updownarrow$

D has a path of length $k-1$

We will construct a colorful path starting at s using dynamic programming:

For each $v \in V$ and $i \in [k]$ we maintain

- a list $C_{i,v}$ of subsets of size i such that some known (s,v)-path of length i uses exactly the colors from that subset. E.g. $i = 3$ and $C \in C_{3,v}$ $C = \{2,4,7\}$

$$\underset{s}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{7}{\circ} \longrightarrow \underset{4}{\overset{v}{\bullet}} \leftarrow P(C)$$

- An (s,v)-path $P(C)$ for each subset $C \in C_{i,v}$

Initially $i = 1$ and $C_{i,v} = \{\{c(v)\}\}$ only one subset in $C_{1,v}$

Make $C_{i+1,v}$ lists from $C_{i,v}$ lists:

$\forall v \in V \ \forall C \in C_{i,v}$ and path $P(C)$ corresponding to C:

$\forall vu \in A$: if $c(u) \notin C$

$C := C + c(u)$

$P(C) = P(C)u$ $\quad \underset{s}{\circ} \rightsquigarrow \underset{v}{\bullet} \to \underset{u}{\circ}$

Clearly D' has colorful path of length k from s

$\Updownarrow$ $\exists v$ s.t. $C_{k,v} \neq \emptyset$

Complexity:

- checking whether $c(u) \in C$ in round i is $O(i)$
- Total number of possible sets in round i $\binom{k}{i}$
- we check for each arc so $i\binom{k}{i}m$ in round i

$$\Rightarrow O\left(\sum_{i=0}^{k} i\binom{k}{i}m\right) = O\left(k \cdot m \cdot \sum_{i=0}^{k}\binom{k}{i}\right) = O(k2^k m)$$

Lemma 3.2 Let $G=(V,E)$ be a (di)graph and let $c: V \to [k]$ be a k-col of V. In time $2^{O(k)} nm$ we can find all pairs $x,y \in V$ which are connected by a colorful path of length $k-1$

Proof For a given $x \in V$ we can find all $y \in V-x$ s.t G has a colorful (x,y)-path of length $k-1$ in time $2^{O(k)} m$ by using the previous algorithm when s only has an arc to x so as there are n choices for x we get total time $2^{O(k)} n \cdot m$.

(second part of proof in Alon paper not person)

Now we are ready to show the power of color coding.

Theorem 3.3

If a (di)graph $G=(V,E)$ has a path of length $k-1$ then we can find such a path in $2^{O(k)} m$ expected time

Proof

A path $P = v_1 v_2 \cdots v_k$ of length $k-1$ has a chance of $\frac{k!}{k^k} > e^{-k}$ of becoming colorful under a random k-colouring of V

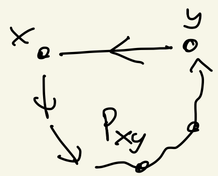
Thus the expected number of times we have to choose a random coloring before P becomes colorful (and thus G has a colorful path of length $k-1$) is at most $e^k = 2^{k \cdot \log_2 e}$

Running the algorithm of Lemma 3.1 $2^{k \log_2 e}$ times take $2^{O(k)} m$ time $\square$.

To look for k-cycles we use Lemma 3.2.

Theorem 3.4 If $G=(V,E)$ is a (di)graph with a k-cycle. Then we can find such a cycle in $2^{O(k)} n \cdot m$ expected time

proof consider a k-cycle of G (assume directed)



As before P_{xy} has a chance of $\frac{k!}{k^k} > e^{-k}$ of becoming colorful on the c so we check for all $x \neq y$ and all colourful (x,y)-paths whether $y \to x$ if yes we have found a k-cycle

Expected # repetitions before a fixed k-cycle is found is e^k and time to check all (x,y)-paths for one k-col c is $2^{O(k)} \cdot n \cdot m$ so since we expect to repeat at most e^k times, the expected time before some C_k is found is

$$2^{O(k)} n \cdot m$$

Note: for $k \in O(\log n)$ this is polynomial!

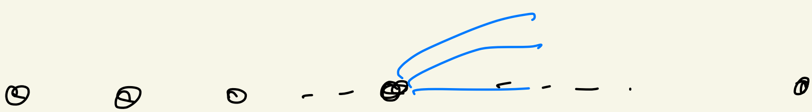
All algorithms above can be derandomized to give polynomial deterministic algorithms! not random!

Section 5. Finding cycles in minor closed graphs

Definition $G=(V,E)$ is *d-degenerate* if every subgraph of G has a vertex of degree at most d.

Lemma 5.1 Given a graph $G=(V,E)$ of degeneracy d we can find an acyclic orientation D of G s.t. $d^+(v) \le d$ $\forall v$ in time $O(E \log V)$

P: Repeat choosing a vertex v of minimum degree in current graph and remove this.
As G is k-degenerate the vertex v has at most d neighbours left so we get an ordering



at most k edges

now orient from left to right.

Minors A subgraph H of $G=(V,E)$ is a **minor** of G if H can be obtained from G by removal and contraction of edges

A class $\mathcal{C}$ of graphs is **minor-closed** if $G \in \mathcal{C}$ and H is a minor of G

$\Rightarrow H \in \mathcal{C}$

Example: Planar graphs

Theorem (Bollobás)

For every non-trivial minor-closed class of graphs there exist a constant $d_{\mathcal{C}}$ such that every graph $G \in \mathcal{C}$ has degeneracy at most $d_{\mathcal{C}}$

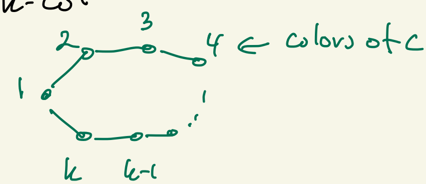
For example $d_{planar} \leq 5$ as every planar graph has a vertex of degree at most 5

Theorem 5.2 Let $\mathcal{C}$ be a non-trivial minor-closed class of graphs and let $k \geq 3$ be a fixed integer.
Then exists a randomized algorithm which given $G = (V, E)$ from $\mathcal{C}$ find a k-cycle C_k in G (if one exists) in $O(|V|)$ expected time.

Proof We may assume that G contains a C_k

Let $c : V \to [k]$ be a random k-col

A k-cycle C_k is well-colored if



$$\boxed{P\left(C_k \text{ well colored} = \frac{2}{k^{k-1}}\right)}$$

Remarks

The randomized alg A will run in time $O(k|V|)$ and for a given $c : V \to [k]$ A will find some well-colored C_k with probability at least $\frac{1}{(2d)^k}$ when d is the degeneracy of G, provided that some k-cycle is well-col under c.

Combining this with the initial random coloring phase we can find some C_k with probability at least

$$\boxed{\frac{1}{(2d)^k} \cdot \frac{2}{k^{k-1}}} \Rightarrow$$

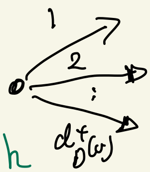
Expected no of repetitions color + A before a C_k is found is $O\left((2dk)^k |V|\right) = O(|V|)$ as k, d are constants

Algorithm $\mathcal{A}$:

Input $G=(V,E)$ and $c: V \to [k]$

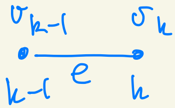
1. Delete all edges uv for which we do not have $|c(v)-c(u)| = 1 \bmod k$ (consecutive colors)
   (such edges cannot belong to a well-colored C_k)

2. Find acyclic orientation D of remaining graph such that $d^+_D(v) \le d$ for all v



3. $\forall v$ label the outgoing arcs by $1, 2, \dots d^+_D(v)$
   This takes $O(V)$ time as G is d degenerate

4. If C_k is well colored, then it contains an edge color



Now $\mathcal{A}$ guesses (by flipping fair coins)

- the orientation of e $k-1 \to k$ or $k \to k-1$ $p = \frac{1}{2}$
- the index of e in the ordering of arcs with the same tail ($k-1$ if $k-1 \to k$ else k) $p(\text{index} = i) = \frac{1}{d}$

let $i \in \{1, 2, \dots, d\}$ be the guessed index

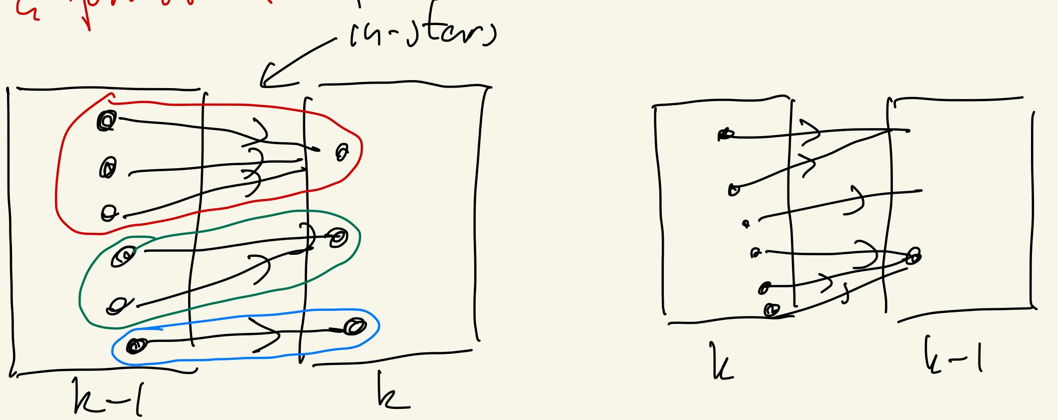
- If $\mathcal{A}$ guessed $\underset{k-1}{\bullet} \xrightarrow{i} \underset{k}{\circ}$ then all arcs from color $k-1$ to k with index $\neq i$ are removed

- If $\mathcal{A}$ guessed $\underset{k-1}{\circ} \xleftarrow{i} \underset{k}{\bullet}$ then remove all arcs from color k to $k-1$ which have index $\neq i$

Here we use that $\mathcal{C}$ is minor-closed to ensure that the resulting graph G' is also in $\mathcal{C}$

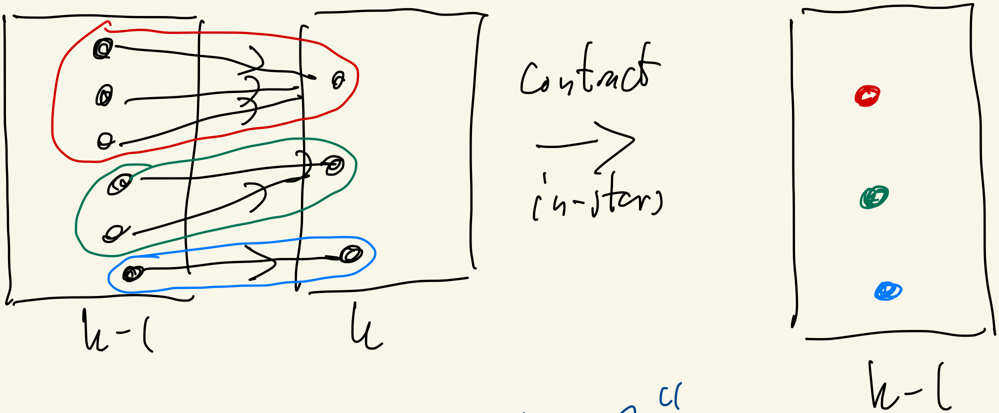
Assuming that G contains a well-colored C_k the new graph G' contains a well-colored C_k with probability at least $\frac{1}{2} \cdot \frac{1}{d} = \frac{1}{2d}$

In G' the subdigraph induced by colors $k-1$ and k is a forest of rooted stars:



Now contract each in-star to a single vertex and give it color $(k-1)$

Here we use that $\mathcal{C}$ is minor-closed so new $G'' \in \mathcal{C}$



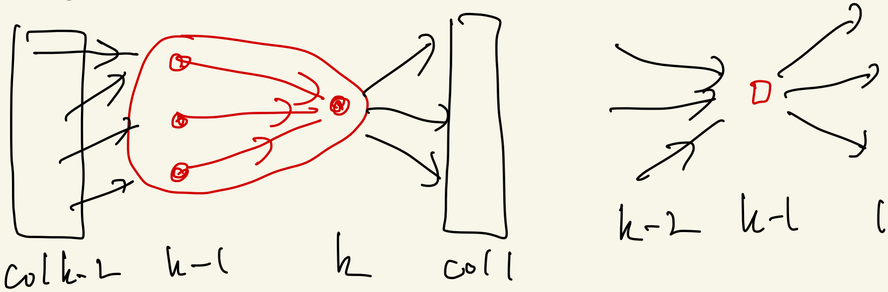
Call the new graph G'' and the new colors c''

G'' contains a well-colored C_{k-1}

$\Updownarrow$

G' contains a well-colored C_k

when we contract:



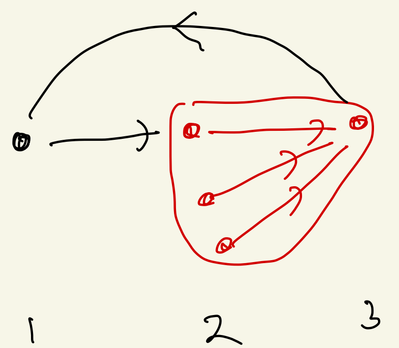
Recursive call: look for well-colored C_{k-1} in G''

In time $O((k-1)V)$ we will find a well-colored C_{k-1} in G'' with probability at least $\frac{1}{(2d)^{k-1}}$

From such a C_{k-1} we get a well-colored C_k in G' (and thus in G) if G' contains one and we saw that this happens with probability at least $\frac{1}{2d}$.

Total time spend is $O(kV)$ and we find a well-colored C_k with probability at least $\frac{1}{(2d)^k}$

How to stop recursion when $k=3$ before contraction



when we contract in-stars between cols 2 and 3 a well-colored C_3 becomes a well-colored C_2
easy to check for such a cycle in time $O(V)$ as G is d-degenerate

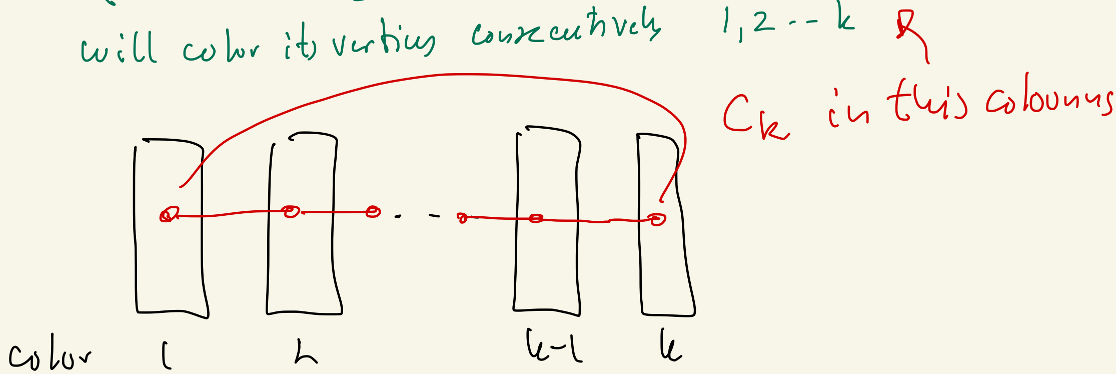
Derandomization

Theorem 5.3 Let $\mathcal{C}$ be minor-closed and let $k \geq 3$ be fixed. Checking for a C_k in $G \in \mathcal{C}$ can be done in deterministic time $O(V \log V)$

proof (simpler

- Replace the random colorings by a list $\mathcal{L}$ of $k^{O(k)} \log V$ colorings with the property that every ordered sequence of k vertices $v_1, v_2, \dots, v_k \in V$ receives colors $1, 2 \dots k \Leftrightarrow c(v_i) = i$ in at least one of the colorings from the list $\mathcal{L}$

- If G has a C_k then at least one of the colourings from $\mathcal{L}$ will color its vertices consecutively $1, 2 \dots k$



- Instead of guessing the direction and index of an edge e in a C_k we try all $2d$ possibilities for e.
  This give $(2d)^k$ possibilities for the k edges of a C_k

  when A reduces $k \to k-1 \to \dots \to 2$

Recall that $\mathcal{A}$ only keeps arcs of the current index i between color $k-1$ and color k and only arcs agreeing with the orientation of e

Since we consider all possible choices for i and the orientation in each step of the recursion, we will find at least one C_k if G has any