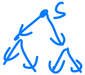


T_s^+

An **out-tree** in a digraph $D = (V, A)$ is a connected subdigraph T_s^+ of D in which every vertex of $V(T_s^+)$, except one vertex s (called the **root**) has exactly one arc entering. This is equivalent to saying that s can reach every other vertex of $V(T_s^+)$ by a directed path using only arcs of T_s^+.

An **out-branching** in a digraph $D = (V, A)$ is a spanning out-tree, that is, every vertex of V is in the tree. We use the notation B_s^+ for an out-branching rooted at the vertex s.

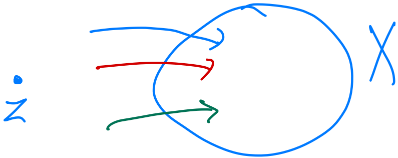
The following classical result due to Edmonds and the algorithmic proof due to Lovász, which we will give implies that one can check the existence of k arc-disjoint out-branchings in polynomial time.

Theorem 10 (Edmonds' branching theorem)

[Edmonds, 1973] A directed multigraph $D = (V, A)$ with a special vertex z has k arc-disjoint spanning out-branchings rooted at z if and only if

$$d^{-}(X) \geq k \qquad \text{for all } X \subseteq V - z. \tag{4}$$

By Menger's theorem, (4) is equivalent to the existence of k arc-disjoint-paths from z to every other vertex of D.



Checking whether $(\square)$ $d^-(X) \geq k \ \forall X \subseteq V-z$:

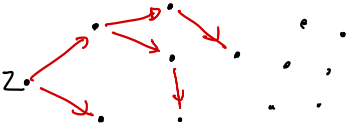
Checking for $\geq k$ (z,t)-paths in D:

- $D \to N_D = (V' \cup \{z,t\}, A, l \equiv 0, u \equiv 1)$ $V' = V \setminus \{z,t\}$
- Run Dinic's algorithm until we have a max flow or current (z,t)-flow x has value k. $O(n^{2/3} m)$
- If $|x| < k$ then let $(X, \bar{X})$ be a (z,t)-cut of capacity $r < k$, then $d^-(\bar{X}) = r < k$, showing that $(\square)$ does not hold.
- In time $(n-1) \cdot O(n^{2/3} m) = O(n^{5/3} m)$ we can check whether $(\square)$ holds.

Proof: (Lovász) The necessity is clear, so we concentrate on sufficiency. The idea is to grow an out-tree F from z in such a way that the following condition is satisfied:

$$d^-_{D-A(F)}(U) \geq k-1 \quad \text{for all } U \subseteq V - z. \tag{5}$$

If we can keep on growing F until it becomes spanning while always preserving (5), then the theorem follows by induction on k. To show that we can do this, it suffices to prove that we can add one more arc at a time to F until it is spanning.



Let us call a set $X \subseteq V - z$ **problematic** if $d^-_{D-A(F)}(X) = k - 1$.
It follows from the submodularity of $d^-_{D-A(F)}$ (recall Corollary 8) that, if X, Y are problematic and $X \cap Y \neq \emptyset$, then so are $X \cap Y, X \cup Y$ as we have

$$\begin{aligned}(k-1)+(k-1) &= d^-_{D-A(F)}(X) + d^-_{D-A(F)}(Y)\\ &\geq d^-_{D-A(F)}(X \cup Y) + d^-_{D-A(F)}(X \cap Y)\\ &\geq (k-1)+(k-1)\end{aligned}$$

Here the last in-equality follows from the fact that we have grown F so that (5) holds.
Observe also that, if X is problematic, then $X \cap V(F) \neq \emptyset$, because X has in-degree at least k in D.

If all problematic sets are contained in $V(F)$, then let $T = V$. Otherwise let T be a minimal (with respect to inclusion) problematic set which is not contained in $V(F)$.

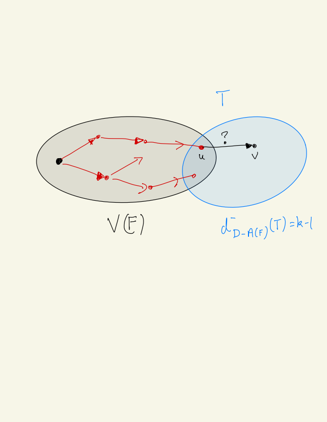


Figure: The situation when a problematic set exists

We claim that there exists an arc uv in D such that $u \in V(F) \cap T$ and $v \in T - V(F)$. Indeed if this was not the case then every arc that enters $T - V(F)$ also enters T and we would have

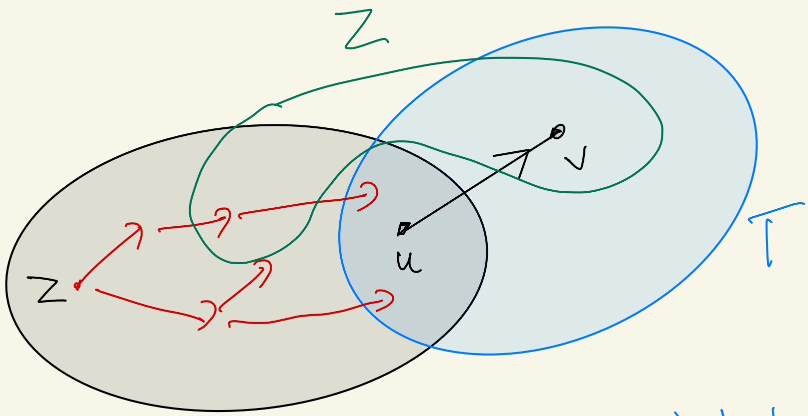
$$d^-_D(T - V(F)) = d^-_{D-A(F)}(T - V(F)) \leq d^-_{D-A(F)}(T) \leq k-1, \quad (6)$$

contradicting the assumption of the theorem. Hence there is an arc uv from $V(F) \cap T$ to $T - V(F)$.

Suppose the arc uv enters a problematic set Z. Then we have

$$\begin{aligned}
(k-1) + (k-1) &= d^-_{D-A(F)}(Z) + d^-_{D-A(F)}(T) \\
&\geq d^-_{D-A(F)}(Z \cup T) + d^-_{D-A(F)}(Z \cap T) \\
&\geq (k-1) + (k-1)
\end{aligned}$$

Thus $Z \cap T$ is problematic and size it is smaller that T (it does not contain u), we obtain a contradiction to the minimality of T.



$V(F)$

$$d^-_{D-A(F)}(T) = k-1$$

$$d^-_{D-A(F)}(Z) = k-1$$

$$\begin{aligned}(k-1)+(k-1) &= d^-_{D-A(F)}(T) + d^-_{D-A(F)}(Z) \\ &\geq d^-_{D-A(F)}(T\cap Z) + d^-_{D-A(F)}(T\cup Z) \\ &\geq (k-1) + (k-1)\end{aligned}$$

$\Downarrow$

$$d^-_{D-A(F)}(T\cap Z) = k-1$$

$\rightarrow \Leftarrow$ choice of T

Hence we can add the arc $u \to v$ to F and proceed. So the claim follows by induction. $\square$.

How to find a good arc $u \to v$ to add?

1) let F be the current out-tree and let $V' = V(F)$

2) For a given arc $u \to v$ with $u \in V'$ and $v \in V \setminus V'$ check whether there are $(k-1)$-arc-disjoint (z, v)-paths

one max flow calc with source z and sink v

3) we know that if $d^-(U) \geq k \;\forall\; U \subseteq V - z$ then there is at least one such arc $u \to v$ which can be added to F

Total time $(n-1)$ arcs added to F starting from $F = \emptyset$
$O(m)$ arcs out of $V(F)$ to check in each iteration in time $O(n^{2/3} m)$

so $O(n \cdot m \cdot n^{2/3} m) = O(n^{5/3} m^2)$

Implications of Edmonds' Branching Theorem

Corollary 11 (Even 1979)

Let $D = (V, A)$ be a k-arc-strong directed multigraph and let x, y be arbitrary distinct vertices of V. Then for every $0 \leq r \leq k$ there exist paths $P_1, P_2, \ldots, P_k$ in D which are arc-disjoint and such that the first r paths are (x, y)-paths and the last $k - r$ paths are (y, x)-paths.

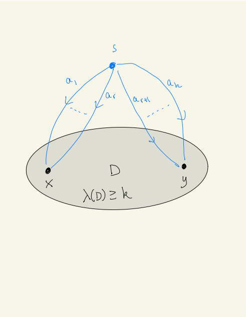


Figure: Proof of Corollary 11

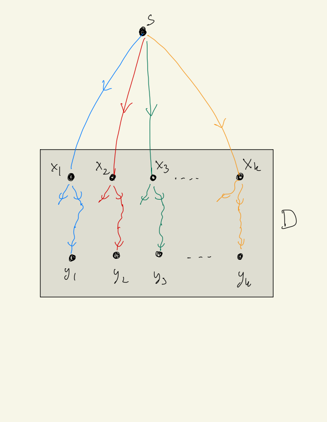
A directed multigraph $D = (V, A)$ is **weakly-k-linked** if it has a collection of arc-disjoint paths $P_1, \dots, P_k$ such that P_i is an (x_i, y_i)-path for every choice of (not necessarily distinct vertices $x_1, x_2, \dots, x_k, y_1, \dots, y_k \in V$.

Note that if D is weakly-k-linked then it is k-arc-strong since we can take $x_1 = \dots = x_k = x$ and $y_1 = \dots = y_k = y$ for arbitrarily chosen x, y, showing that $\lambda_D(x, y) \geq k$ and hence, by Menger's theorem, D is k-arc-strong.

Shiloach observed that Edmonds'branching theorem implies that the other direction also holds:

Theorem 12 (Shiloach 1979)

A directed multigraph D is weakly k-linked if and only if $\lambda(D) \geq k$.



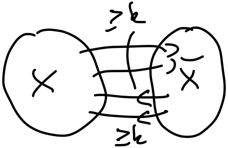
D+s satisfies the condition in Edmonds theorem as D is k-arc-strong

$\Downarrow$ $\exists$ k arc-disjoint branchings

$B^+_{s,1}, B^+_{s,2}, \dots, B^+_{s,k}$

Figure: Proof of Theorem 12

Edge-disjoint spanning trees



Recall Robbins'theorem

Theorem 13 (Robbins, 1939)

A graph G has a strongly connected orientattion if and only if G is connected and has no cut-edge (that is, $\lambda(G) \geq 2$.

Nash-Williams generalized this to the following.

Theorem 14 (Nash-Williams, 1960)

A graph G has a k-arc-strong orientation if and only if $\lambda(G) \geq 2k$.

Theorem 15 (Nash-Williams 1961, Tutte 1961)

Every graph G with $\lambda(G) \geq 2k$ has k edge-disjoint spanning trees.

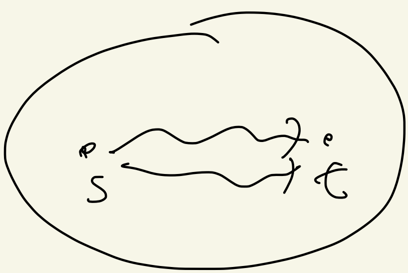
Proof: :

- Let $G = (V, E)$ satisfy that $\lambda(G) \geq 2k$.
- By Nash-Williams' theorem, G has an orientation $D = (V, A)$ with $\lambda(D) \geq k$
- Let z be an arbitrary vertex of D.
- As $d^-(X) \geq k$ for every proper subset X of V we also have $d^-(X) \geq k$ for every $X \subset V - z$.
- By Edmonds' branching theorem D has k arc-disjoint out-branchings $B^+_{z,1}, \ldots, B^+_{z,k}$.
- Back in G each of these correspond to a spanning tree.

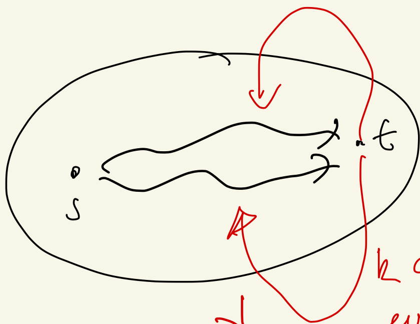
Edmonds branching theorem

$\Downarrow$

Menger's theorem (arc-version)



D



k arcs from t to every $v \neq s,t$

D'

D has k arc-disjoint (s,t)-paths

$\Updownarrow$

D' has k arc-disjoint out-branchings rooted at s

$\Updownarrow$ Edmonds branching theorem

$d^-_{D'}(U) \geq k \quad \forall U \subseteq V-s$

$\Updownarrow$

$d^-_{D}(U) \geq k \quad \forall U \subseteq V-s$ s.t. $t \in U$