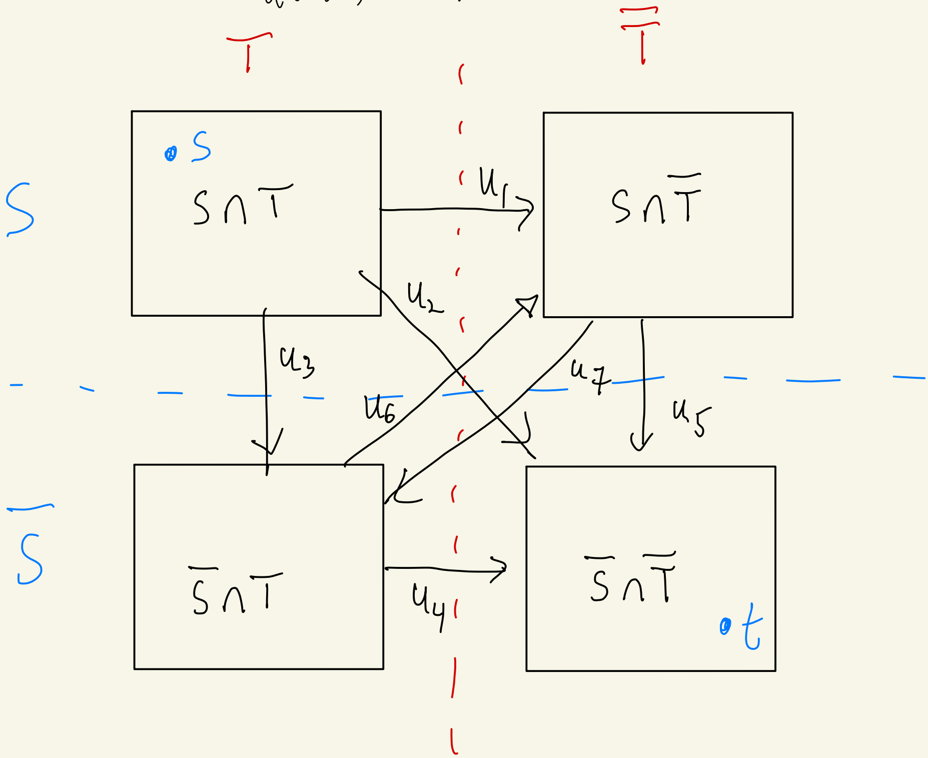
BJG 3.33

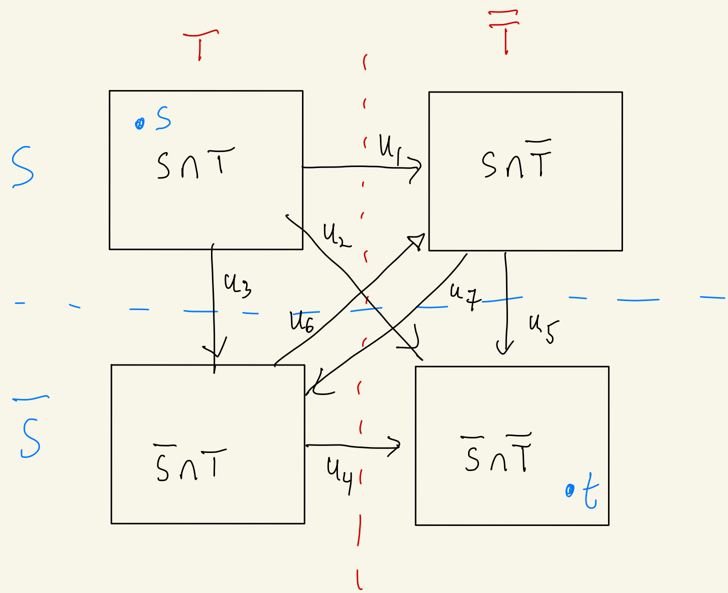
$N = (V \cup \{s,t\}, A, \ell \equiv 0, u)$

Claim: $(S,\bar{S})$ and $(T,\bar{T})$ (s,t)-cuts

$$\Downarrow$$

$$u(S,\bar{S}) + u(T,\bar{T}) \geq u(S\cap T, \overline{S\cap T}) + u(S\cup T, \overline{S\cup T})$$





$$u(S\cap T, \bar{S}\cap\bar{T}) + u(S\cup T, \overline{S\cup T})$$

$$= (u_1 + \underline{u_2} + u_3) + (\underline{u_2} + u_4 + u_5)$$

$$\leq (\underline{u_2} + u_3 + u_5 + u_7) + (u_1 + \underline{u_2} + u_4 + u_6)$$

$$= u(S, \bar{S}) + u(T, \bar{T})$$

BJG 3.34

If $(S, \bar{S})$ and $(T, \bar{T})$ are minimum (s,t)-cuts then $(S \cap T, \overline{S \cap T})$ and $(S \cup T, \overline{S \cup T})$ are also minimum (s,t)-cuts

Let $K = u(S, \bar{S}) = u(T, \bar{T})$

then

$$K + K = u(S, \bar{S}) + u(T, \bar{T})$$

by 3.33

$$\geq u(S \cap T, \overline{S \cap T}) + u(S \cup T, \overline{S \cup T})$$

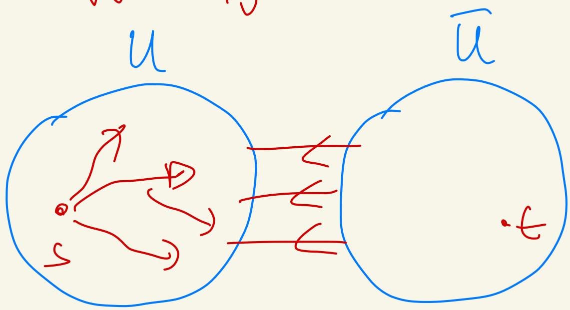
$$\geq K + K$$

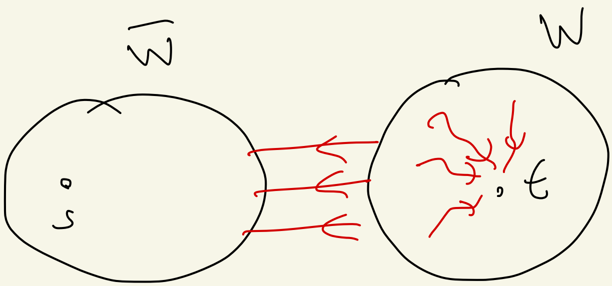
B) G 3.35 $\quad N = (V \cup \{s,t\}, A, \ell = 0, w)$

Let x be a max flow in N

Let $\mathcal{U} = \{ i \mid \exists\, (s,i)\text{-path in } N(x) \}$

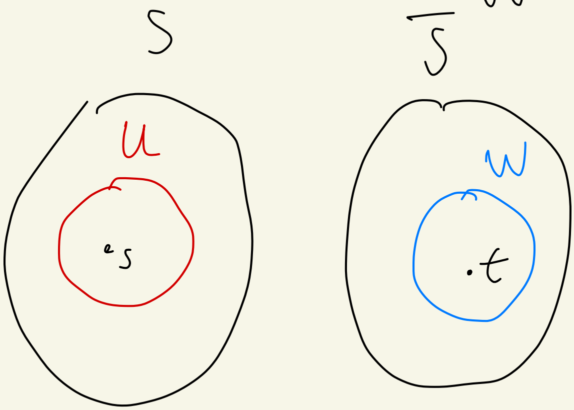
$W = \{ j \mid \exists\, (j,t)\text{-path in } N(x) \}$





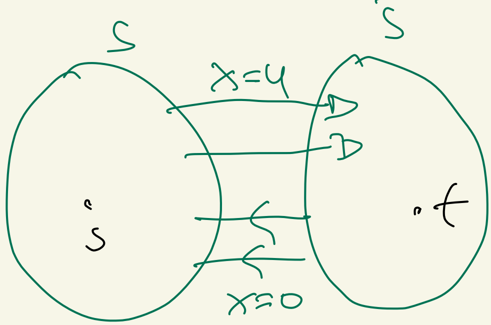
MFMC thm $\Rightarrow$ $(\mathcal{U}, \bar{\mathcal{U}})$ and $(\bar{W}, W)$ are min cuts

Claim $\forall$ min cut $(S, \bar{S})$ we have $U \subseteq S$ and $W \subseteq \bar{S}$



Note if x is a max flow and $(S, \bar{S})$ is a min cut then

$$|x| = x(S, \bar{S}) - x(\bar{S}, S)$$

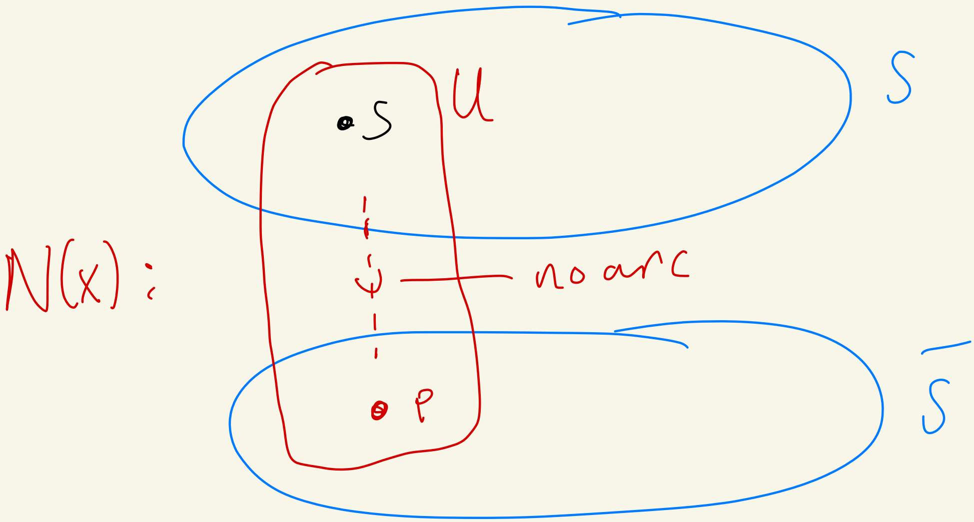


Sufficient to show that

$$U \subseteq S$$

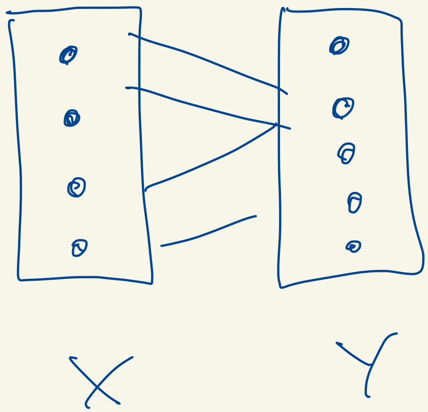
Suppose $U \not\subseteq S$ then $U \cap \bar{S} \neq \emptyset$

($U \cap S \neq \emptyset$ as $x \in U \cap S$)



f-factor problem for bipartite $G \to$ flow problem

G

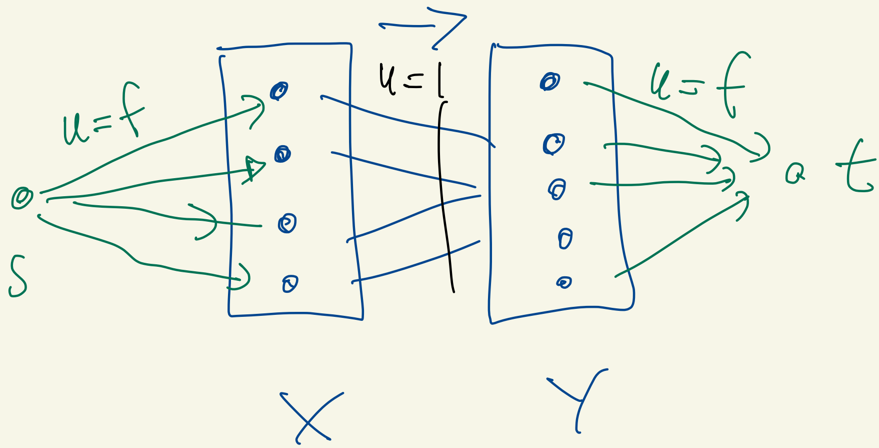


must have

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(y)$$

or no solution

N_G



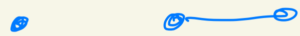
Easy: N_G has an (s,t)-flow of value $\sum_{x \in X} f(x)$ if and only if G has an f-factor

Schrijver 5.1

(i) A tree T has ≤ 1 perfect matchings

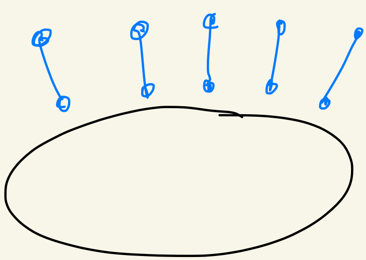
proof by induction

$|V(T)| \le 2$ ✓



Suppose claim holds for trees with $< n$ vertices and let T have n vertices

Each leaf must be matched to its parent so if two leaves have the same parent then is no solution. otherwise we have



← $T' = T$ minus leaves and their parents

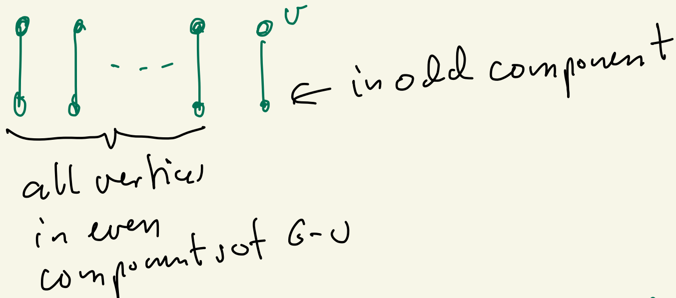
By induction, T' has at most one perfect matching

So T has at most one p.m. $\square$.

(ii) T has a p.m $\Updownarrow$ $\forall v$ $T-v$ has exactly one odd component

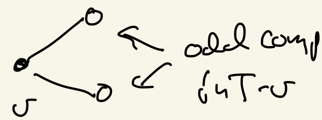
Proof (without using Tutte)

$\Downarrow$



$\Uparrow$ suppose $T-v$ has exactly one odd comp. $\forall v \in V$

- $|V(T)|$ is even:
  Look at arbitrary v and components of $T-v$ one is odd so $T-v$ has an odd # vertices $\Rightarrow |V(T)|$ even
- no vertex is parent to 2 leaves



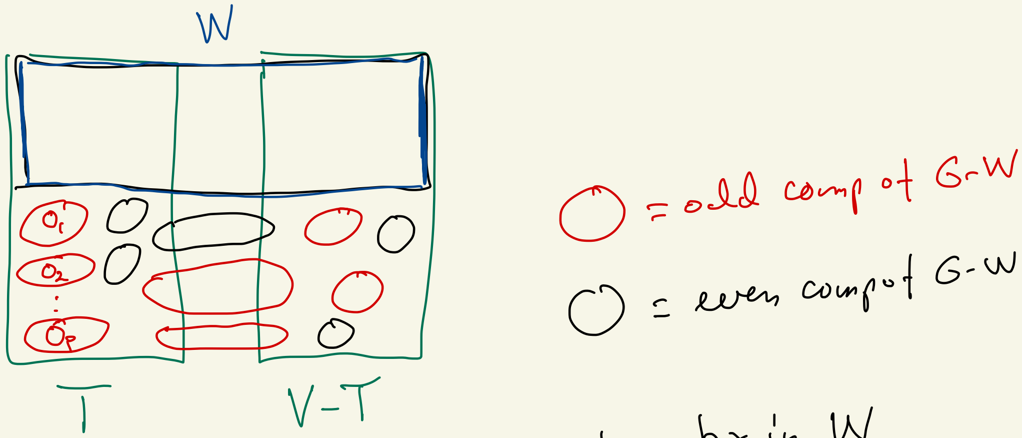
- If $T = \bullet\!-\!\circ$ ok so apply induction to $T' = T - uu'$ when u' is a leaf and u is its parent
- $T'-v$ has exactly one odd comp $\forall v \in V(T')$ so by induction it has a p.m M and $M + uu'$ is a p.m of T

Schrijver 5.4 $G=(V,E)$ $T \subseteq V$

Claim: G has a matching M covering T ($T \subseteq V(M)$)

$\Updownarrow$

(*) $\forall W \subseteq V$: # odd components of $G-W$ with all vertices in T is at most $|W|$

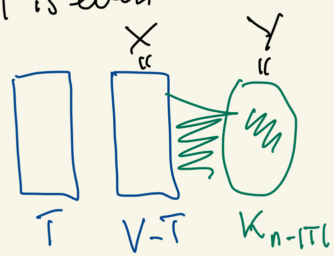


$\Downarrow$: each of $O_1 \cdots O_p$ need a private vertex in W

so must have (*)

$\Uparrow$ Case 1: $|T|$ is even

Build G':



$n = |V-T|$

Easy to see that G' has perfect matching

$\Updownarrow$

G has matching covering T

Consider $W' \subseteq V(G')$ such that $odd(W') - |W'|$ is maximized. Here $odd(W')$ is # odd components in $G'-W'$

If $odd(W') \leq |W'|$ then G' has p.m by Tutte

so assume $odd(W') > |W'|$

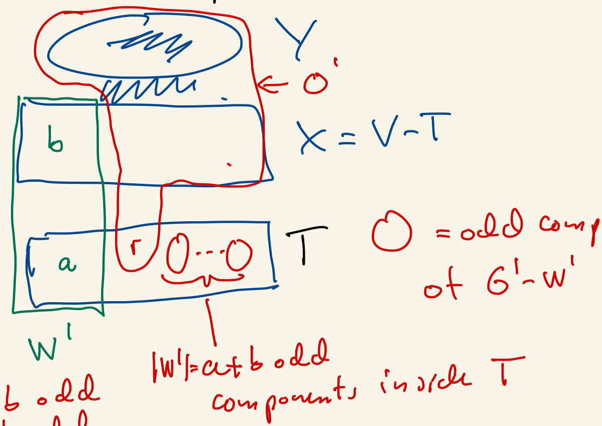
Then $odd(W') = |W'|+1$ as G satisfies (*) and there is at most one component in $G'[X \cup Y - W']$ (zero if $X \cup Y \subseteq W'$)

As $G'[X \cup Y - W']$ is connected (or empty) at least $|W'|$ of the odd components of $G'-W'$ are contained in T

Let $W = W' \cap V$ and note that if O is an odd component of $G'-W'$ with $O \subseteq T$ then O is also an odd comp of $G-W$

This and (*) implies that $W = W'$ so $W' \cap Y = \phi$

By (*) at least one of the odd components of $G'-W'$ intersects X and this contains all of $Y \cup (X-W')$



$|O'| = |Y| + |X-W'| + r$

$= n-t+n-t-b+r$

$= 2(n-t)+r-b \implies r-b$ odd $\implies$ $\underline{r+b \text{ odd}}$

on the other hand t is even so ↔ contradiction

$r+a+a+b = r+b+2a$ is even $\implies$ $\underline{r+b \text{ even}}$

<u>Case 2 $|T|$ is odd</u> then the picture is the same instead of when we have taken $Y = K_{n-t+1}$ instead of $Y = K_{n-t+1}$. Thus we get

$|O'| = |Y| + |X-W'| + r = n-t+1+n-t-b+r = 2(n-t)+r-b+1$

$\implies r-b+1$ odd $\implies r+b+1$ odd $\implies r+b$ even

But t is odd so $r+a+a+b = r+b+2a$ is odd

$\implies r+b$ odd $\quad ↯$

PS problem 5 p.243 and Schrijver appl. 5.2

2-processor scheduling

Given a set of jobs J and an acyclic digraph D (called a DAG) on the jobs such that $i \to j$ in $D \Leftrightarrow$ job i must be done before job j can be done

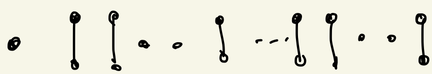
From $D=(J,A)$ we can build an undirected graph $G_P=(J,E_P)$ where $E_P = \{ij \mid i,j \in J$ and D has no (i,j)-path and no (j,i)-path$\}$

So $ij \in E_P \Leftrightarrow$ i and j may be scheduled at the same time

Let $s_1, s_2, \ldots, s_t$ be a schedule of the jobs in J such that

- $|\{j \in J \mid s(j) = i\}| \le 2 \quad \forall i = 1,2,\ldots,t$
- $ij \in A \Rightarrow s(i) < s(j)$

Then those indices $i \in [t]$ for which $|\{j \in J \mid s(j)=i\}| = 2$ form a matching in G_p so in G_p the schedule s corresponds to a matching



and $t = |J| - |M|$

$i \quad 1 \quad 2 \quad 3 \quad \ldots \quad t$

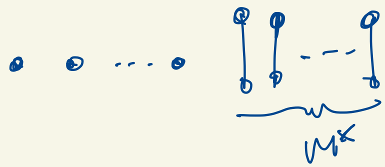
This shows that in particular the schedule length t is at least $|J| - |M^*|$ when M^* is a maximum matching of G_p

Part 2 (see Schrijver appl 5.2)

Claim: there exists a schedule $S: J \to [t]$

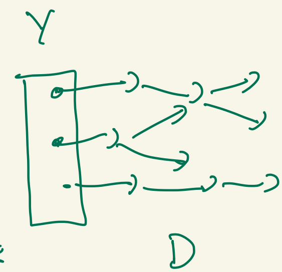
with $t = |J| - |M^*|$

Let Q be a minimum partition of J into vertices and edges of G, that is, Q comes from a maximum matching M^* and the unpaired vertices



Let Y be the sources of D

(in-degree $= 0$)



- Suppose $q \in Y$ and q is not matched by M^*

Then consider $J' = J - \{q\}$

By induction J' can be scheduled in time t' where

$t' = |J'| - \nu'(G') = (|J|-1) - \nu(G) = |J| - \nu(G) - 1$

so $t = |J| - \nu(G)$ as we can schedule q at time 1.

$\boxed{\text{So every vertex in } Y \text{ is matched by } M^*}$

(☐) • Suppose next that some $q \in Y$ is matched in M^* to some $q' \in Y$

Then remove q, q' and apply induction

$J'' = J - \{q, q'\}$ has a schedule taking time t'' where

$t'' = |J''| - \nu(G'')$

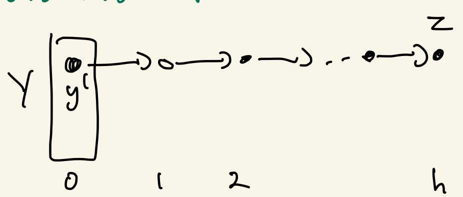
Here $\nu(G'') = |M^*| - 1 = \nu(G) - 1$ so

$t'' = |J| - 2 - (\nu(G) - 1) = |J| - 1 - \nu(G)$ and by scheduling qq' at time 1, we get a schedule of length

$t = t'' + 1 = |J| - \nu(G)$

Hence we can assume that every $q \in Y$ is matched to some $q' \notin Y$

Now choose $yz \in M$ such that $y \in Y$ and z has minimum height, where height of $v \in D$ is the length of a longest path ending in z:



Let $y' \in Y$ be the initial vertex of a longest path from Y to z

Then $y' \neq y$ as $yz \in E_P$

Let $z'y'$ be the matching edge for y' in M^*

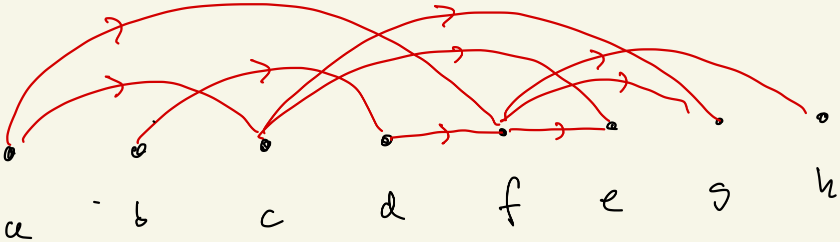
Then there is no path $z' \leadsto z$ by the choice of z and there is no path $z \leadsto z$ since $q'z' \in M^*$ and $q' \leadsto z$ is a path in D. Thus $zz' \in E_P$

$yy' \in E_P$ as both y and y' are minimal

Now let M' be the matching $M' = M^* - \{yz, y'z'\} + \{yy', zz'\}$

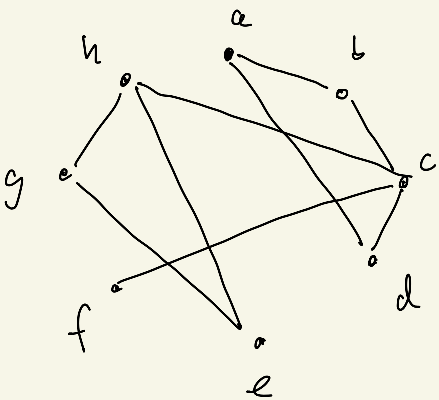
and finish as above (☐)

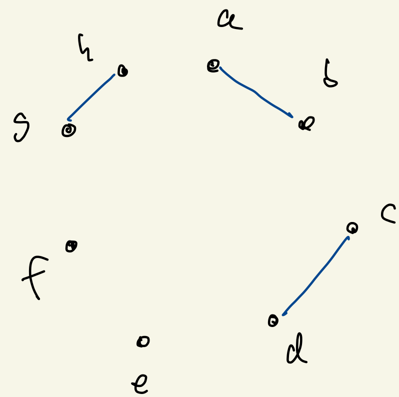
2-processor scheduling example on Weekly note 5



D

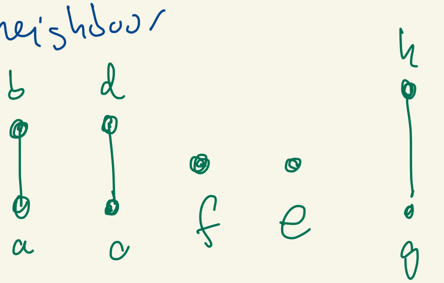
G_D





$|M^*| = 3$

No matching of size 4: If yes then it has to contain fc and then da but now b has no neighbour



optimum schedule as every schedule takes time at least

$|J| - |M^*| = 8 - 3 = 5$

PS problem 11 p 245

Given $D=(V,A)$ check if it has a spanning subdigraph $D'=(V,A')$ with $d^+_{D'}(v)=d^-_{D'}(v)=1$

Let $B_D=(V',V'',E)$ when $u'v''\in E$ if and only if $u\to v\in A$

Then B_D has a perfect matching

$\Updownarrow$

D' exists

We know how to check whether B_D has a perfect matching via flows

PS problem 10 p 303

$G=(V,E)$ and $w: V \to \mathbb{Z}_+$

Find matching M such that $w(M) = \sum_{v \in V(M)} w(v)$ is maximized

Claim The greedy algorithm finds a maximum weight matching

(a) Greedy algorithm :

$\mathcal{G}$

$X \leftarrow \emptyset$; $V' \leftarrow V$; $M \leftarrow \emptyset$

while $V' \neq \emptyset$

let $v \in V'$ have max weight

$V' \leftarrow V' - v$

(I) If v is matched by M

$X \leftarrow X + v$

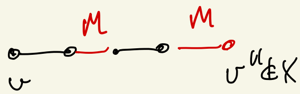
Else

if $\exists$ M-alternating path Q from v to some v''

(II) with $w(v'') < w(v)$:

$M \leftarrow M \Delta Q$

$X \leftarrow X + v$

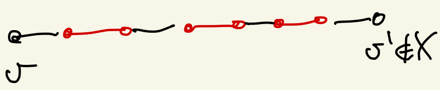


Else

if $\exists$ an M augmenting path P from v to some v' ($\notin X$)

(III) $M \leftarrow M \Delta P$

$X \leftarrow X + v$



end

Return X, M

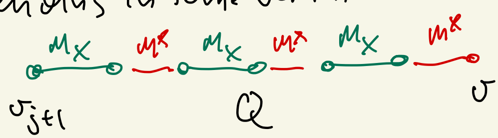
We want to prove that $M = M_X$ is a max weight matching

order the element of X as $v_1, v_2, \dots v_{|X|}$ in the order they are added.

Claim $\exists$ optimum weight matching M^* which covers all of X.

suppose not and let M^* be chosen s.t $\{v_1, v_2 \dots v_j\} \subseteq$ is matched by M^* by $\{v_1, v_2, \dots v_{j+1}\}$ is not matched by any optimal matching.

Consider $M_X \Delta M^*$: The optimality of M^* and the fact that all vertices have positive weight, implies that $M_X \Delta M^*$ has an alternating path Q starting in v_{j+1} and ending in some vertex v not matched by M_X:



If $w(v) \leq w(v_{j+1})$ then $w(M^* \Delta Q) \geq w(M^*)$

and $M' = M^* \Delta Q$ matches all of $\{v_1, v_2, \dots v_{j+1}\}$

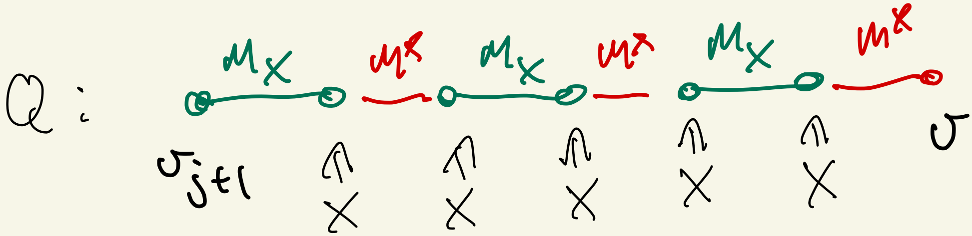
contradiction

Hence we must have $w(v) > w(v_{j+1})$

consider the step t when we extract v from V'

If some vertex of $Q - v_{j+1}$ is not in X at that time we would have added v to X via one of (I), (III), (II)

so we have at step t



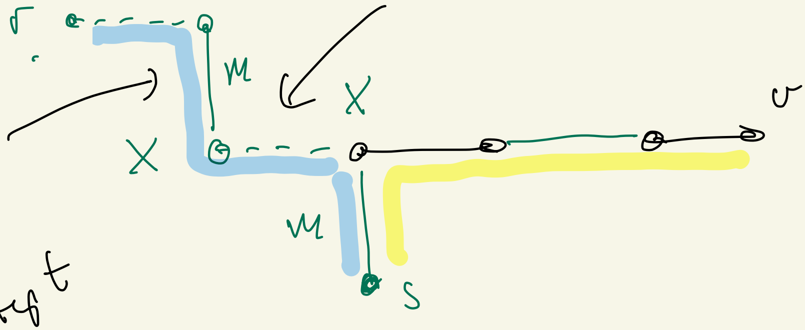
This implies that we should have added v to X in step t as the current X contains all vertices of $Q - \{v_{j+1}, v\}$ so one of (I), (II), (III) will apply

See next page!

Either

first edge of Q from v not in M at step t

path used to add r after step t



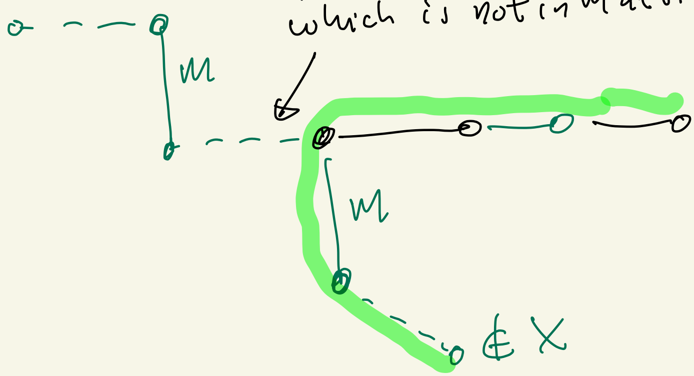
$w(r) > w(s)$ and $w(v) > w(r)$

$\Rightarrow$ (II) applies to at step t

or

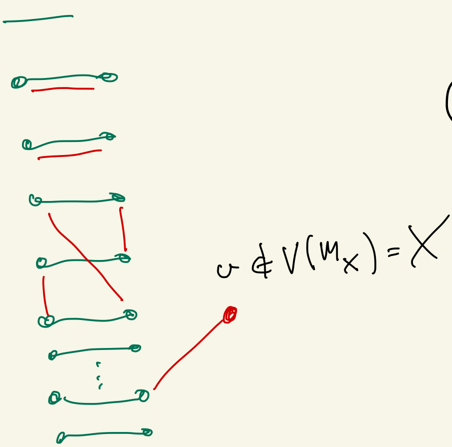
$w(r) < w(v)$

first edge of Q from v which is not in M at step t



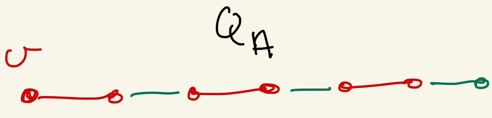
can be used in (III)

Suppose M^* is optimum weight and $V(M_X) < V(M^*)$



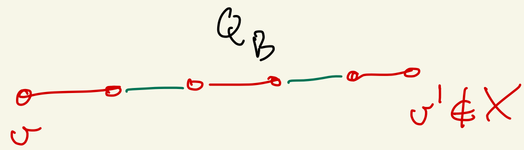
in $M_X \Delta M^*$ there's either

(A)



or

(B)



again consider the step t when v is removed from V' and we can argue as before that all vertices of $Q_A - v$ and all vertices of $Q_B - \{v, v'\}$ are in X after step $t-1$. This implies that we could have added v using either (II) or (III) contradiction.

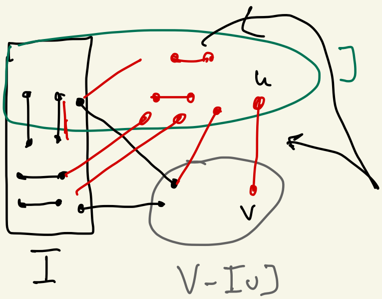
Hence $M^* = M_X$ and M_X is optimum.

$\square$

(b) Say that $U \subseteq V$ is independent if G has a matching that covers U and let $\mathcal{M}$ be the independent subsets of V

Suppose $I, J \in \mathcal{M}$ with $|J| = |I| + 1$ and let M_J cover J and M_I cover I

If M_I covers some $x \in J - I$ then $I + x \in \mathcal{M}$ (using M_I again). So assume M_I covers no vertex of $J - I$.



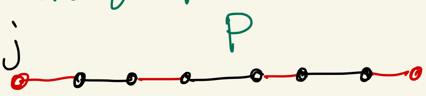
$M_I =$ o—o

$M_J =$ o—o

If $\exists\, uv \in M_J$ st $u \in J - I$ and $v \notin V(M_I)$ then $I + u$ good via $M_I + uv$

Consider $M_I \Delta M_J$ as $|M_J| > |M_I|$

this graph contains an M_I alternating path as follows



← not in I (unmatched by M_I)

$j \in J - I$

$M_I \Delta P$ covers $I + j$

(c) Let $\mathcal{U}, \mathcal{U}' \subseteq W \subseteq V$ be maximal independent subsets of W
(so $\mathcal{U}, \mathcal{U}' \in \mathcal{M}$ and are matched by $M_{\mathcal{U}}$ resp. $M_{\mathcal{U}'}$)

Suppose $|\mathcal{U}'| > |\mathcal{U}|$

Then as in (b) we can find some $u' \in \mathcal{U}'$ s.t. $\mathcal{U} + u' \in \mathcal{M}$. But $\mathcal{U} + u' \subseteq W$ $\lightning$ maxof $\mathcal{U}$

(d) skipped!