13. Matroids

Many combinatorial optimization problems can be formulated as follows. Given a set system $(E, \mathcal{F})$, i.e. a finite set E and some $\mathcal{F} \subseteq 2^E$, and a cost function $c : \mathcal{F} \to \mathbb{R}$, find an element of $\mathcal{F}$ whose cost is minimum or maximum. In the following we assume that c is a modular set function, i.e. we have $c : E \to \mathbb{R}$ and $c(X) = \sum_{e \in X} c(e)$.

In this chapter we restrict ourselves to those combinatorial optimization problems where $\mathcal{F}$ describes an independence system (i.e. is closed under subsets) or even a matroid. The results of this chapter generalize several results obtained in previous chapters.

In Section 13.1 we introduce independence systems and matroids and show that many combinatorial optimization problems can be described in this context. There are several equivalent axiom systems for matroids (Section 13.2) and an interesting duality relation discussed in Section 13.3. The main reason why matroids are important is that a simple greedy algorithm can be used for optimization over matroids. We analyze greedy algorithms in Section 13.4 before turning to the problem of optimizing over the intersection of two matroids. As shown in Sections 13.5 and 13.7 this problem can be solved in polynomial time. This also solves the problem of covering a matroid by independent sets as discussed in Section 13.6.

13.1 Independence Systems and Matroids

Definition 13.1. *A set system* $(E, \mathcal{F})$ *is an* **independence system** *if*

(M1) $\emptyset \in \mathcal{F}$;
(M2) *If* $X \subseteq Y \in \mathcal{F}$ *then* $X \in \mathcal{F}$.

The elements of $\mathcal{F}$ *are called* **independent**, *the elements of* $2^E \setminus \mathcal{F}$ **dependent**. *Minimal dependent sets are called* **circuits**, *maximal independent sets are called* **bases**. *For* $X \subseteq E$, *the maximal independent subsets of* X *are called bases of* X.

Definition 13.2. *Let* $(E, \mathcal{F})$ *be an independence system. For* $X \subseteq E$ *we define the* **rank** *of* X *by* $r(X) := \max\{|Y| : Y \subseteq X,\ Y \in \mathcal{F}\}$. *Moreover, we define the* **closure** *of* X *by* $\sigma(X) := \{y \in E : r(X \cup \{y\}) = r(X)\}$.

Throughout this chapter, $(E, \mathcal{F})$ will be an independence system, and $c : E \to \mathbb{R}$ will be a cost function. We shall concentrate on the following two problems:

Maximization Problem For Independence Systems	
Instance:	An independence system $(E, \mathcal{F})$ and $c : E \to \mathbb{R}$.
Task:	Find an $X \in \mathcal{F}$ such that $c(X) := \sum_{e \in X} c(e)$ is maximum.

Minimization Problem For Independence Systems	
Instance:	An independence system $(E, \mathcal{F})$ and $c : E \to \mathbb{R}$.
Task:	Find a basis B such that $c(B)$ is minimum.

The instance specification is somewhat vague. The set E and the cost function c are given explicitly as usual. However, the set $\mathcal{F}$ is usually not given by an explicit list of its elements. Rather one assumes an oracle which – given a subset $F \subseteq E$ – decides whether $F \in \mathcal{F}$. We shall return to this question in Section 13.4.

The following list shows that many combinatorial optimization problems actually have one of the above two forms:

(1) Maximum Weight Stable Set Problem
Given a graph G and weights $c : V(G) \to \mathbb{R}$, find a stable set X in G of maximum weight.
Here $E = V(G)$ and $\mathcal{F} = \{F \subseteq E : F \text{ is stable in } G\}$.

(2) TSP
Given a complete undirected graph G and weights $c : E(G) \to \mathbb{R}_+$, find a minimum weight Hamiltonian circuit in G.
Here $E = E(G)$ and $\mathcal{F} = \{F \subseteq E : F \text{ is subset of a Hamiltonian circuit in } G\}$.

(3) Shortest Path Problem
Given a digraph G, $c : E(G) \to \mathbb{R}$ and $s, t \in V(G)$ such that t is reachable from s, find a shortest s-t-path in G with respect to c.
Here $E = E(G)$ and $\mathcal{F} = \{F \subseteq E : F \text{ is subset of an } s\text{-}t\text{-path}\}$.

(4) Knapsack Problem
Given nonnegative numbers c_i, w_i $(1 \le i \le n)$, and k, find a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{j \in S} w_j \le k$ and $\sum_{j \in S} c_j$ is maximum.
Here $E = \{1, \dots, n\}$ and $\mathcal{F} = \left\{F \subseteq E : \sum_{j \in F} w_j \le k\right\}$.

(5) Minimum Spanning Tree Problem
Given a connected undirected graph G and weights $c : E(G) \to \mathbb{R}$, find a minimum weight spanning tree in G.
Here $E = E(G)$ and $\mathcal{F}$ is the set of forests in G.

(6) Maximum Weight Forest Problem
Given an undirected graph G and weights $c : E(G) \to \mathbb{R}$, find a maximum weight forest in G.
Here again $E = E(G)$ and $\mathcal{F}$ is the set of forests in G.

(7) Minimum Steiner Tree Problem
Given a connected undirected graph G, weights $c : E(G) \to \mathbb{R}_+$, and a set

$T \subseteq V(G)$ of terminals, find a Steiner tree for T, i.e. a tree S with $T \subseteq V(S)$ and $E(S) \subseteq E(G)$ all whose leaves are elements of T, such that $c(E(S))$ is minimum.
Here $E = E(G)$ and $\mathcal{F} = \{F \subseteq E : F$ is a subset of a Steiner tree for $T\}$.

(8) MAXIMUM WEIGHT BRANCHING PROBLEM
Given a digraph G and weights $c : E(G) \to \mathbb{R}$, find a maximum weight branching in G.
Here $E = E(G)$ and $\mathcal{F}$ is the set of branchings in G.

(9) MAXIMUM WEIGHT MATCHING PROBLEM
Given an undirected graph G and weights $c : E(G) \to \mathbb{R}$, find a maximum weight matching in G.
Here $E = E(G)$ and $\mathcal{F}$ is the set of matchings in G.

This list contains *NP*-hard problems ((1),(2),(4),(7)) as well as polynomially solvable problems ((5),(6),(8),(9)). Problem (3) is *NP*-hard in the above form but polynomially solvable for nonnegative weights. (See Chapter 15.)

Definition 13.3. *An independence system is a* **matroid** *if*

(M3) *If $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.*

The name matroid points out that the structure is a generalization of matrices. This will become clear by our first example:

Proposition 13.4. *The following independence systems $(E, \mathcal{F})$ are matroids:*

(a) *E is a set of columns of a matrix A over some field, and $\mathcal{F} := \{F \subseteq E :$ The columns in F are linearly independent over that field$\}$.*
(b) *E is a set of edges of some undirected graph G and $\mathcal{F} := \{F \subseteq E : (V(G), F)$ is a forest$\}$.*
(c) *E is a finite set, k an integer and $\mathcal{F} := \{F \subseteq E : |F| \leq k\}$.*
(d) *E is a set of edges of some undirected graph G, S a stable set in G, k_s integers ($s \in S$) and $\mathcal{F} := \{F \subseteq E : |\delta_F(s)| \leq k_s$ for all $s \in S\}$.*
(e) *E is a set of edges of some digraph G, $S \subseteq V(G)$, k_s integers ($s \in S$) and $\mathcal{F} := \{F \subseteq E : |\delta_F^-(s)| \leq k_s$ for all $s \in S\}$.*

Proof: In all cases it is obvious that $(E, \mathcal{F})$ is indeed an independence system. So it remains to show that (M3) holds. For (a) this is well known from Linear Algebra.

To prove (M3) for (b), let $X, Y \in \mathcal{F}$ and suppose $Y \cup \{x\} \notin \mathcal{F}$ for all $x \in X \setminus Y$. We show that $|X| \leq |Y|$. For each edge $x = \{v, w\} \in X$, v and w are in the same connected component of $(V(G), Y)$. Hence each connected component $Z \subseteq V(G)$ of $(V(G), X)$ is a subset of a connected component of $(V(G), Y)$. So the number p of connected components of the forest $(V(G), X)$ is greater than or equal to the number q of connected components of the forest $(V(G), Y)$. But then $|V(G)| - |X| = p \geq q = |V(G)| - |Y|$, implying $|X| \leq |Y|$.

For (c), (d) and (e) the proof of (M3) is trivial. □

Some of these matroids have special names: The matroid in (a) is called the **vector matroid** of A. Let $\mathcal{M}$ be a matroid. If there is a matrix A over the field F such that $\mathcal{M}$ is the vector matroid of A, then $\mathcal{M}$ is called **representable over** F. There are matroids that are not representable over any field.

The matroid in (b) is called the **cycle matroid of** G and will sometimes be denoted by $\mathcal{M}(G)$. A matroid which is the cycle matroid of some graph is called a **graphic matroid**.

The matroids in (c) are called **uniform matroids**.

In our list of independence systems at the beginning of this section, the only matroids are the graphic matroids in (5) and (6). To check that all the other independence systems in the above list are not matroids in general is easily proved with the help of the following theorem (Exercise 1):

Theorem 13.5. *Let $(E, \mathcal{F})$ be an independence system. Then the following statements are equivalent:*

(M3) *If $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.*
(M3′) *If $X, Y \in \mathcal{F}$ and $|X| = |Y|+1$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.*
(M3″) *For each $X \subseteq E$, all bases of X have the same cardinality.*

Proof: Trivially, (M3)⇒(M3′)⇒(M3″). To prove (M3″)⇒(M3), let $X, Y \in \mathcal{F}$ and $|X| > |Y|$. By (M3″), Y cannot be a basis of $X \cup Y$. So there must be an $x \in (X \cup Y) \setminus Y = X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$. □

Sometimes it is useful to have a second rank function:

Definition 13.6. *Let $(E, \mathcal{F})$ be an independence system. For $X \subseteq E$ we define the* **lower rank** *by*

$$\rho(X) := \min\{|Y| : Y \subseteq X,\ Y \in \mathcal{F} \text{ and } Y \cup \{x\} \notin \mathcal{F} \text{ for all } x \in X \setminus Y\}.$$

The **rank quotient** *of $(E, \mathcal{F})$ is defined by*

$$q(E, \mathcal{F}) := \min_{F \subseteq E} \frac{\rho(F)}{r(F)}.$$

Proposition 13.7. *Let $(E, \mathcal{F})$ be an independence system. Then $q(E, \mathcal{F}) \leq 1$. Furthermore, $(E, \mathcal{F})$ is a matroid if and only if $q(E, \mathcal{F}) = 1$.*

Proof: $q(E, \mathcal{F}) \leq 1$ follows from the definition. $q(E, \mathcal{F}) = 1$ is obviously equivalent to (M3″). □

To estimate the rank quotient, the following statement can be used:

Theorem 13.8. (Hausmann, Jenkyns and Korte [1980]) *Let $(E, \mathcal{F})$ be an independence system. If, for any $A \in \mathcal{F}$ and $e \in E$, $A \cup \{e\}$ contains at most p circuits, then $q(E, \mathcal{F}) \geq \frac{1}{p}$.*

Proof: Let $F \subseteq E$ and J, K two bases of F. We show $\frac{|J|}{|K|} \geq \frac{1}{p}$.

Let $J \setminus K = \{e_1, \dots, e_t\}$. We construct a sequence $K = K_0, K_1, \dots, K_t$ of independent subsets of $J \cup K$ such that $K_i \cap \{e_1, \dots, e_t\} = \{e_1, \dots, e_i\}$ and $|K_{i-1} \setminus K_i| \leq p$.

Since $K_i \cup \{e_{i+1}\}$ contains at most p circuits and each such circuit must meet $K_i \setminus J$ (because J is independent), there is an $X \subseteq K_i \setminus J$ such that $|X| \leq p$ and $(K_i \setminus X) \cup \{e_{i+1}\} \in \mathcal{F}$. We set $K_{i+1} := (K_i \setminus X) \cup \{e_{i+1}\}$.

Now $J \subseteq K_t \in \mathcal{F}$. Since J is a basis of F, $J = K_t$. We conclude that

$$|K \setminus J| = \sum_{i=1}^{t} |K_{i-1} \setminus K_i| \leq pt = p\,|J \setminus K|,$$

proving $|K| \leq p\,|J|$. $\square$

This shows that in example (9) we have $q(E, \mathcal{F}) \geq \frac{1}{2}$ (see also Exercise 1 of Chapter 10). In fact $q(E, \mathcal{F}) = \frac{1}{2}$ iff G contains a path of length 3 as a subgraph (otherwise $q(E, \mathcal{F}) = 1$). For the independence system in example (1) of our list, the rank quotient can become arbitrarily small (choose G to be a star). In Exercise 5, the rank quotients for other independence systems will be discussed.

13.2 Other Matroid Axioms

In this section we consider other axiom systems defining matroids. They characterize fundamental properties of the family of bases, the rank function, the closure operator and the family of circuits of a matroid.

Theorem 13.9. *Let E be a finite set and $\mathcal{B} \subseteq 2^E$. $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if the following holds:*

(B1) *$\mathcal{B} \neq \emptyset$;*
(B2) *For any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ there exists a $y \in B_2 \setminus B_1$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

Proof: The set of bases of a matroid satisfies (B1) (by (M1)) and (B2): For bases B_1, B_2 and $x \in B_1 \setminus B_2$ we have that $B_1 \setminus \{x\}$ is independent. By (M3) there is some $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is independent. Indeed, it must be a basis, because all bases of a matroid have the same cardinality.

On the other hand, let $\mathcal{B}$ satisfy (B1) and (B2). We first show that all elements of $\mathcal{B}$ have the same cardinality: Otherwise let $B_1, B_2 \in \mathcal{B}$ with $|B_1| > |B_2|$ such that $|B_1 \cap B_2|$ is maximum. Let $x \in B_1 \setminus B_2$. By (B2) there is a $y \in B_2 \setminus B_1$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$, contradicting the maximality of $|B_1 \cap B_2|$.

Now let

$$\mathcal{F} := \{F \subseteq E : \text{there exists a } B \in \mathcal{B} \text{ with } F \subseteq B\}.$$

$(E, \mathcal{F})$ is an independence system, and $\mathcal{B}$ is the family of its bases. To show that $(E, \mathcal{F})$ satisfies (M3), let $X, Y \in \mathcal{F}$ with $|X| > |Y|$. Let $X \subseteq B_1 \in \mathcal{B}$ and

$Y \subseteq B_2 \in \mathcal{B}$, where B_1 and B_2 are chosen such that $|B_1 \cap B_2|$ is maximum. If $B_2 \cap (X \setminus Y) \neq \emptyset$, we are done because we can augment Y.

We claim that the other case, $B_2 \cap (X \setminus Y) = \emptyset$, is impossible. Namely with this assumption we get

$$|B_1 \cap B_2| + |Y \setminus B_1| + |(B_2 \setminus B_1) \setminus Y| \;=\; |B_2| \;=\; |B_1| \;\geq\; |B_1 \cap B_2| + |X \setminus Y|.$$

Since $|X \setminus Y| > |Y \setminus X| \geq |Y \setminus B_1|$, this implies $(B_2 \setminus B_1) \setminus Y \neq \emptyset$. So let $y \in (B_2 \setminus B_1) \setminus Y$. By (B2) there exists an $x \in B_1 \setminus B_2$ with $(B_2 \setminus \{y\}) \cup \{x\} \in \mathcal{B}$, contradicting the maximality of $|B_1 \cap B_2|$. □

A very important property of matroids is that the rank function is submodular:

Theorem 13.10. *Let E be a finite set and $r : 2^E \to \mathbb{Z}_+$. Then the following statements are equivalent:*

(a) *r is the rank function of a matroid $(E, \mathcal{F})$ (and $\mathcal{F} = \{F \subseteq E : r(F) = |F|\}$).*
(b) *For all $X, Y \subseteq E$:*
  (R1) *$r(X) \leq |X|$;*
  (R2) *If $X \subseteq Y$ then $r(X) \leq r(Y)$;*
  (R3) *$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.*
(c) *For all $X \subseteq E$ and $x, y \in E$:*
  (R1′) *$r(\emptyset) = 0$;*
  (R2′) *$r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;*
  (R3′) *If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$ then $r(X \cup \{x, y\}) = r(X)$.*

Proof: (a)⇒(b): If r is a rank function of an independence system $(E, \mathcal{F})$, (R1) and (R2) evidently hold. If $(E, \mathcal{F})$ is a matroid, we can also show (R3):

Let $X, Y \subseteq E$, and let A be a basis of $X \cap Y$. By (M3), A can be extended to a basis $A \,\dot\cup\, B$ of X and to a basis $(A \cup B) \,\dot\cup\, C$ of $X \cup Y$. Then $A \cup C$ is an independent subset of Y, so

$$\begin{aligned} r(X) + r(Y) &\geq |A \cup B| + |A \cup C| \\ &= 2|A| + |B| + |C| \;=\; |A \cup B \cup C| + |A| \\ &= r(X \cup Y) + r(X \cap Y). \end{aligned}$$

(b)⇒(c): (R1′) is implied by (R1). $r(X) \leq r(X \cup \{y\})$ follows from (R2). By (R3) and (R2),

$$r(X \cup \{y\}) \;\leq\; r(X) + r(\{y\}) - r(X \cap \{y\}) \;\leq\; r(X) + r(\{y\}) \;\leq\; r(X) + 1,$$

proving (R2′).

(R3′) is trivial for $x = y$. For $x \neq y$ we have, by (R2) and (R3),

$$2r(X) \;\leq\; r(X) + r(X \cup \{x, y\}) \;\leq\; r(X \cup \{x\}) + r(X \cup \{y\}),$$

implying (R3′).

(c)⇒(a): Let $r : 2^E \to \mathbb{Z}_+$ be a function satisfying (R1′)–(R3′). Let

$$\mathcal{F} := \{F \subseteq E : r(F) = |F|\}.$$

We claim that $(E, \mathcal{F})$ is a matroid. (M1) follows from (R1′). (R2′) implies $r(X) \leq |X|$ for all $X \subseteq E$. If $Y \in \mathcal{F}$, $y \in Y$ and $X := Y \setminus \{y\}$, we have

$$|X| + 1 = |Y| = r(Y) = r(X \cup \{y\}) \leq r(X) + 1 \leq |X| + 1,$$

so $X \in \mathcal{F}$. This implies (M2).

Now let $X, Y \in \mathcal{F}$ and $|X| = |Y| + 1$. Let $X \setminus Y = \{x_1, \dots, x_k\}$. Suppose that (M3′) is violated, i.e. $r(Y \cup \{x_i\}) = |Y|$ for $i = 1, \dots, k$. Then by (R3′) $r(Y \cup \{x_1, x_i\}) = r(Y)$ for $i = 2, \dots, k$. Repeated application of this argument yields $r(Y) = r(Y \cup \{x_1, \dots, x_k\}) = r(X \cup Y) \geq r(X)$, a contradiction.

So $(E, \mathcal{F})$ is indeed a matroid. To show that r is the rank function of this matroid, we have to prove that $r(X) = \max\{|Y| : Y \subseteq X, r(Y) = |Y|\}$ for all $X \subseteq E$. So let $X \subseteq E$, and let Y a maximum subset of X with $r(Y) = |Y|$. For all $x \in X \setminus Y$ we have $r(Y \cup \{x\}) < |Y| + 1$, so by (R2′) $r(Y \cup \{x\}) = |Y|$. Repeated application of (R3′) implies $r(X) = |Y|$. □

Theorem 13.11. *Let E be a finite set and $\sigma : 2^E \to 2^E$ a function. σ is the closure operator of a matroid $(E, \mathcal{F})$ if and only if the following conditions hold for all $X, Y \subseteq E$ and $x, y \in E$:*

(S1) $X \subseteq \sigma(X)$;
(S2) $X \subseteq Y \subseteq E$ *implies* $\sigma(X) \subseteq \sigma(Y)$;
(S3) $\sigma(X) = \sigma(\sigma(X))$;
(S4) *If* $y \notin \sigma(X)$ *and* $y \in \sigma(X \cup \{x\})$ *then* $x \in \sigma(X \cup \{y\})$.

Proof: If σ is the closure operator of a matroid, then (S1) holds trivially.

For $X \subseteq Y$ and $z \in \sigma(X)$ we have by (R3) and (R2)

$$\begin{aligned} r(X) + r(Y) &= r(X \cup \{z\}) + r(Y) \\ &\geq r((X \cup \{z\}) \cap Y) + r(X \cup \{z\} \cup Y) \\ &\geq r(X) + r(Y \cup \{z\}), \end{aligned}$$

implying $z \in \sigma(Y)$ and thus proving (S2).

By repeated application of (R3′) we have $r(\sigma(X)) = r(X)$ for all X, which implies (S3).

To prove (S4), suppose that there are X, x, y with $y \notin \sigma(X)$, $y \in \sigma(X \cup \{x\}$ and $x \notin \sigma(X \cup \{y\})$. Then $r(X \cup \{y\}) = r(X) + 1$, $r(X \cup \{x, y\}) = r(X \cup \{x\})$ and $r(X \cup \{x, y\}) = r(X \cup \{y\}) + 1$. Thus $r(X \cup \{x\}) = r(X) + 2$, contradicting (R2′).

To show the converse, let $\sigma : 2^E \to 2^E$ be a function satisfying (S1)–(S4). Let

$$\mathcal{F} := \{X \subseteq E : x \notin \sigma(X \setminus \{x\}) \text{ for all } x \in X\}.$$

We claim that $(E, \mathcal{F})$ is a matroid.

(M1) is trivial. For $X \subseteq Y \in \mathcal{F}$ and $x \in X$ we have $x \notin \sigma(Y \setminus \{x\}) \supseteq \sigma(X \setminus \{x\})$, so $X \in \mathcal{F}$ and (M2) holds. To prove (M3) we need the following statement:

Claim: For $X \in \mathcal{F}$ and $Y \subseteq E$ with $|X| > |Y|$ we have $X \not\subseteq \sigma(Y)$.

We prove the claim by induction on $|Y \setminus X|$. If $Y \subset X$, then let $x \in X \setminus Y$. Since $X \in \mathcal{F}$ we have $x \notin \sigma(X \setminus \{x\}) \supseteq \sigma(Y)$ by (S2). Hence $x \in X \setminus \sigma(Y)$ as required.

If $|Y \setminus X| > 0$, then let $y \in Y \setminus X$. By the induction hypothesis there exists an $x \in X \setminus \sigma(Y \setminus \{y\})$. If $x \notin \sigma(Y)$, then we are done. Otherwise $x \notin \sigma(Y \setminus \{y\})$ but $x \in \sigma(Y) = \sigma((Y \setminus \{y\}) \cup \{y\})$, so by (S4) $y \in \sigma((Y \setminus \{y\}) \cup \{x\})$. By (S1) we get $Y \subseteq \sigma((Y \setminus \{y\}) \cup \{x\})$ and thus $\sigma(Y) \subseteq \sigma((Y \setminus \{y\}) \cup \{x\})$ by (S2) and (S3). Applying the induction hypothesis to X and $(Y \setminus \{y\}) \cup \{x\}$ (note that $x \neq y$) yields $X \not\subseteq \sigma((Y \setminus \{y\}) \cup \{x\})$, so $X \not\subseteq \sigma(Y)$ as required.

Having proved the claim we can easily verify (M3). Let $X, Y \in \mathcal{F}$ with $|X| > |Y|$. By the claim there exists an $x \in X \setminus \sigma(Y)$. Now for each $z \in Y \cup \{x\}$ we have $z \notin \sigma(Y \setminus \{z\})$, because $Y \in \mathcal{F}$ and $x \notin \sigma(Y) \supseteq \sigma(Y \setminus \{z\})$. By (S4) $z \notin \sigma(Y \setminus \{z\})$ and $x \notin \sigma(Y)$ imply $z \notin \sigma((Y \setminus \{z\}) \cup \{x\}) \supseteq \sigma((Y \cup \{x\}) \setminus \{z\})$. Hence $Y \cup \{x\} \in \mathcal{F}$.

So (M3) indeed holds and $(E, \mathcal{F})$ is a matroid, say with rank function r and closure operator σ'. It remains to prove that $\sigma = \sigma'$.

By definition, $\sigma'(X) = \{y \in E : r(X \cup \{y\}) = r(X)\}$ and

$$r(X) \;=\; \max\{|Y| : Y \subseteq X,\ y \notin \sigma(Y \setminus \{y\}) \text{ for all } y \in Y\}$$

for all $X \subseteq E$.

Let $X \subseteq E$. To show $\sigma'(X) \subseteq \sigma(X)$, let $z \in \sigma'(X) \setminus X$. Let Y be a set attaining

$$\max\{|Y| : Y \subseteq X,\ y \notin \sigma(Y \setminus \{y\}) \text{ for all } y \in Y\}.$$

Since $r(Y \cup \{z\}) \le r(X \cup \{z\}) = r(X) = |Y| < |Y \cup \{z\}|$ we have $y \in \sigma((Y \cup \{z\}) \setminus \{y\})$ for some $y \in Y \cup \{z\}$. If $y = z$, then we have $z \in \sigma(Y)$. Otherwise (S4) and $y \notin \sigma(Y \setminus \{y\})$ also yield $z \in \sigma(Y)$. Hence by (S2) $z \in \sigma(X)$. Together with (S1) this implies $\sigma'(X) \subseteq \sigma(X)$.

Now let $z \notin \sigma'(X)$, i.e. $r(X \cup \{z\}) > r(X)$. Let Y be a set attaining

$$\max\{|Y| : Y \subseteq X \cup \{z\},\ y \notin \sigma(Y \setminus \{y\}) \text{ for all } y \in Y\}.$$

Then $z \in Y$ and $|Y \setminus \{z\}| = |Y| - 1 = r(X \cup \{z\}) - 1 = r(X)$. Therefore $Y \setminus \{z\}$ attains

$$\max\{|W| : W \subseteq X,\ y \notin \sigma(W \setminus \{y\}) \text{ for all } y \in W\},$$

implying $\sigma((Y \setminus \{z\}) \cup \{x\}) = \sigma(Y \setminus \{z\})$ for all $x \in X$ and hence $X \subseteq \sigma(Y \setminus \{z\})$. We conclude that $z \notin \sigma(Y \setminus \{z\}) = \sigma(X)$. □

Theorem 13.12. *Let E be a finite set and $\mathcal{C} \subseteq 2^E$. $\mathcal{C}$ is set of circuits of an independence system $(E, \mathcal{F})$, where $\mathcal{F} = \{F \subset E :$ there exists no $C \in \mathcal{C}$ with $C \subseteq F\}$, if and only if the following conditions hold:*

(C1) *$\emptyset \notin \mathcal{C}$;*

(C2) *For any $C_1, C_2 \in \mathcal{C}$, $C_1 \subseteq C_2$ implies $C_1 = C_2$.*

Moreover, if $\mathcal{C}$ is set of circuits of an independence system $(E, \mathcal{F})$, then the following statements are equivalent:

(a) *$(E, \mathcal{F})$ is a matroid.*
(b) *For any $X \in \mathcal{F}$ and $e \in E$, $X \cup \{e\}$ contains at most one circuit.*
(C3) *For any $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ there exists a $C_3 \in \mathcal{C}$ with $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*
(C3′) *For any $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ there exists a $C_3 \in \mathcal{C}$ with $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*

Proof: By definition, the family of circuits of any independence system satisfies (C1) and (C2). If $\mathcal{C}$ satisfies (C1), then $(E, \mathcal{F})$ is an independence system. If $\mathcal{C}$ also satisfies (C2), it is the set of circuits of this independence system.

(a)⇒(C3′): Let $\mathcal{C}$ be the family of circuits of a matroid, and let $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$. By applying (R3) twice we have

$$\begin{aligned}
& |C_1| - 1 + r((C_1 \cup C_2) \setminus \{e, f\}) + |C_2| - 1 \\
= \; & r(C_1) + r((C_1 \cup C_2) \setminus \{e, f\}) + r(C_2) \\
\geq \; & r(C_1) + r((C_1 \cup C_2) \setminus \{f\}) + r(C_2 \setminus \{e\}) \\
\geq \; & r(C_1 \setminus \{f\}) + r(C_1 \cup C_2) + r(C_2 \setminus \{e\}) \\
= \; & |C_1| - 1 + r(C_1 \cup C_2) + |C_2| - 1.
\end{aligned}$$

So $r((C_1 \cup C_2) \setminus \{e, f\}) = r(C_1 \cup C_2)$. Let B be a basis of $(C_1 \cup C_2) \setminus \{e, f\}$. Then $B \cup \{f\}$ contains a circuit C_3, with $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$ as required.

(C3′)⇒(C3): trivial.

(C3)⇒(b): If $X \in \mathcal{F}$ and $X \cup \{e\}$ contains two circuits C_1, C_2, (C3) implies $(C_1 \cup C_2) \setminus \{e\} \notin \mathcal{F}$. However, $(C_1 \cup C_2) \setminus \{e\}$ is a subset of X.

(b)⇒(a): Follows from Theorem 13.8 and Proposition 13.7. □

Especially property (b) will be used often. For $X \in \mathcal{F}$ and $e \in E$ such that $X \cup \{e\} \notin \mathcal{F}$ we write $C(X, e)$ for the unique circuit in $X \cup \{e\}$. If $X \cup \{e\} \in \mathcal{F}$ we write $C(X, e) := \emptyset$.

13.3 Duality

Another basic concept in matroid theory is duality.

Definition 13.13. *Let $(E, \mathcal{F})$ be an independence system. We define the* **dual** *of $(E, \mathcal{F})$ by $(E, \mathcal{F}^*)$, where*

$$\mathcal{F}^* = \{F \subseteq E : \text{there is a basis } B \text{ of } (E, \mathcal{F}) \text{ such that } F \cap B = \emptyset\}.$$

It is obvious that the dual of an independence system is again an independence system.

Proposition 13.14. $(E, \mathcal{F}^{**}) = (E, \mathcal{F})$.

Proof: $F \in \mathcal{F}^{**} \Leftrightarrow$ there is a basis B^* of $(E, \mathcal{F}^*)$ such that $F \cap B^* = \emptyset \Leftrightarrow$ there is a basis B of $(E, \mathcal{F})$ such that $F \cap (E \setminus B) = \emptyset \Leftrightarrow F \in \mathcal{F}$. □

Theorem 13.15. *Let $(E, \mathcal{F})$ be an independence system, $(E, \mathcal{F}^*)$ its dual, and r resp. r^* the corresponding rank functions.*

(a) *$(E, \mathcal{F})$ is a matroid if and only if $(E, \mathcal{F}^*)$ is a matroid.* (Whitney [1935])
(b) *If $(E, \mathcal{F})$ is a matroid, then $r^*(F) = |F| + r(E \setminus F) - r(E)$ for $F \subseteq E$.*

Proof: Due to Proposition 13.14 we have to show only one direction of (a). So let $(E, \mathcal{F})$ be a matroid. We define $q : 2^E \to \mathbb{Z}_+$ by $q(F) := |F| + r(E \setminus F) - r(E)$. We claim that q satisfies (R1), (R2) and (R3). By this claim and Theorem 13.10, q is the rank function of a matroid. Since obviously $q(F) = |F|$ if and only if $F \in \mathcal{F}^*$, we conclude that $q = r^*$, and (a) and (b) are proved.

Now we prove the above claim: q satisfies (R1) because r satisfies (R2). To check that q satisfies (R2), let $X \subseteq Y \subseteq E$. Since $(E, \mathcal{F})$ is a matroid, (R3) holds for r, so

$$r(E \setminus X) + 0 \;=\; r((E \setminus Y) \cup (Y \setminus X)) + r(\emptyset) \;\leq\; r(E \setminus Y) + r(Y \setminus X).$$

We conclude that

$$r(E \setminus X) - r(E \setminus Y) \;\leq\; r(Y \setminus X) \;\leq\; |Y \setminus X| \;=\; |Y| - |X|$$

(note that r satisfies (R1)), so $q(X) \leq q(Y)$.

It remains to show that q satisfies (R3). Let $X, Y \subseteq E$. Using the fact that r satisfies (R3) we have

$$\begin{aligned}
& q(X \cup Y) + q(X \cap Y) \\
= \;& |X \cup Y| + |X \cap Y| + r(E \setminus (X \cup Y)) + r(E \setminus (X \cap Y)) - 2r(E) \\
= \;& |X| + |Y| + r((E \setminus X) \cap (E \setminus Y)) + r((E \setminus X) \cup (E \setminus Y)) - 2r(E) \\
\leq \;& |X| + |Y| + r(E \setminus X) + r(E \setminus Y) - 2r(E) \\
= \;& q(X) + q(Y).
\end{aligned}$$

□

For any graph G we have introduced the cycle matroid $\mathcal{M}(G)$ which of course has a dual. For an embedded planar graph G there is also a planar dual G^* (which in general depends on the embedding of G). It is interesting that the two concepts of duality coincide:

Theorem 13.16. *Let G be a connected planar graph with an arbitrary planar embedding, and G^* the planar dual. Then*

$$\mathcal{M}(G^*) \;=\; (\mathcal{M}(G))^* .$$

Proof: For $T \subseteq E(G)$ we write $\overline{T}^* := \{e^* : e \in E(G) \setminus T\}$, where e^* is the dual of edge e. We have to prove the following:

Claim: T is the edge set of a spanning tree in G iff $\overline{T}^*$ is the edge set of a spanning tree in G^*.

Since $(G^*)^* = G$ (by Proposition 2.42) and $\overline{(\overline{T}^*)}^* = T$ it suffices to prove one direction of the claim.

So let $T \subseteq E(G)$, where $\overline{T}^*$ is the edge set of a spanning tree in G^*. $(V(G), T)$ must be connected, for otherwise a connected component would define a cut, the dual of which contains a circuit in $\overline{T}^*$ (Theorem 2.43). On the other hand, if $(V(G), T)$ contains a circuit, then the dual edge set is a cut and $(V(G^*), \overline{T}^*)$ is disconnected. Hence $(V(G), T)$ is indeed a spanning tree in G. $\square$

This implies that if G is planar then $(\mathcal{M}(G))^*$ is a graphic matroid. If, for any graph G, $(\mathcal{M}(G))^*$ is a graphic matroid, say $(\mathcal{M}(G))^* = \mathcal{M}(G')$, then G' is evidently an abstract dual of G. By Exercise 33 of Chapter 2, the converse is also true: G is planar if and only if G has an abstract dual (Whitney [1933]). This implies that $(\mathcal{M}(G))^*$ is graphic if and only if G is planar.

Note that Theorem 13.16 quite directly implies Euler's formula (Theorem 2.32): Let G be a connected planar graph with a planar embedding, and let $\mathcal{M}(G)$ be the cycle matroid of G. By Theorem 13.15 (b), $r(E(G))+r^*(E(G)) = |E(G)|$. Since $r(E(G)) = |V(G)| - 1$ (the number of edges in a spanning tree) and $r^*(E(G)) = |V(G^*)| - 1$ (by Theorem 13.16), we obtain that the number of faces of G is $|V(G^*)| = |E(G)| - |V(G)| + 2$, Euler's formula.

Duality of independence systems has also some nice applications in polyhedral combinatorics. A set system $(E, \mathcal{F})$ is called a **clutter** if $X \not\subset Y$ for all $X, Y \in \mathcal{F}$. If $(E, \mathcal{F})$ is a clutter, then we define its **blocking clutter** by

$$BL(E, \mathcal{F}) \quad := \quad (E, \{X \subseteq E : X \cap Y \neq \emptyset \text{ for all } Y \in \mathcal{F}, \\ X \text{ minimal with this property}\}).$$

For an independence system $(E, \mathcal{F})$ and its dual $(E, \mathcal{F}^*)$ let $\mathcal{B}$ resp. $\mathcal{B}^*$ be the family of bases, and $\mathcal{C}$ resp. $\mathcal{C}^*$ the family of circuits. (Every clutter arises in both of these ways except for $\mathcal{F} = \emptyset$ or $\mathcal{F} = \{\emptyset\}$.) It follows immediately from the definitions that $(E, \mathcal{B}^*) = BL(E, \mathcal{C})$ and $(E, \mathcal{C}^*) = BL(E, \mathcal{B})$. Together with Proposition 13.14 this implies $BL(BL(E, \mathcal{F})) = (E, \mathcal{F})$ for every clutter $(E, \mathcal{F})$. We give some examples for clutters $(E, \mathcal{F})$ and their blocking clutters $(E, \mathcal{F}')$. In each case $E = E(G)$ for some graph G:

(1) $\mathcal{F}$ is the set of spanning trees, $\mathcal{F}'$ is the set of minimal cuts;
(2) $\mathcal{F}$ is the set of arborescences rooted at r, $\mathcal{F}'$ is the set of minimal r-cuts;
(3) $\mathcal{F}$ is the set of s-t-paths, $\mathcal{F}'$ is the set of minimal cuts separating s and t (this example works in undirected graphs and in digraphs);
(4) $\mathcal{F}$ is the set of circuits in an undirected graph, $\mathcal{F}'$ is the set of complements of maximal forests;
(5) $\mathcal{F}$ is the set of circuits in a digraph, $\mathcal{F}'$ is the set of minimal feedback edge sets;

(6) $\mathcal{F}$ is the set of minimal edge sets whose contraction makes the digraph strongly connected, $\mathcal{F}'$ is the set of minimal directed cuts;
(7) $\mathcal{F}$ is the set of minimal T-joins, $\mathcal{F}'$ is the set of minimal T-cuts.

All these blocking relations can be verified easily: (1) and (2) follow directly from Theorems 2.4 and 2.5, (3), (4) and (5) are trivial, (6) follows from Corollary 2.7, and (7) from Proposition 12.6.

In some cases, the blocking clutter gives a polyhedral characterization of the Minimization Problem For Independence Systems for nonnegative cost functions:

Definition 13.17. *Let $(E, \mathcal{F})$ be a clutter, $(E, \mathcal{F}')$ its blocking clutter and P the convex hull of the incidence vectors of the elements of $\mathcal{F}$. We say that $(E, \mathcal{F})$ has the* **Max-Flow-Min-Cut property** *if*

$$\left\{x + y : x \in P,\ y \in \mathbb{R}_+^E\right\} = \left\{x \in \mathbb{R}_+^E : \sum_{e \in B} x_e \geq 1 \text{ for all } B \in \mathcal{F}'\right\}.$$

Examples are (2) and (7) of our list above (by Theorems 6.12 and 12.16), but also (3) and (6) (see Exercise 10). The following theorem relates the above covering-type formulation to a packing formulation of the dual problem and allows to derive certain min-max theorems from others:

Theorem 13.18. (Fulkerson [1971], Lehman [1979]) *Let $(E, \mathcal{F})$ be a clutter and $(E, \mathcal{F}')$ its blocking clutter. Then the following statements are equivalent:*

(a) *$(E, \mathcal{F})$ has the Max-Flow-Min-Cut property;*
(b) *$(E, \mathcal{F}')$ has the Max-Flow-Min-Cut property;*
(c) $\min\{c(A) : A \in \mathcal{F}\} = \max\left\{\mathbb{1}y : y \in \mathbb{R}_+^{\mathcal{F}'},\ \sum_{B \in \mathcal{F}' : e \in B} y_B \leq c(e)\right.$ *for all* $\left. e \in E\right\}$ *for every* $c : E \to \mathbb{R}_+$.

Proof: Since $BL(E, \mathcal{F}') = BL(BL(E, \mathcal{F})) = (E, \mathcal{F})$ it suffices to prove (a)$\Rightarrow$(c)$\Rightarrow$(b). The other implication (b)$\Rightarrow$(a) then follows by exchanging the roles of $\mathcal{F}$ and $\mathcal{F}'$.

(a)$\Rightarrow$(c): By Corollary 3.28 we have for every $c : E \to \mathbb{R}_+$

$$\min\{c(A) : A \in \mathcal{F}\} = \min\{cx : x \in P\} = \min\left\{c(x + y) : x \in P,\ y \in \mathbb{R}_+^E\right\},$$

where P is the convex hull of the incidence vectors of elements of $\mathcal{F}$. From this, the Max-Flow-Min-Cut property and the LP Duality Theorem 3.16 we get (c).

(c)$\Rightarrow$(b): Let P' denote the convex hull of the incidence vectors of the elements of $\mathcal{F}'$. We have to show that

$$\left\{x + y : x \in P',\ y \in \mathbb{R}_+^E\right\} = \left\{x \in \mathbb{R}_+^E : \sum_{e \in A} x_e \geq 1 \text{ for all } A \in \mathcal{F}\right\}.$$

Since "$\subseteq$" is trivial from the definition of blocking clutters we only show the other inclusion. So let $c \in \mathbb{R}_+^E$ be a vector with $\sum_{e \in A} c_e \geq 1$ for all $A \in \mathcal{F}$. By (c) we have

$$\begin{aligned} 1 &\le \min\{c(A) : A \in \mathcal{F}\} \\ &= \max\left\{ \mathbb{1}y : y \in \mathbb{R}_+^{\mathcal{F}'}, \sum_{B \in \mathcal{F}' : e \in B} y_B \le c(e) \text{ for all } e \in E \right\}, \end{aligned}$$

so let $y \in \mathbb{R}_+^{\mathcal{F}'}$ be a vector with $\mathbb{1}y = 1$ and $\sum_{B \in \mathcal{F}' : e \in B} y_B \le c(e)$ for all $e \in E$. Then $x_e := \sum_{B \in \mathcal{F}' : e \in B} y_B$ $(e \in E)$ defines a vector $x \in P'$ with $x \le c$, proving that $c \in \{x + y : x \in P', y \in \mathbb{R}_+^E\}$. □

For example, this theorem implies the Max-Flow-Min-Cut Theorem 8.6 quite directly: Let (G, u, s, t) be a network. By Exercise 1 of Chapter 7 the minimum length of an s-t-path in (G, u) equals the maximum number of s-t-cuts such that each edge e is contained in at most $u(e)$ of them. Hence the clutter of s-t-paths (example (3) in the above list) has the Max-Flow-Min-Cut Property, and so has its blocking clutter. Now (c) applied to the clutter of minimal s-t-cuts implies the Max-Flow-Min-Cut Theorem.

Note however that Theorem 13.18 does not guarantee an integral vector attaining the maximum in (c), even if c is integral. The clutter of T-joins for $G = K_4$ and $T = V(G)$ shows that this does not exist in general.

13.4 The Greedy Algorithm

Again, let $(E, \mathcal{F})$ be an independence system and $c : E \to \mathbb{R}_+$. We consider the MAXIMIZATION PROBLEM for $(E, \mathcal{F}, c)$ and formulate two "greedy algorithms". We do not have to consider negative weights since elements with negative weight never appear in an optimum solution.

We assume that $(E, \mathcal{F})$ is given by an oracle. For the first algorithm we simply assume an **independence oracle**, i.e. an oracle which, given a set $F \subseteq E$, decides whether $F \in \mathcal{F}$ or not.

BEST-IN-GREEDY ALGORITHM

Input: An independence system $(E, \mathcal{F})$, given by an independence oracle. Weights $c : E \to \mathbb{R}_+$.

Output: A set $F \in \mathcal{F}$.

① Sort $E = \{e_1, e_2, \dots, e_n\}$ such that $c(e_1) \ge c(e_2) \ge \dots \ge c(e_n)$.

② Set $F := \emptyset$.

③ **For** $i := 1$ **to** n **do**: **If** $F \cup \{e_i\} \in \mathcal{F}$ **then** set $F := F \cup \{e_i\}$.

The second algorithm requires a more complicated oracle. Given a set $F \subseteq E$, this oracle decides whether F contains a basis. Let us call such an oracle a **basis-superset oracle**.

Worst-Out-Greedy Algorithm

Input: An independence system $(E, \mathcal{F})$, given by a basis-superset oracle. Weights $c : E \to \mathbb{R}_+$.

Output: A basis F of $(E, \mathcal{F})$.

① Sort $E = \{e_1, e_2, \dots, e_n\}$ such that $c(e_1) \le c(e_2) \le \dots \le c(e_n)$.

② Set $F := E$.

③ **For** $i := 1$ **to** n **do**: **If** $F \setminus \{e_i\}$ contains a basis **then** set $F := F \setminus \{e_i\}$.

Before we analyse these algorithms, let us take a closer look at the oracles required. It is an interesting questions whether such oracles are polynomially equivalent, i.e. whether one can be simulated by polynomial-time oracle algorithm using the other. The independence oracle and the basis-superset oracle do not seem to be polynomially equivalent:

If we consider the independence system for the TSP (example (2) of the list in Section 13.1), it is easy (and the subject of Exercise 13) to decide whether a set of edges is independent, i.e. the subset of a Hamiltonian circuit (recall that we are working with a complete graph). On the other hand, it is a difficult problem to decide whether a set of edges contains a Hamiltonian circuit (this is *NP*-complete; cf. Theorem 15.25).

Conversely, in the independence system for the Shortest Path Problem (example (3)), it is easy to decide whether a set of edges contains an s-t-path. Here it is not known how to decide whether a given set is independent (i.e. subset of an s-t-path) in polynomial time (Korte and Monma [1979] proved *NP*-completeness).

For matroids, both oracles are polynomially equivalent. Other equivalent oracles are the **rank oracle** and **closure oracle**, which return the rank resp. the closure of a given subset of E (Exercise 16).

However, even for matroids there are other natural oracles that are not polynomially equivalent. For example, the oracle deciding whether a given set is a basis is weaker than the independence oracle. The oracle which for a given $F \subseteq E$ returns the minimum cardinality of a dependent subset of F is stronger than the independence oracle (Hausmann and Korte [1981]).

One can analogously formulate both greedy algorithms for the Minimization Problem. It is easy to see that the Best-In-Greedy for the Maximization Problem for $(E, \mathcal{F}, c)$ corresponds to the Worst-Out-Greedy for the Minimization Problem for $(E, \mathcal{F}^*, c)$: adding an element to F in the Best-In-Greedy corresponds to removing an element from F in the Worst-Out-Greedy. Observe that Kruskal's Algorithm (see Section 6.1) is a Best-In-Greedy algorithm for the Minimization Problem in a cycle matroid.

The rest of this section contains some results concerning the quality of a solution found by the greedy algorithms.

Theorem 13.19. (Jenkyns [1976], Korte and Hausmann [1978]) *Let $(E, \mathcal{F})$ be an independence system. For $c : E \to \mathbb{R}_+$ we denote by $G(E, \mathcal{F}, c)$ the cost of* *some solution found by the* BEST-IN-GREEDY *for the* MAXIMIZATION PROBLEM. *Then*

$$q(E, \mathcal{F}) \leq \frac{G(E, \mathcal{F}, c)}{\mathrm{OPT}(E, \mathcal{F}, c)} \leq 1$$

for all $c : E \to \mathbb{R}_+$. There is a cost function where the lower bound is attained.

Proof: Let $E = \{e_1, e_2, \dots, e_n\}$, $c : E \to \mathbb{R}_+$, and $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$. Let G_n be the solution found by the BEST-IN-GREEDY (when sorting E like this), while O_n is an optimum solution. We define $E_j := \{e_1, \dots, e_j\}$, $G_j := G_n \cap E_j$ and $O_j := O_n \cap E_j$ $(j = 0, \dots, n)$. Set $d_n := c(e_n)$ and $d_j := c(e_j) - c(e_{j+1})$ for $j = 1, \dots, n-1$.

Since $O_j \in \mathcal{F}$, we have $|O_j| \leq r(E_j)$. Since G_j is a basis of E_j, we have $|G_j| \geq \rho(E_j)$. With these two inequalities we conclude that

$$\begin{aligned}
c(G_n) &= \sum_{j=1}^{n} (|G_j| - |G_{j-1}|)\, c(e_j) \\
&= \sum_{j=1}^{n} |G_j|\, d_j \\
&\geq \sum_{j=1}^{n} \rho(E_j)\, d_j \\
&\geq q(E, \mathcal{F}) \sum_{j=1}^{n} r(E_j)\, d_j \qquad (13.1) \\
&\geq q(E, \mathcal{F}) \sum_{j=1}^{n} |O_j|\, d_j \\
&= q(E, \mathcal{F}) \sum_{j=1}^{n} (|O_j| - |O_{j-1}|)\, c(e_j) \\
&= q(E, \mathcal{F})\, c(O_n).
\end{aligned}$$

Finally we show that the lower bound is sharp. Choose $F \subseteq E$ and bases B_1, B_2 of F such that

$$\frac{|B_1|}{|B_2|} = q(E, \mathcal{F}).$$

Define

$$c(e) := \begin{cases} 1 & \text{for } e \in F \\ 0 & \text{for } e \in E \setminus F \end{cases}$$

and sort $e_1, \dots, e_n$ such that $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$ and $B_1 = \{e_1, \dots, e_{|B_1|}\}$. Then $G(E, \mathcal{F}, c) = |B_1|$ and $\mathrm{OPT}(E, \mathcal{F}, c) = |B_2|$, and the lower bound is attained. $\square$

In particular we have the so-called Edmonds-Rado Theorem:

Theorem 13.20. (Rado [1957], Edmonds [1971]) *An independence system $(E, \mathcal{F})$ is a matroid if and only if the* Best-In-Greedy *finds an optimum solution for the* Maximization Problem *for $(E, \mathcal{F}, c)$ for all cost functions $c : E \to \mathbb{R}_+$.*

Proof: By Theorem 13.19 we have $q(E, \mathcal{F}) < 1$ if and only if there exists a cost function $c : E \to \mathbb{R}_+$ for which the Best-In-Greedy does not find an optimum solution. By Proposition 13.7 we have $q(E, \mathcal{F}) < 1$ if and only if $(E, \mathcal{F})$ is not a matroid. $\square$

This is one of the rare cases where we can define a structure by its algorithmic behaviour. We also obtain a polyhedral description:

Theorem 13.21. (Edmonds [1970]) *Let $(E, \mathcal{F})$ be a matroid and $r : E \to \mathbb{Z}_+$ its rank function. Then the* **matroid polytope** *of $(E, \mathcal{F})$, i.e. the convex hull of the incidence vectors of all elements of $\mathcal{F}$, is equal to*

$$\left\{x \in \mathbb{R}^E : x \geq 0, \sum_{e \in A} x_e \leq r(A) \textit{ for all } A \subseteq E\right\}.$$

Proof: Obviously, this polytope contains all incidence vectors of independent sets. By Corollary 3.27 it remains to show that all vertices of this polytope are integral. By Theorem 5.12 this is equivalent to showing that

$$\max\left\{cx : x \geq 0, \sum_{e \in A} x_e \leq r(A) \text{ for all } A \subseteq E\right\} \tag{13.2}$$

has an integral optimum solution for any $c : E \to \mathbb{R}$. W.l.o.g. $c(e) \geq 0$ for all e, since for $e \in E$ with $c(e) < 0$ any optimum solution x of (13.2) has $x_e = 0$.

Let x be an optimum solution of (13.2). In (13.1) we replace $|O_j|$ by $\sum_{e \in E_j} x_e$ ($j = 0, \dots, n$). We obtain $c(G_n) \geq \sum_{e \in E} c(e)x_e$. So the Best-In-Greedy produces a solution whose incidence vector is another optimum solution of (13.2). $\square$

When applied to graphic matroids, this also yields Theorem 6.10. As in this special case, we also have total dual integrality in general. A generalization of this result will be proved in Section 14.2.

The above observation that the Best-In-Greedy for the Maximization Problem for $(E, \mathcal{F}, c)$ corresponds to the Worst-Out-Greedy for the Minimization Problem for $(E, \mathcal{F}^*, c)$ suggests the following dual counterpart of Theorem 13.19:

Theorem 13.22. (Korte and Monma [1979]) *Let $(E, \mathcal{F})$ be an independence system. For $c : E \to \mathbb{R}_+$ let $G(E, \mathcal{F}, c)$ denote a solution found by the* Worst-Out-Greedy *for the* Minimization Problem. *Then*

$$1 \leq \frac{G(E, \mathcal{F}, c)}{\mathrm{OPT}(E, \mathcal{F}, c)} \leq \max_{F \subseteq E} \frac{|F| - \rho^*(F)}{|F| - r^*(F)} \tag{13.3}$$

for all $c : E \to \mathbb{R}_+$. There is a cost function where the upper bound is attained.

Proof: We use the same notation as in the proof of Theorem 13.19. By construction, $G_j \cup (E \setminus E_j)$ contains a basis of E, but $(G_j \cup (E \setminus E_j)) \setminus \{e\}$ does not contain a basis of E for any $e \in G_j$ $(j = 1, \dots, n)$. In other words, $E_j \setminus G_j$ is a basis of E_j with respect to $(E, \mathcal{F}^*)$, so $|E_j| - |G_j| \geq \rho^*(E_j)$.

Since $O_n \subseteq E \setminus (E_j \setminus O_j)$ and O_n is a basis, $E_j \setminus O_j$ is independent in $(E, \mathcal{F}^*)$, so $|E_j| - |O_j| \leq r^*(E_j)$.

We conclude that

$$\begin{aligned} |G_j| &\leq |E_j| - \rho^*(E_j) \qquad \text{and} \\ |O_j| &\geq |E_j| - r^*(E_j), \end{aligned}$$

where ρ^* and r^* are the rank functions of $(E, \mathcal{F}^*)$. Now the same calculation as (13.1) provides the upper bound. To see that this bound is tight, consider

$$c(e) := \begin{cases} 1 & \text{for } e \in F \\ 0 & \text{for } e \in E \setminus F \end{cases},$$

where $F \subseteq E$ is a set where the maximum in (13.3) is attained. Let B_1 be a basis of F with respect to $(E, \mathcal{F}^*)$, with $|B_1| = \rho^*(F)$. If we sort $e_1, \dots, e_n$ such that $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$ and $B_1 = \{e_1, \dots, e_{|B_1|}\}$, we have $G(E, \mathcal{F}, c) = |F| - |B_1|$ and $\mathrm{OPT}(E, \mathcal{F}, c) = |F| - r^*(F)$. $\square$

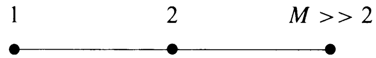

Fig. 13.1.

If we apply the WORST-OUT-GREEDY to the MAXIMIZATION PROBLEM or the BEST-IN-GREEDY to the MINIMIZATION PROBLEM, there is no lower resp. upper bound for $\frac{G(E,\mathcal{F},c)}{\mathrm{OPT}(E,\mathcal{F},c)}$. To see this, consider the problem of finding a maximal stable set of minimum weight or a minimal vertex cover of maximum weight in the simple graph shown in Figure 13.1.

However in the case of matroids, it does not matter whether we use the BEST-IN-GREEDY or the WORST-OUT-GREEDY: since all bases have the same cardinality, the MINIMIZATION PROBLEM for $(E, \mathcal{F}, c)$ is equivalent to the MAXIMIZATION PROBLEM for $(E, \mathcal{F}, c')$, where $c'(e) := M - c(e)$ and $M := 1 + \max\{c(e) : e \in E\}$. Therefore KRUSKAL'S ALGORITHM (Section 6.1) solves the MINIMUM SPANNING TREE PROBLEM optimally.

The Edmonds-Rado Theorem 13.20 also yields the following characterization of optimum k-element solutions of the MAXIMIZATION PROBLEM.

Theorem 13.23. *Let $(E, \mathcal{F})$ be a matroid, $c : E \to \mathbb{R}$, $k \in \mathbb{N}$ and $X \in \mathcal{F}$ with $|X| = k$. Then $c(X) = \max\{c(Y) : Y \in \mathcal{F}, |Y| = k\}$ if and only if the following two conditions hold:*

(a) *For all $y \in E \setminus X$ with $X \cup \{y\} \notin \mathcal{F}$ and all $x \in C(X, y)$ we have $c(x) \geq c(y)$;*
(b) *For all $y \in E \setminus X$ with $X \cup \{y\} \in \mathcal{F}$ and all $x \in X$ we have $c(x) \geq c(y)$.*

Proof: The necessity is trivial: if one of the conditions is violated for some y and x, the k-element set $X' := (X \cup \{y\}) \setminus \{x\} \in \mathcal{F}$ has greater cost than X.

To see the sufficiency, let $\mathcal{F}' := \{F \in \mathcal{F} : |F| \leq k\}$ and $c'(e) := c(e) + M$ for all $e \in E$, where $M = \max\{|c(e)| : e \in E\}$. Sort $E = \{e_1, \dots, e_n\}$ such that $c'(e_1) \geq \dots \geq c'(e_n)$ and, for any i, $c'(e_i) = c'(e_{i+1})$ and $e_{i+1} \in X$ imply $e_i \in X$ (i.e. elements of X come first among those of equal weight).

Let X' be the solution found by the BEST-IN-GREEDY for the instance $(E, \mathcal{F}', c')$ (sorted like this). Since $(E, \mathcal{F}')$ is a matroid, the Edmonds-Rado Theorem 13.20 implies:

$$\begin{aligned} c(X') + kM &= c'(X') = \max\{c'(Y) : Y \in \mathcal{F}'\} \\ &= \max\{c(Y) : Y \in \mathcal{F}, |Y| = k\} + kM. \end{aligned}$$

We conclude the proof by showing that $X = X'$. We know that $|X| = k = |X'|$. So suppose $X \neq X'$, and let $e_i \in X' \setminus X$ with i minimum. Then $X \cap \{e_1, \dots, e_{i-1}\} = X' \cap \{e_1, \dots, e_{i-1}\}$. Now if $X \cup \{e_i\} \notin \mathcal{F}$, then (a) implies $C(X, e_i) \subseteq X'$, a contradiction. If $X \cup \{e_i\} \in \mathcal{F}$, then (b) implies $X \subseteq X'$ which is also impossible. $\square$

We shall need this theorem in Section 13.7. The special case that $(E, \mathcal{F})$ is a graphic matroid and $k = r(E)$ is part of Theorem 6.2.

13.5 Matroid Intersection

Definition 13.24. *Given two independence systems $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$, we define their* **intersection** *by $(E, \mathcal{F}_1 \cap \mathcal{F}_2)$.*

The intersection of a finite number of independence systems is defined analogously. It is clear that the result is again an independence system.

Proposition 13.25. *Any independence system $(E, \mathcal{F})$ is the intersection of a finite number of matroids.*

Proof: Each circuit C of $(E, \mathcal{F})$ defines a matroid $(E, \{F \subseteq E : C \setminus F \neq \emptyset\})$ by Theorem 13.12. The intersection of all these matroids is of course $(E, \mathcal{F})$. $\square$

Since the intersection of matroids is not a matroid in general, we cannot hope to get an optimum common independent set by a greedy algorithm. However, the following result, together with Theorem 13.19, implies a bound for the solution found by the BEST-IN-GREEDY:

Proposition 13.26. *If $(E, \mathcal{F})$ is the intersection of p matroids, then $q(E, \mathcal{F}) \geq \frac{1}{p}$.*

Proof: By Theorem 13.12(b), $X \cup \{e\}$ contains at most p circuits for any $X \in \mathcal{F}$ and $e \in E$. The statement now follows from Theorem 13.8. $\square$

Of particular interest are independence systems that are the intersection of two matroids. The prime example here is the matching problem in a bipartite graph $G = (A \dot{\cup} B, E(G))$. If $E = E(G)$ and $\mathcal{F} := \{F \subseteq E : F$ is a matching in $G\}$, $(E, \mathcal{F})$ is the intersection of two matroids. Namely, let

$$\mathcal{F}_1 := \{F \subseteq E : |\delta_F(x)| \leq 1 \text{ for all } x \in A\} \quad \text{and}$$
$$\mathcal{F}_2 := \{F \subseteq E : |\delta_F(x)| \leq 1 \text{ for all } x \in B\}.$$

$(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$ are matroids by Proposition 13.4(d). Clearly, $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$.

A second example is the independence system consisting of all branchings in a digraph G (Example 8 of the list at the beginning of Section 13.1). Here one matroid contains all sets of edges such that each vertex has at most one entering edge (see Proposition 13.4(e)), while the second matroid is the cycle matroid $\mathcal{M}(G)$ of the underlying undirected graph.

We shall now describe Edmonds' algorithm for the following problem:

Matroid Intersection Problem	
Instance:	Two matroids $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$, given by independence oracles.
Task:	Find a set $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that $|F|$ is maximum.

We start with the following lemma. Recall that, for $X \in \mathcal{F}$ and $e \in E$, $C(X, e)$ denotes the unique circuit in $X \cup \{e\}$ if $X \cup \{e\} \notin \mathcal{F}$, and $C(X, e) = \emptyset$ otherwise.

Lemma 13.27. (Frank [1981]) *Let $(E, \mathcal{F})$ be a matroid and $X \in \mathcal{F}$. Let $x_1, \ldots, x_s \in X$ and $y_1, \ldots, y_s \notin X$ with*

(a) *$x_k \in C(X, y_k)$ for $k = 1, \ldots, s$ and*
(b) *$x_j \notin C(X, y_k)$ for $1 \leq j < k \leq s$.*

Then $(X \setminus \{x_1, \ldots, x_s\}) \cup \{y_1, \ldots, y_s\} \in \mathcal{F}$.

Proof: Let $X_r := (X \setminus \{x_1, \ldots, x_r\}) \cup \{y_1, \ldots, y_r\}$. We show that $X_r \in \mathcal{F}$ for all r by induction. For $r = 0$ this is trivial. Let us assume that $X_{r-1} \in \mathcal{F}$ for some $r \in \{1, \ldots, s\}$. If $X_{r-1} \cup \{y_r\} \in \mathcal{F}$ then we immediately have $X_r \in \mathcal{F}$. Otherwise $X_{r-1} \cup \{y_r\}$ contains a unique circuit C (by Theorem 13.12(b)). Since $C(X, y_r) \subseteq X_{r-1} \cup \{y_r\}$ (by (b)), we must have $C = C(X, y_r)$. But then by (a) $x_r \in C(X, y_r) = C$, so $X_r = (X_{r-1} \cup \{y_r\}) \setminus \{x_r\} \in \mathcal{F}$. □

The idea behind Edmonds' Matroid Intersection Algorithm is the following. Starting with $X = \emptyset$, we augment X by one element in each iteration. Since in general we cannot hope for an element e such that $X \cup \{e\} \in \mathcal{F}_1 \cap \mathcal{F}_2$, we shall look for "alternating paths". To make this convenient, we define an auxiliary graph. We apply the notion $C(X, e)$ to $(E, \mathcal{F}_i)$ and write $C_i(X, e)$ $(i = 1, 2)$.

Given a set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$, we define a directed auxiliary graph G_X by

$$A_X^{(1)} := \{ (x, y) : y \in E \setminus X,\ x \in C_1(X, y) \setminus \{y\} \},$$
$$A_X^{(2)} := \{ (y, x) : y \in E \setminus X,\ x \in C_2(X, y) \setminus \{y\} \},$$
$$G_X := (E, A_X^{(1)} \cup A_X^{(2)}).$$

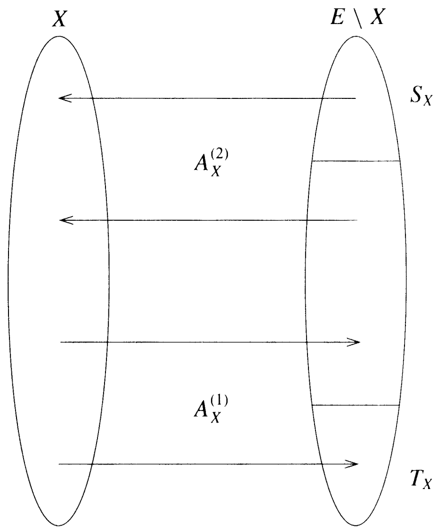

Fig. 13.2.

We set

$$S_X \;:=\; \{y \in E \setminus X : X \cup \{y\} \in \mathcal{F}_1\},$$
$$T_X \;:=\; \{y \in E \setminus X : X \cup \{y\} \in \mathcal{F}_2\}$$

(see Figure 13.2) and look for a shortest path from S_X to T_X. Such a path will enable us to augment the set X. (If $S_X \cap T_X \neq \emptyset$, we have a path of length zero and we can augment X by any element in $S_X \cap T_X$.)

Lemma 13.28. *Let $X \in \mathcal{F}_1 \cap \mathcal{F}_2$. Let $y_0, x_1, y_1, \dots, x_s, y_s$ be the vertices of a shortest y_0-y_s-path in G_X (in this order), with $y_0 \in S_X$ and $y_s \in T_X$. Then*

$$X' \;:=\; (X \cup \{y_0, \dots, y_s\}) \setminus \{x_1, \dots, x_s\} \in \mathcal{F}_1 \cap \mathcal{F}_2.$$

Proof: First we show that $X \cup \{y_0\}$, $x_1, \dots, x_s$ and $y_1, \dots, y_s$ satisfy the requirements of Lemma 13.27 with respect to $\mathcal{F}_1$. Observe that $X \cup \{y_0\} \in \mathcal{F}_1$ because $y_0 \in S_X$. (a) is satisfied because $(x_j, y_j) \in A_X^{(1)}$ for all j, and (b) is satisfied because otherwise the path could be shortcut. We conclude that $X' \in \mathcal{F}_1$.

Secondly, we show that $X \cup \{y_s\}$, $x_s, x_{s-1}, \dots, x_1$ and $y_{s-1}, \dots, y_1, y_0$ satisfy the requirements of Lemma 13.27 with respect to $\mathcal{F}_2$. Observe that $X \cup \{y_s\} \in \mathcal{F}_2$ because $y_s \in T_X$. (a) is satisfied because $(y_{j-1}, x_j) \in A_X^{(2)}$ for all j, and (b) is satisfied because otherwise the path could be shortcut. We conclude that $X' \in \mathcal{F}_2$. □

We shall now prove that if there exists no S_X-T_X-path in G_X, then X is already maximum. We need the following simple fact:

Proposition 13.29. *Let $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ be two matroids with rank functions r_1 and r_2. Then for any $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ and any $Q \subseteq E$ we have*

$$|F| \le r_1(Q) + r_2(E \setminus Q).$$

Proof: $F \cap Q \in \mathcal{F}_1$ implies $|F \cap Q| \le r_1(Q)$. Similarly $F \setminus Q \in \mathcal{F}_2$ implies $|F \setminus Q| \le r_2(E \setminus Q)$. Adding the two inequalities completes the proof. □

Lemma 13.30. *$X \in \mathcal{F}_1 \cap \mathcal{F}_2$ is maximum if and only if there is no S_X-T_X-path in G_X.*

Proof: If there is an S_X-T_X-path, there is also a shortest one. We apply Lemma 13.28 and obtain a set $X' \in \mathcal{F}_1 \cap \mathcal{F}_2$ of greater cardinality.

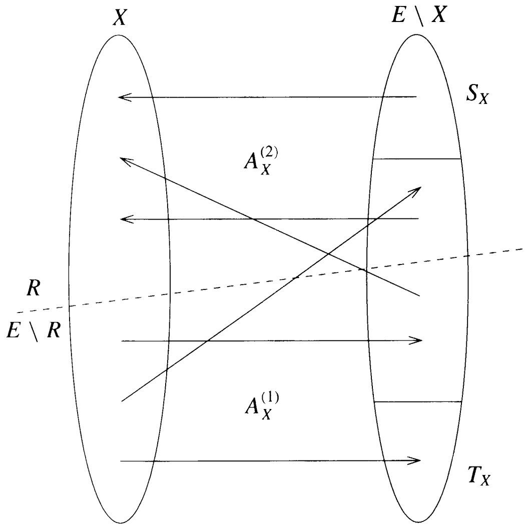

Fig. 13.3.

Otherwise let R be the set of vertices reachable from S_X in G_X (see Figure 13.3). We have $R \cap T_X = \emptyset$. Let r_1 resp. r_2 be the rank function of $\mathcal{F}_1$ resp. $\mathcal{F}_2$.

We claim that $r_2(R) = |X \cap R|$. If not, there would be a $y \in R \setminus X$ with $(X \cap R) \cup \{y\} \in \mathcal{F}_2$. Since $X \cup \{y\} \notin \mathcal{F}_2$ (because $y \notin T_X$), the circuit $C_2(X, y)$ must contain an element $x \in X \setminus R$. But then $(y, x) \in A_X^{(2)}$ means that there is an edge leaving R. This contradicts the definition of R.

Next we prove that $r_1(E \setminus R) = |X \setminus R|$. If not, there would be a $y \in (E \setminus R) \setminus X$ with $(X \setminus R) \cup \{y\} \in \mathcal{F}_1$. Since $X \cup \{y\} \notin \mathcal{F}_1$ (because $y \notin S_X$), the circuit $C_1(X, y)$ must contain an element $x \in X \cap R$. But then $(x, y) \in A_X^{(1)}$ means that there is an edge leaving R. This contradicts the definition of R.

Altogether we have $|X| = r_2(R) + r_1(E \setminus R)$. By Proposition 13.29, this implies optimality. □

The last paragraph of this proof yields the following min-max-equality:

Theorem 13.31. (Edmonds [1970]) *Let $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ be two matroids with rank functions r_1 and r_2. Then*

$$\max\{|X| : X \in \mathcal{F}_1 \cap \mathcal{F}_2\} = \min\{r_1(Q) + r_2(E \setminus Q) : Q \subseteq E\}. \qquad \square$$

We are now ready for a detailed description of the algorithm.

EDMONDS' MATROID INTERSECTION ALGORITHM

Input: Two matroids $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$, given by independence oracles.

Output: A set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ of maximum cardinality.

① Set $X := \emptyset$.

② **For** each $y \in E \setminus X$ and $i \in \{1, 2\}$ **do**: Compute
$C_i(X, y) := \{x \in X \cup \{y\} : X \cup \{y\} \notin \mathcal{F}_i, (X \cup \{y\}) \setminus \{x\} \in \mathcal{F}_i\}$.

③ Compute S_X, T_X, and G_X as defined above.

④ Apply BFS to find a shortest S_X-T_X-path P in G_X.
If none exists **then stop**.

⑤ Set $X := X \triangle V(P)$ and **go to** ②.

Theorem 13.32. EDMONDS' MATROID INTERSECTION ALGORITHM *correctly solves the* MATROID INTERSECTION PROBLEM *in $O(|E|^3\theta)$ time, where θ is the maximum complexity of the two independence oracles.*

Proof: The correctness follows from Lemma 13.28 and 13.30. ② and ③ can be done in $O(|E|^2\theta)$, ④ in $O(|E|)$ time. Since there are at most $|E|$ augmentations, the overall complexity is $O(|E|^3\theta)$. $\square$

Faster matroid intersection algorithms are discussed by Cunningham [1986] and Gabow and Xu [1996]. We remark that the problem of finding a maximum cardinality set in the intersection of three matroids is an *NP*-hard problem; see Exercise 14(c) of Chapter 15.

13.6 Matroid Partitioning

Instead of the intersection of matroids we now consider their union which is defined as follows:

Definition 13.33. *Let $(E, \mathcal{F}_1), \dots, (E, \mathcal{F}_k)$ be k matroids. A set $X \subseteq E$ is called* **partitionable** *if there exists a partition $X = X_1 \dot{\cup} \cdots \dot{\cup} X_k$ with $X_i \in \mathcal{F}_i$ for $i = 1, \dots, k$. Let $\mathcal{F}$ be the family of partitionable subsets of E. Then $(E, \mathcal{F})$ is called the* **union** *or* **sum** *of $(E, \mathcal{F}_1), \dots, (E, \mathcal{F}_k)$.*

We shall prove that the union of matroids is a matroid again. Moreover, we solve the following problem via matroid intersection:

MATROID PARTITIONING PROBLEM	
Instance:	A number $k \in \mathbb{N}$, k matroids $(E, \mathcal{F}_1), \dots, (E, \mathcal{F}_k)$, given by independence oracles.
Task:	Find a partitionable set $X \subseteq E$ of maximum cardinality.

The main theorem with respect to matroid partitioning is:

Theorem 13.34. (Nash-Williams [1967]) *Let $(E, \mathcal{F}_1), \dots, (E, \mathcal{F}_k)$ be matroids with rank functions $r_1, \dots, r_k$, and let $(E, \mathcal{F})$ be their union. Then $(E, \mathcal{F})$ is a matroid, and its rank function r is given by $r(X) = \min_{A \subseteq X} \left(|X \setminus A| + \sum_{i=1}^k r_i(A) \right)$.*

Proof: $(E, \mathcal{F})$ is obviously an independence system. Let $X \subseteq E$. We first prove $r(X) = \min_{A \subseteq X} \left(|X \setminus A| + \sum_{i=1}^k r_i(A) \right)$.

For any $Y \subseteq X$ such that Y is partitionable, i.e. $Y = Y_1 \,\dot\cup\, \cdots \,\dot\cup\, Y_k$ with $Y_i \in \mathcal{F}_i$ $(i = 1, \dots, k)$, and any $A \subseteq X$ we have

$$|Y| = |Y \setminus A| + |Y \cap A| \le |X \setminus A| + \sum_{i=1}^k |Y_i \cap A| \le |X \setminus A| + \sum_{i=1}^k r_i(A),$$

so $r(X) \le \min_{A \subseteq X} \left(|X \setminus A| + \sum_{i=1}^k r_i(A) \right)$.

On the other hand, let $X' := X \times \{1, \dots, k\}$. We define two matroids on X'. For $Q \subseteq X'$ and $i \in \{1, \dots, k\}$ we write $Q_i := \{e \in X : (e, i) \in Q\}$. Let

$$\mathcal{I}_1 := \{Q \subseteq X' : Q_i \in \mathcal{F}_i \text{ for all } i = 1, \dots, k\}$$

and

$$\mathcal{I}_2 := \{Q \subseteq X' : Q_i \cap Q_j = \emptyset \text{ for all } i \ne j\}.$$

Evidently, both $(X', \mathcal{I}_1)$ and $(X', \mathcal{I}_2)$ are matroids, and their rank functions are given by $s_1(Q) := \sum_{i=1}^k r_i(Q_i)$ resp. $s_2(Q) := \left| \bigcup_{i=1}^k Q_i \right|$ for $Q \subseteq X'$.

Now the family of partitionable subsets of X can be written as

$$\{A \subseteq X : \text{there is a function } f : A \to \{1, \dots, k\} \text{ with } \{(e, f(e)) : e \in A\} \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

So the maximum cardinality of a partitionable set is the maximum cardinality of a common independent set in $\mathcal{I}_1$ and $\mathcal{I}_2$. By Theorem 13.31 this maximum cardinality equals $\min \left\{ s_1(Q) + s_2(X' \setminus Q) : Q \subseteq X' \right\}$. If $Q \subseteq X'$ attains this minimum, then for $A := Q_1 \cap \cdots \cap Q_k$ we have

$$r(X) = s_1(Q) + s_2(X' \setminus Q) = \sum_{i=1}^k r_i(Q_i) + \left| X \setminus \bigcap_{i=1}^k Q_i \right| \ge \sum_{i=1}^k r_i(A) + |X \setminus A|.$$

So we have found a set $A \subseteq X$ with $\sum_{i=1}^k r_i(A) + |X \setminus A| \le r(X)$.

Having proved the formula for the rank function r, we finally show that r is submodular. By Theorem 13.10, this implies that $(E, \mathcal{F})$ is a matroid. To show the submodularity, let $X, Y \subseteq E$, and let $A \subseteq X$, $B \subseteq Y$ with $r(X) = |X \setminus A| + \sum_{i=1}^{k} r_i(A)$ and $r(Y) = |Y \setminus B| + \sum_{i=1}^{k} r_i(B)$. Then

$$
\begin{aligned}
& r(X) + r(Y) \\
= \quad & |X \setminus A| + |Y \setminus B| + \sum_{i=1}^{k} (r_i(A) + r_i(B)) \\
\geq \quad & |(X \cup Y) \setminus (A \cup B)| + |(X \cap Y) \setminus (A \cap B)| + \sum_{i=1}^{k} (r_i(A \cup B) + r_i(A \cap B)) \\
\geq \quad & r(X \cup Y) + r(X \cap Y).
\end{aligned}
$$

□

The construction in the above proof (Edmonds [1970]) reduces the MATROID PARTITIONING PROBLEM to the MATROID INTERSECTION PROBLEM. A reduction in the other direction is also possible (Exercise 20), so both problems can be regarded as equivalent.

Note that we find a maximum independent set in the union of an arbitrary number of matroids, while the intersection of more than two matroids is intractable.

13.7 Weighted Matroid Intersection

We now consider a generalization of the above algorithm to the weighted case.

WEIGHTED MATROID INTERSECTION PROBLEM	
Instance:	Two matroids $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$, given by independence oracles. Weights $c : E \to \mathbb{R}$.
Task:	Find a set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ whose weight $c(X)$ is maximum.

We shall describe a primal-dual algorithm due to Frank [1981] for this problem. It generalizes EDMONDS' MATROID INTERSECTION ALGORITHM. Again we start with $X := X_0 = \emptyset$ and increase the cardinality in each iteration by one. We obtain sets $X_0, \dots, X_m \in \mathcal{F}_1 \cap \mathcal{F}_2$ with $|X_k| = k$ $(k = 0, \dots, m)$ and $m = \max\{|X| : X \in \mathcal{F}_1 \cap \mathcal{F}_2\}$. Each X_k will be optimum, i.e.

$$c(X_k) = \max\{c(X) : X \in \mathcal{F}_1 \cap \mathcal{F}_2,\ |X| = k\}. \qquad (13.4)$$

Hence at the end we just choose the optimum set among $X_0, \dots, X_m$.

The main idea is to split up the weight function. At any stage we have two functions $c_1, c_2 : E \to \mathbb{R}$ with $c_1(e) + c_2(e) = c(e)$ for all $e \in E$. For each k we shall guarantee

$$c_i(X_k) = \max\{c_i(X) : X \in \mathcal{F}_i,\ |X| = k\} \qquad (i = 1, 2). \qquad (13.5)$$

This condition obviously implies (13.4). To obtain (13.5) we use the optimality criterion of Theorem 13.23. Instead of G_X, S_X and T_X only a subgraph $\bar{G}$ and subsets $\bar{S}$, $\bar{T}$ are considered.

WEIGHTED MATROID INTERSECTION ALGORITHM	
Input:	Two matroids $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$, given by independence oracles. Weights $c : E \to \mathbb{R}$.
Output:	A set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ of maximum weight.

① Set $k := 0$ and $X_0 := \emptyset$. Set $c_1(e) := c(e)$ and $c_2(e) = 0$ for all $e \in E$.

② **For** each $y \in E \setminus X_k$ and $i \in \{1, 2\}$ **do**: Compute
$C_i(X_k, y) := \{x \in X_k \cup \{y\} : X_k \cup \{y\} \notin \mathcal{F}_i,\ (X_k \cup \{y\}) \setminus \{x\} \in \mathcal{F}_i\}$.

③ Compute

$$\begin{aligned}
A^{(1)} &:= \{(x, y) : y \in E \setminus X_k,\ x \in C_1(X_k, y) \setminus \{y\}\},\\
A^{(2)} &:= \{(y, x) : y \in E \setminus X_k,\ x \in C_2(X_k, y) \setminus \{y\}\},\\
S &:= \{y \in E \setminus X_k : X_k \cup \{y\} \in \mathcal{F}_1\},\\
T &:= \{y \in E \setminus X_k : X_k \cup \{y\} \in \mathcal{F}_2\}.
\end{aligned}$$

④ Compute

$$\begin{aligned}
m_1 &:= \max\{c_1(y) : y \in S\}\\
m_2 &:= \max\{c_2(y) : y \in T\}\\
\bar{S} &:= \{y \in S : c_1(y) = m_1\}\\
\bar{T} &:= \{y \in T : c_2(y) = m_2\}\\
\bar{A}^{(1)} &:= \{(x, y) \in A^{(1)} : c_1(x) = c_1(y)\},\\
\bar{A}^{(2)} &:= \{(y, x) \in A^{(2)} : c_2(x) = c_2(y)\},\\
\bar{G} &:= (E, \bar{A}^{(1)} \cup \bar{A}^{(2)}).
\end{aligned}$$

⑤ Apply BFS to compute the set R of vertices reachable from $\bar{S}$ in $\bar{G}$.

⑥ **If** $R \cap \bar{T} \neq \emptyset$ **then**: Find an $\bar{S}$-$\bar{T}$-path P in $\bar{G}$ with a minimum number of edges, set $X_{k+1} := X_k \triangle V(P)$ and $k := k + 1$ and **go to** ②.

⑦ Compute

$$\begin{aligned}
\varepsilon_1 &:= \min\{c_1(x) - c_1(y) : (x, y) \in A^{(1)} \cap \delta^+(R)\};\\
\varepsilon_2 &:= \min\{c_2(x) - c_2(y) : (y, x) \in A^{(2)} \cap \delta^+(R)\};\\
\varepsilon_3 &:= \min\{m_1 - c_1(y) : y \in S \setminus R\};\\
\varepsilon_4 &:= \min\{m_2 - c_2(y) : y \in T \cap R\};\\
\varepsilon &:= \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}
\end{aligned}$$

(where $\min \emptyset := \infty$).

⑧ **If** $\varepsilon < \infty$ **then**:
Set $c_1(x) := c_1(x) - \varepsilon$ and $c_2(x) := c_2(x) + \varepsilon$ for all $x \in R$. **Go to** ④.
If $\varepsilon = \infty$ **then**:
Among $X_0, X_1, \ldots, X_k$, let X be the one with maximum weight. **Stop**.

See Edmonds [1979] and Lawler [1976] for earlier versions of this algorithm.

Theorem 13.35. (Frank [1981]) *The* WEIGHTED MATROID INTERSECTION ALGORITHM *correctly solves the* WEIGHTED MATROID INTERSECTION PROBLEM *in* $O(|E|^4 + |E|^3\theta)$ *time, where* θ *is the maximum complexity of the two independence oracles.*

Proof: Let m be the final value of k. The algorithm computes sets $X_0, X_1, \ldots, X_m$. We first prove that $X_k \in \mathcal{F}_1 \cap \mathcal{F}_2$ for $k = 0, \ldots, m$, by induction on k. This is trivial for $k = 0$. If we are working with $X_k \in \mathcal{F}_1 \cap \mathcal{F}_2$ for some k, $\bar{G}$ is a subgraph of $(E, A^{(1)} \cup A^{(2)}) = G_{X_k}$. So if a path P is found in ⑤, Lemma 13.28 ensures that $X_{k+1} \in \mathcal{F}_1 \cap \mathcal{F}_2$.

When the algorithm stops, we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \infty$, so T is not reachable from S in G_{X_m}. Then by Lemma 13.30 $m = |X_m| = \max\{|X| : X \in \mathcal{F}_1 \cap \mathcal{F}_2\}$.

To prove correctness, we show that for $k = 0, \ldots, m$, $c(X_k) = \max\{c(X) : X \in \mathcal{F}_1 \cap \mathcal{F}_2, |X| = k\}$. Since we always have $c = c_1 + c_2$, it suffices to prove that at any stage of the algorithm (13.5) holds. This is clearly true when the algorithm starts (for $k = 0$); we show that (13.5) is never violated. We use Theorem 13.23.

When we set $X_{k+1} := X_k \triangle V(P)$ in ⑥ we have to check that (13.5) holds. Let P be an s-t-path, $s \in \bar{S}$, $t \in \bar{T}$. By definition of $\bar{G}$ we have $c_1(X_{k+1}) = c_1(X_k) + c_1(s)$ and $c_2(X_{k+1}) = c_2(X_k) + c_2(t)$. Since X_k satisfies (13.5), conditions (a) and (b) of Theorem 13.23 must hold with respect to X_k and each of $\mathcal{F}_1$ and $\mathcal{F}_2$.

By definition of $\bar{S}$ both conditions continue to hold for $X_k \cup \{s\}$ and $\mathcal{F}_1$. Therefore $c_1(X_{k+1}) = c_1(X_k \cup \{s\}) = \max\{c_1(Y) : Y \in \mathcal{F}_1, |Y| = k+1\}$. Moreover, by definition of $\bar{T}$, (a) and (b) of Theorem 13.23 continue to hold for $X_k \cup \{t\}$ and $\mathcal{F}_2$, implying $c_2(X_{k+1}) = c_2(X_k \cup \{t\}) = \max\{c_2(Y) : Y \in \mathcal{F}_2, |Y| = k + 1\}$. In other words, (13.5) indeed holds for X_{k+1}.

Now suppose we change c_1 and c_2 in ⑧. We first show that $\varepsilon > 0$. By (13.5) and Theorem 13.23 we have $c_1(x) \geq c_1(y)$ for all $y \in E \setminus X_k$ and $x \in C_1(X_k, y) \setminus \{y\}$. So for any $(x, y) \in A^{(1)}$ we have $c_1(x) \geq c_1(y)$. Moreover, by the definition of R no edge $(x, y) \in \delta^+(R)$ belongs to $\bar{A}^{(1)}$. This implies $\varepsilon_1 > 0$.

$\varepsilon_2 > 0$ is proved analogously. $m_1 \geq c_1(y)$ holds for all $y \in S$. If in addition $y \notin R$ then $y \notin \bar{S}$, so $m_1 > c_1(y)$. Therefore $\varepsilon_3 > 0$. Similarly, $\varepsilon_4 > 0$ (using $\bar{T} \cap R = \emptyset$). We conclude that $\varepsilon > 0$.

We can now prove that ⑧ preserves (13.5). Let c_1' be the modified c_1, i.e. $c_1'(x) := \begin{cases} c_1(x) - \varepsilon & \text{if } x \in R \\ c_1(x) & \text{if } x \notin R \end{cases}$. We prove that X_k and c_1' satisfy the conditions of Theorem 13.23 with respect to $\mathcal{F}_1$.

To prove (a), let $y \in E \setminus X_k$ and $x \in C_1(X_k, y) \setminus \{y\}$. Suppose $c'_1(x) < c'_1(y)$. Since $c_1(x) \geq c_1(y)$ and $\varepsilon > 0$, we must have $x \in R$ and $y \notin R$. Since also $(x, y) \in A^{(1)}$, we have $\varepsilon \leq \varepsilon_1 \leq c_1(x) - c_1(y) = (c'_1(x) + \varepsilon) - c'_1(y)$, a contradiction.

To prove (b), let $x \in X_k$ and $y \in E \setminus X_k$ with $X_k \cup \{y\} \in \mathcal{F}_1$. Now suppose $c'_1(y) > c'_1(x)$. Since $c_1(y) \leq m_1 \leq c_1(x)$, we must have $x \in R$ and $y \notin R$. Since $y \in S$ we have $\varepsilon \leq \varepsilon_3 \leq m_1 - c_1(y) \leq c_1(x) - c_1(y) = (c'_1(x) + \varepsilon) - c'_1(y)$, a contradiction.

Let c'_2 be the modified c_2, i.e. $c'_2(x) := \begin{cases} c_2(x) + \varepsilon & \text{if } x \in R \\ c_2(x) & \text{if } x \notin R \end{cases}$. We show that X_k and c'_2 satisfy the conditions of Theorem 13.23 with respect to $\mathcal{F}_2$.

To prove (a), let $y \in E \setminus X_k$ and $x \in C_2(X_k, y) \setminus \{y\}$. Suppose $c'_2(x) < c'_2(y)$. Since $c_2(x) \geq c_2(y)$, we must have $y \in R$ and $x \notin R$. Since also $(y, x) \in A^{(2)}$, we have $\varepsilon \leq \varepsilon_2 \leq c_2(x) - c_2(y) = c'_2(x) - (c'_2(y) - \varepsilon)$, a contradiction.

To prove (b), let $x \in X_k$ and $y \in E \setminus X_k$ with $X_k \cup \{y\} \in \mathcal{F}_2$. Now suppose $c'_2(y) > c'_2(x)$. Since $c_2(y) \leq m_2 \leq c_2(x)$, we must have $y \in R$ and $x \notin R$. Since $y \in T$ we have $\varepsilon \leq \varepsilon_4 \leq m_2 - c_2(y) \leq c_2(x) - c_2(y) = c'_2(x) - (c'_2(y) - \varepsilon)$, a contradiction.

So we have proved that (13.5) is not violated during ⑧, and thus the algorithm works correctly.

We now consider the running time. Observe that after ⑧, the new sets $\bar{S}$, $\bar{T}$ and R, as computed subsequently in ④ and ⑤, are supersets of the old $\bar{S}$, $\bar{T}$ resp. R. If $\varepsilon = \varepsilon_4 < \infty$, an augmentation (increase of k) follows. Otherwise the cardinality of R increases immediately (in ⑤) by at least one. So ④ – ⑧ are repeated less than $|E|$ times between two augmentations.

Since the running time of ④ – ⑧ is $O(|E|^2)$, the total running time between two augmentations is $O(|E|^3)$ plus $O(|E|^2)$ oracle calls (in ②). Since there are $m \leq |E|$ augmentations, the stated overall running time follows. □

The running time can easily be improved to $O(|E|^3\theta)$ (Exercise 22).

Exercises

1. Prove that all the independence systems apart from (5) and (6) in the list at the beginning of Section 13.1 are – in general – not matroids.
2. Show that the uniform matroid with four elements and rank 2 is not a graphic matroid.
3. Prove that every graphic matroid is representable over every field.
4. Let G be an undirected graph, $K \in \mathbb{N}$, and let $\mathcal{F}$ contain those subsets of $E(G)$ that are the union of K forests. Prove that $(E(G), \mathcal{F})$ is a matroid.
5. Compute a tight lower bound for the rank quotients of the independence systems listed at the beginning of Section 13.1.
6. Let $\mathcal{S}$ be a family of sets. A set T is a transversal of $\mathcal{S}$ if there is a bijection $\Phi : T \to \mathcal{S}$ with $t \in \Phi(t)$ for all $t \in T$. (For a necessary and sufficient

condition for the existence of a transversal, see Exercise 6 of Chapter 10.) Assume that $\mathcal{S}$ has a transversal. Prove that the family of transversals of $\mathcal{S}$ is the family of bases of a matroid.

7. Let E be a finite set and $\mathcal{B} \subseteq 2^E$. Show that $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if the following holds:
   (B1) $\mathcal{B} \neq \emptyset$;
   (B2) For any $B_1, B_2 \in \mathcal{B}$ and $y \in B_2 \setminus B_1$ there exists an $x \in B_1 \setminus B_2$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.
8. Let G be a graph. Let $\mathcal{F}$ be the family of sets $X \subseteq V(G)$, for which a maximum matching exists that covers no vertex in X. Prove that $(V(G), \mathcal{F})$ is a matroid. What is the dual matroid?
9. Show that $\mathcal{M}(G^*) = (\mathcal{M}(G))^*$ also holds for disconnected graphs G, extending Theorem 13.16.
   Hint: Use Exercise 30(a) of Chapter 2.
10. Show that the clutters in (3) and (6) in the list of Section 13.3 have the Max-Flow-Min-Cut property. (Use Theorem 19.10.) Show that the clutters in (1), (4) and (5) do not have the Max-Flow-Min-Cut property.

* 11. A clutter $(E, \mathcal{F})$ is called binary if for all $X_1, \dots, X_k \in \mathcal{F}$ with k odd there exists a $Y \in \mathcal{F}$ with $Y \subseteq X_1 \triangle \cdots \triangle X_k$. Prove that the clutter of minimal T-joins and the clutter of minimal T-cuts (example (7) of the list in Section 13.3) are binary. Prove that a clutter is binary if and only if $|A \cap B|$ is odd for all $A \in \mathcal{F}$ and all $B \in \mathcal{F}^*$, where $(E, \mathcal{F}^*)$ is the blocking clutter. Conclude that a clutter is binary if and only if its blocking clutter is binary.
   Note: Seymour [1977] classified the binary clutters with the Max-Flow-Min-Cut property.

* 12. Let P be a polyhedron of blocking type, i.e. we have $x + y \in P$ for all $x \in P$ and $y \geq 0$. The blocking polyhedron of P is defined to be $B(P) := \{z : z^\top x \geq 1 \text{ for all } x \in P\}$. Prove that $B(P)$ is again a polyhedron of blocking type and that $B(B(P)) = P$.
   Note: Compare this with Theorem 4.22.

13. How can one check (in polynomial time) whether a given set of edges of a complete graph G is a subset of some Hamiltonian circuit in G?
14. Prove that if $(E, \mathcal{F})$ is a matroid, then the Best-In-Greedy maximizes any bottleneck function $c(F) = \min\{c_e : e \in F\}$ over the bases.
15. Let $(E, \mathcal{F})$ be a matroid and $c : E \to \mathbb{R}$ such that $c(e) \neq c(e')$ for all $e \neq e'$ and $c(e) \neq 0$ for all e. Prove that both the Maximization and the Minimization Problem for $(E, \mathcal{F}, c)$ have a unique optimum solution.

* 16. Prove that for matroids the independence, basis-superset, closure and rank oracles are polynomially equivalent.
   Hint: To show that the rank oracle reduces to the independence oracle, use the Best-In-Greedy. To show that the independence oracle reduces to the basis-superset oracle, use the Worst-Out-Greedy.
   (Hausmann and Korte [1981])

17. Given an undirected graph G, we wish to colour the edges with a minimum number of colours such that for any circuit C of G, the edges of C do not all have the same colour. Show that there is a polynomial-time algorithm for this problem.
18. Let $(E, \mathcal{F}_1), \dots, (E, \mathcal{F}_k)$ be matroids with rank functions $r_1, \dots, r_k$. Prove that a set $X \subseteq E$ is partitionable if and only if $|A| \le \sum_{i=1}^{k} r_i(A)$ for all $A \subseteq X$. Show that Theorem 6.17 is a special case.
(Edmonds and Fulkerson [1965])
19. Let $(E, \mathcal{F})$ be a matroid with rank function r. Prove (using Theorem 13.34):
    (a) $(E, \mathcal{F})$ has k pairwise disjoint bases if and only if $kr(A)+|E \setminus A| \ge kr(E)$ for all $A \subseteq E$.
    (b) $(E, \mathcal{F})$ has k independent sets whose union is E if and only if $kr(A) \ge |A|$ for all $A \subseteq E$.

    Show that Theorem 6.17 and Theorem 6.14 are special cases.
20. Let $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ be two matroids. Let X be a maximal partitionable subset with respect to $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2^*)$: $X = X_1 \dot\cup X_2$ with $X_1 \in \mathcal{F}_1$ and $X_2 \in \mathcal{F}_2^*$. Let $B_2 \supseteq X_2$ be a basis of $\mathcal{F}_2^*$. Prove that then $X \setminus B_2$ is a maximum-cardinality set in $\mathcal{F}_1 \cap \mathcal{F}_2$.
(Edmonds [1970])
21. Let $(E, \mathcal{S})$ be a set system, and let $(E, \mathcal{F})$ be a matroid with rank function r. Show that $\mathcal{S}$ has a transversal that is independent in $(E, \mathcal{F})$ if and only if $r\left(\bigcup_{B \in \mathcal{B}} B\right) \ge |\mathcal{B}|$ for all $\mathcal{B} \subseteq \mathcal{S}$.
Hint: First describe the rank function of the matroid whose independent sets are all transversals (Exercise 6), using Theorem 13.34. Then apply Theorem 13.31.
(Rado [1942])
22. Show that the running time of the Weighted Matroid Intersection Algorithm (cf. Theorem 13.35) can be improved to $O(|E|^3\theta)$.
23. Let $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ be two matroids, and $c : E \to \mathbb{R}$. Let $X_0, \dots, X_m \in \mathcal{F}_1 \cap \mathcal{F}_2$ with $|X_k| = k$ and $c(X_k) = \max\{c(X) : X \in \mathcal{F}_1 \cap \mathcal{F}_2,\ |X| = k\}$ for all k. Prove that for $k = 1, \dots, m-2$

$$c(X_{k+1}) - c(X_k) \ \le\ c(X_k) - c(X_{k-1}).$$

(Krogdahl [unpublished])
24. Consider the following problem. Given a digraph G with edge weights, a vertex $s \in V(G)$, and a number K, find a minimum weight subgraph H of G containing K edge-disjoint paths from s to each other vertex. Show that this reduces to the Weighted Matroid Intersection Problem.
Hint: See Exercise 18 of Chapter 6 and Exercise 4 of this chapter.
(Edmonds [1970]; Frank and Tardos [1989]; Gabow [1991])
25. Let A and B be two finite sets of cardinality $n \in \mathbb{N}$, $\bar{a} \in A$, and $c : \{\{a, b\} : a \in A,\ b \in B\} \to \mathbb{R}$ a cost function. Let $\mathcal{T}$ be the family of edge sets of all trees T with $V(T) = A \dot\cup B$ and $|\delta_T(a)| = 2$ for all $a \in A \setminus \{\bar{a}\}$. Show that

a minimum cost element of $\mathcal{T}$ can be computed in $O(n^7)$ time. How many edges will be incident to $\bar{a}$?

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