Lexicographic Breath-First-search (Golumbic 4.3)

Intuition: Since every chordal graph has at least two simplicial vertices, we may freely choose any vertex v_n to occupy position n in a p.e.s $[v_1, v_2, \ldots v_n]$

The following algorithm, called LexBFS, constructs a p.e.s of any chordal graph G backwards.

Recall that lexicographic order is the same as dictionary order

so $985 > 9761$ and $6432 > 643$

Begin

Assign each vertex $v \in V$ the label set $S_v = \emptyset$

For $i := n$ downto 1 do

Pick an unnumbered vertex v with the largest label S_v (in lex order)

$\sigma(i) \leftarrow v$

$\forall u \in N(v)$: add v to S_u

end

The label S_v of v lists its neighbours seen so far in descending order

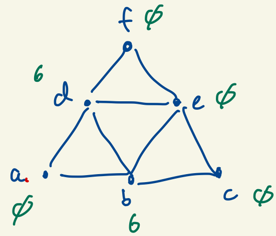
$S_v = 12\,10\,7$

Example

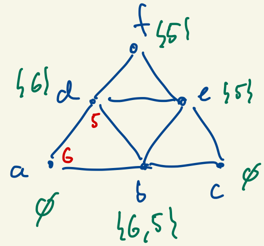
for simplicity we write $S_v = \{9,8,6,3\}$ instead of $S_v = 9863$

start with $v_6 = a$

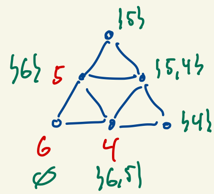
$S_d = \{6\} = S_b$ all other $\emptyset$



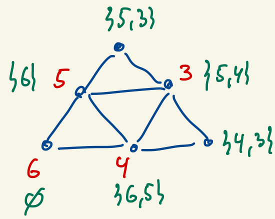
d and b have largest label
arbitrarily choose d as v_5



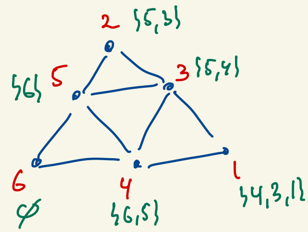
b has largest label so $v_4 = b$



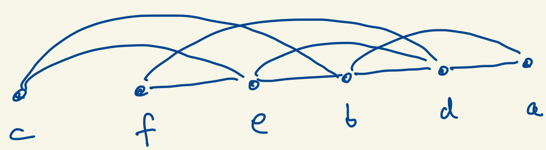
e has largest label so $v_3 = e$



f has largest label so $v_2 = f$
and finally $v_1 = c$



Final order



simplicial order!

Important properties of lexBFS

For $i = 1, 2 \dots n$ and $x \in V$ define

$L_i(x)$ is the set S_x just after the vertex which receives position i is chosen and before its neighbours have their lists updated

So $L_n(x) = \emptyset$

$L_{n-1}(x) = \{n\} \quad \forall x \in N(\sigma(n))$

$\vdots$

Properties of the ordering:

(L1) $L_j(x) \geq L_i(x)$ when $j < i$

Label values increase

(L2) $L_i(x) < L_i(y)$

$\Downarrow$

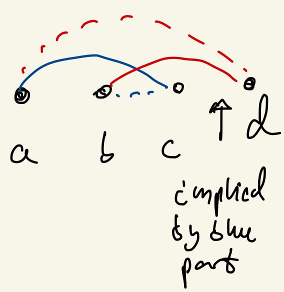
$L_j(x) < L_j(y) \quad \forall j < i$

once x is smaller than y it stays smaller than y

(L3) $\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c) \wedge c \in N(a) - N(b)$

$\Downarrow$

$\exists d \in N(b) - N(a)$ s.t. $\sigma^{-1}(c) < \sigma^{-1}(d)$



If d did not exist then a would have been chosen before b in σ

Theorem 4.3

$G = (V, E)$ is chordal
$\Updownarrow$
lexBFS produces a p.e.s σ for G.

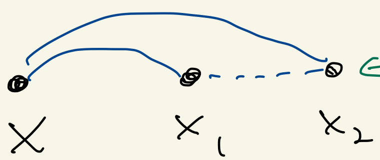
Proof

$\Uparrow$ follows from Thm 4.1

$\Downarrow$: By induction on n (base case trivial)

Enough to show that $x = \sigma(1)$ is a simplicial vertex of G (rest follows by induction)

Suppose x is not simplicial. Then $\exists\, x_1, x_2$ s.t



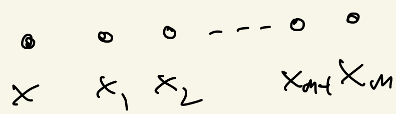
$\leftarrow$ as large as possible wrt σ s.t neighbour of x and $x_1 x_2 \notin E$

Now assume we have found vertices $x_1, x_2, \ldots, x_m$ s.t

(1) $x x_i \in E \iff i \in \{1, 2\}$

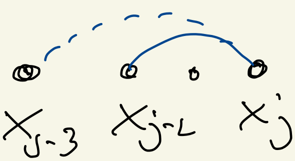
(2) $x_i x_j \in E \iff |i - j| = 2$

(3) $\sigma^{-1}(x_1) < \sigma^{-1}(x_2) < \ldots < \sigma^{-1}(x_m)$



(4) x_j is the largest wrt σ such that

$x_{j-2} x_j \in E$ and $x_{j-3} x_j \notin E$



(1) $xx_i \in E \iff i \in \{1,2\}$

(2) $x_ix_j \in E \iff |i-j| = 2$

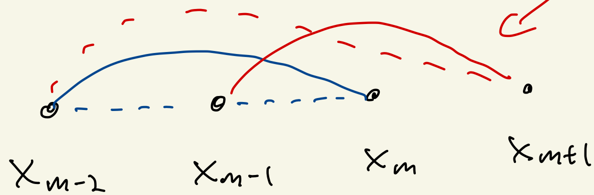
(3) $\sigma^{-1}(x_1) < \sigma^{-1}(x_2) < \dots < \sigma^{-1}(x_m)$

(4) x_j is the largest wrt σ such that

$x_{j-2}x_j \in E$ and $x_{j-3}x_j \notin E$

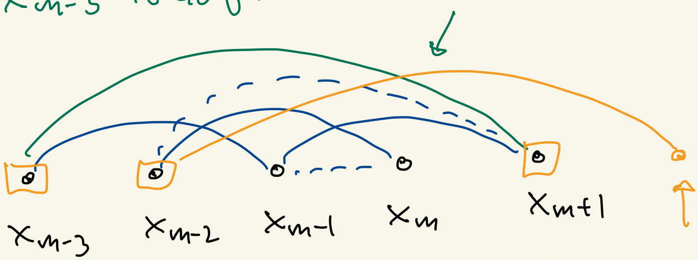
$m=2$ is the starting situation with $x_0 = x$, $x_{-1} = x_1$, x_2

By (L3) there exist a vertex x_{m+1} such that



Choose x_{m+1} as the largest such vertex

Suppose x_{m-3} is adjacent to x_{m+1}

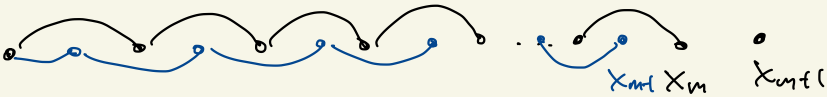


Apply (L3) to x_{m-3}, x_{m-2} and x_{m+1} to conclude that x_{m-2} has a neighbour with a higher number than x_{m+1}

But that contradicts that x_m is chosen before x_{m-2} (by the way we chose x_{m+1})

So x_{m-3} is <u>not</u> adj to x_{m+1}

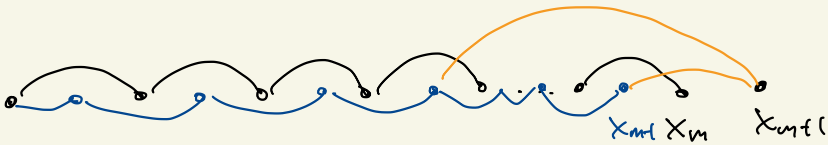
So (1) and (2) imply that the two paths



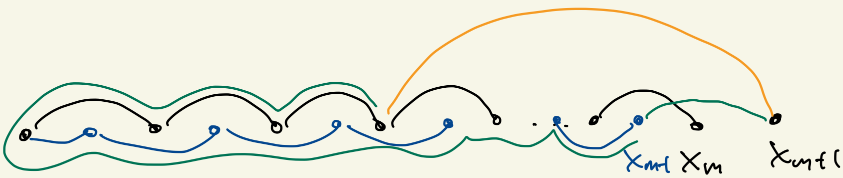
are <u>all</u> the edges between $\{x_0, x_1 \cdots x_m\}$

$\Leftrightarrow$ the path $x_{m-1} x_{m-3} x_{m-5} \cdots x_1 x_0 x_2 x_4 \cdots x_m$ is induced

If x_{m+1} is adjacent to some $x_{m-(2i+1)}$ $i \geq 2$ then we find an induced cycle with more than 3 vertices ↯



If x_{m+1} is adjacent to some x_{m-2i} $i \geq 0$ then we also find an induced cycle with more than 3 vertices ↯



The conclusion is that $x_1 x_2 \cdots \underline{x_{m+1}}$ also satisfies (1)~(4) and we can repeat to find $x_{m+2}, x_{m+3} \cdots$

This contradicts that $|V(G)| < \infty$ D.