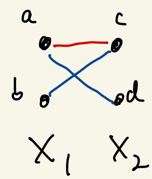
Matroid Intersection

Problem Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$
Find $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|I| \geq |I'|$ $\forall I' \in \mathcal{I}_1 \cap \mathcal{I}_2$

Recall that $X = (S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is not always a matroid
For example, let $B = (X_1, X_2, E)$ be a bipartite graph and
let $\mathcal{I}_i = \{E' \subseteq E \mid d_{E'}(v) \leq 1 \ \forall v \in X_i\}$ $i = 1, 2$
Then $M_i = (E, \mathcal{I}_i)$ is a matroid but $\mathcal{I}_1 \cap \mathcal{I}_2$ does not satisfy
the exchange axiom:

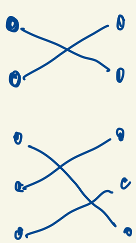


$I' = \{ac\}$ $I = \{bc, ad\}$

$|I| > |I'|$ but cannot add any edge from I to I

But we can solve maximum bipartite matching as a matroid intersection problem for the matroids M_1, M_2 above:

matchings



independent in both M_1, M_2

so $\max\{|M| \mid M \text{ is a matching}\}$

$= \max\{|I| \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

Another example of matroid intersection: out-branchings

Given $D=(V,A)$ and $s\in V$ define

$$\mathcal{I}_1 = \{A' \mid A' \text{ is a forest in } UG(D)\}$$ $UG(D)$ is the underlying undirected graph of D

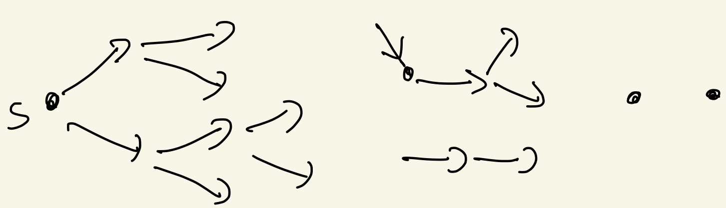
$$\mathcal{I}_2 = \{A'' \mid d^-_{A''}(v) \le 1 \ \forall v\in V-s \text{ and } d^-_{A''}(s)=0\}$$

- We have seen that $M_1=(A,\mathcal{I}_1)$ is a matroid as it is the circuit matroid of $UG(D)$
- $M_2=(A,\mathcal{I}_2)$ is a matroid as $\mathcal{I}_2$ satisfies the three axioms $\emptyset\in\mathcal{I}_2$, $X\in\mathcal{I}_2 \wedge Y\subseteq X \Rightarrow Y\in\mathcal{I}_2$, $X,Y\in\mathcal{I}_2,\ |Y|=|X|+1 \Rightarrow \exists y\in Y\setminus X$ s.t. $X+y\in\mathcal{I}_2$

Now $\max\{|I| \mid I\in\mathcal{I}_1\cap\mathcal{I}_2\} = n-1$

$\Updownarrow$

D has an out-branching from s



independent sets in $\mathcal{I}_1\cap\mathcal{I}_2$

No cycles in underlying graph + indegree ≤ 1

and $d^-(s)=0$

Recall that if $M=(S,\mathcal{I})$ is a matroid and $X\in\mathcal{I}$ but $X+e\notin\mathcal{I}$, then is a unique circuit (minimal dependent set) C in $X+e$

For any $X\in\mathcal{I}$ and $e\in S$ let

- $C(X,e)=\emptyset$ if $X+e\in\mathcal{I}$
- $C(X,e)=$ unique circuit in $X+e$ if $X+e\notin\mathcal{I}$

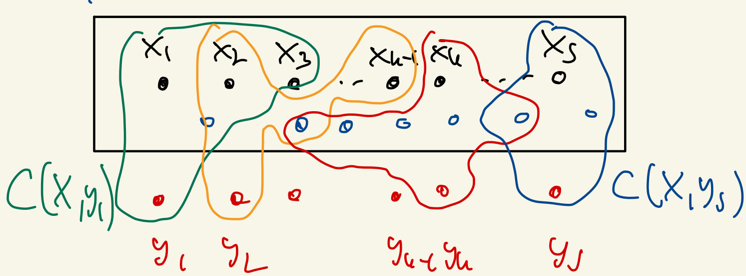
Below we follow Korte & Vygen Section 13.5

<u>Lemma 13.27</u> Let $M=(E,\mathcal{I})$ be a matroid and let $X\in\mathcal{I}$ and suppose that $x_1,x_2,\dots,x_s$ are distinct elements of X and $y_1,y_2,\dots,y_s$ are distinct elements of $E-X$ such that

(a) $x_k\in C(X,y_k)$ for $k=1,2,\dots,s$

(b) $x_j\notin C(X,y_k)$ for all j with $1\le j<k\le s$

Then $X-\{x_1,x_2,\dots,x_s\}+\{y_1,y_2,\dots,y_s\}\in\mathcal{I}$



Proof: we show by induction on r that $X_r \in \mathcal{J}$

when $X_r = X - \{x_1, x_2, \dots, x_r\} + \{y_1, \dots y_r\}$

$r = 0$ ok as $X_r = X$

suppose $X_{r-1} \in \mathcal{J}$

if $X_{r-1} + y_r \in \mathcal{J}$ then $X_r = (X_{r-1} + y_r) - x_r \in \mathcal{J}$

otherwise $\exists$ unique circuit C in $X_{r-1} + y_r$

By (b) none of $x_1, x_2, \dots, x_{r-1}$ belongs to C

so $C(X, y_r) \subseteq X_{r-1} + y_r$ implying $C = C(X, y_r)$

By (a) $x_r \in C(X, y_r)$ so $X_{r-1} + y_r - x_r \in \mathcal{J}$ $\square$.

Edmonds matroid intersection algorithm:

- Start with $X = \emptyset$
- Augment X by one element at a time until no new element can be added

How to do this?

Given current set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ define a digraph $G_X = (E, A_X^{(1)} \cup A_X^{(2)})$

Here $A_X^{(1)} = \{x \to y \mid y \in E - X \wedge x \in C_1(X,y) - y\}$ $\quad$ C_1: circuit of M_1

$A_X^{(2)} = \{y \to x \mid y \in E - X \wedge x \in C_2(X,y) - y\}$ $\quad$ C_2: circuit of M_2

This means that

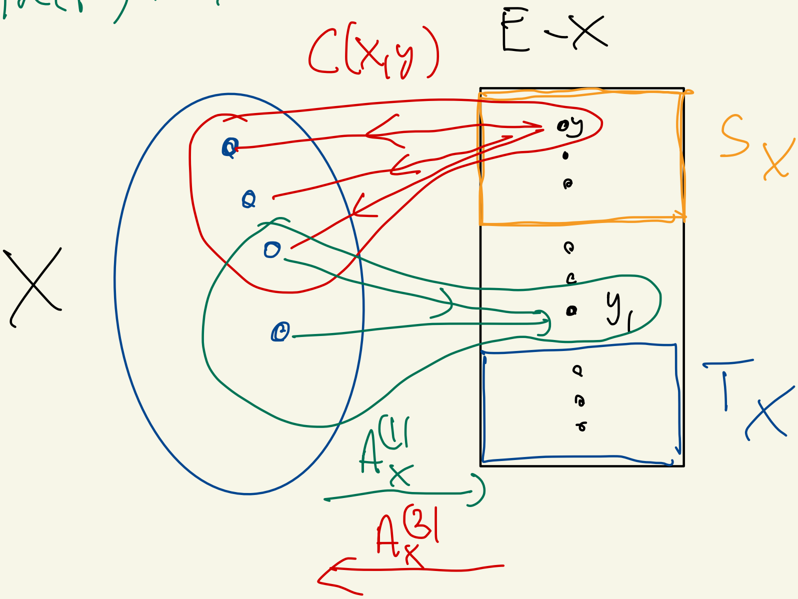
In $A_X^{(1)}$ an arc $x \to y$ indicates that by deleting x we may add y to X so that $X + y - x \in \mathcal{I}_1$

In $A_X^{(2)}$ an arc $y \to x$ indicates that by deleting x we may add y to X so that $X + y - x \in \mathcal{I}_2$

Let $S_X = \{y \in E - X \mid X + y \in \mathcal{I}_1\}$, $T_X = \{y \in E - X \mid X + y \in \mathcal{I}_2\}$

If $S_X = \emptyset$ or $T_X = \emptyset$ then X is maximal in $\mathcal{I}_1 \cap \mathcal{I}_2$

If $y \in S_X \cap T_X$ then $X + y \in \mathcal{I}_1 \cap \mathcal{I}_2$ so we can add y directly to X. So assume $S_X \cap T_X = \emptyset$



Lemma 13.28 Suppose $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ and that

$$P = y_0 \to x_1 \to y_1 \to x_2 \to y_2 \to \cdots \to x_s \to y_s$$

is a shortest (S_X, T_X)-path in $G_X = (E, A_X^{(1)} \cup A_X^2)$

Then $X' = (X + \{y_0, y_1, \ldots, y_s\}) - \{x_1, x_2, \ldots x_s\}$ is in $\mathcal{I}_1 \cap \mathcal{I}_2$ (□)

(note that $|X'| = |X| + 1$)

Proof We show that $X + y_0$, $x_1, x_2, \ldots x_s$ and $y_1, y_2 \ldots y_s$ satisfy the conditions of Lemma 13.27 wrt M_1. This will imply that $X' \in \mathcal{I}_1$

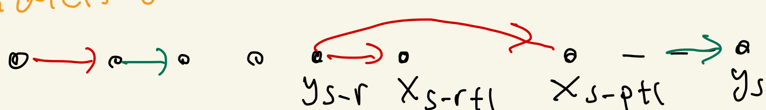
- $X + y_0 \in \mathcal{I}_1$ as $y_0 \in S_X$
- (a) holds as $x_j \to y_j \in A_X^{(1)}$ for $j = 1, 2 \ldots s$
- To see that (b) holds, assume that $x_j \in C_1(X, y_k)$ for some $1 \le j < k \le s$. Then $x_j \to y_k \in A_X^1$ implying that P is not a shortest (S_X, T_X)-path, contradiction

Similarly we can show that $X' \in \mathcal{I}_2$ by showing that

$X + y_s$, $x'_1, x'_2, \ldots x'_s$ and $y'_1, y'_2, \ldots, y'_s$ with $x'_i = x_{s-i+1}$ $y'_i = y_{s-i}$

satisfy the conditions of Lemma 13.27 wrt M_2.

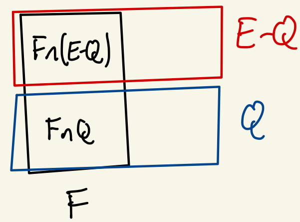
- $X + y_s \in \mathcal{I}_2$ as $y_s \in T_X$
- (a) holds as $y_j \to x_{j+1} \in A_X^2$ for $j = 0, 1, \ldots, s-1$
- To see that (b) holds, assume that $x'_p \in C_2(X, y'_r)$ for some $p < r$ then the arc $y_{s-r} \to x_{s-p+1}$ contradicts that P is shortest



Proposition 13.29 Let $M_i = (E, \mathcal{J}_i)$ $i = 1,2$ be matroids and let r_i, $i=1,2$ be their rank functions. Then

$\forall F \in \mathcal{J}_1 \cap \mathcal{J}_2$ and $\forall Q \subseteq E$: $|F| \leq r_1(Q) + r_2(E-Q)$

Proof:



$F \cap Q \in \mathcal{J}_1$ so $|F \cap Q| \leq r_1(Q)$

$F \cap (E-Q) \in \mathcal{J}_2$ so $|F \cap (E-Q)| \leq r_2(E-Q)$

$\Downarrow$

$|F| = |F \cap Q| + |F \cap (E-Q)| \leq r_1(Q) + r_2(E-Q)$

$\square$.

Lemma 13.30 $X \in \mathcal{J}_1 \cap \mathcal{J}_2$ has maximum size

$\Updownarrow$

G_X has no (S_X, T_X)-path

Proof

$\Downarrow$ follows from Lemma 13.28

$\Uparrow$: Let $R = \{e \in E \mid \exists \text{ path from } S_X \text{ to } e \text{ in } G_X\}$

Then $S_X \subseteq R$ and $R \cap T_X = \emptyset$

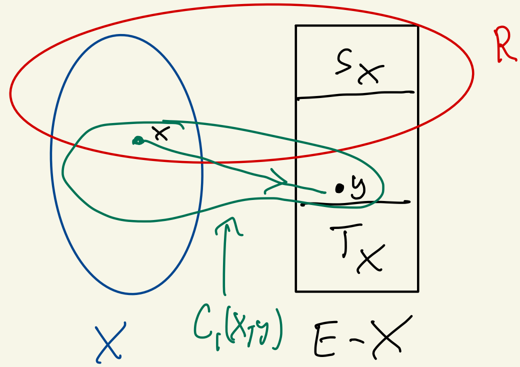
We claim that $r_1(X-R) = |X-R|$ and $r_2(R) = |X \cap R|$

If this holds we can take $Q = E-R$ and get

$$|X| = |X-R| + |X \cap R| = r_1(X-R) + r_2(R)$$
$$= r_1(Q) + r_2(E-Q)$$

so $|X|$ is maximum by Prop 13.29

- Suppose $r_1(E-R) > |X-R|$ then $\exists\, y \in (E-R)-X$ such that $(X-R)+y \in \mathcal{J}_1$



But $X+y \notin \mathcal{J}_1$ as $y \notin S_X \subseteq R$

$(y \in (E-R)-X \subseteq E-X)$

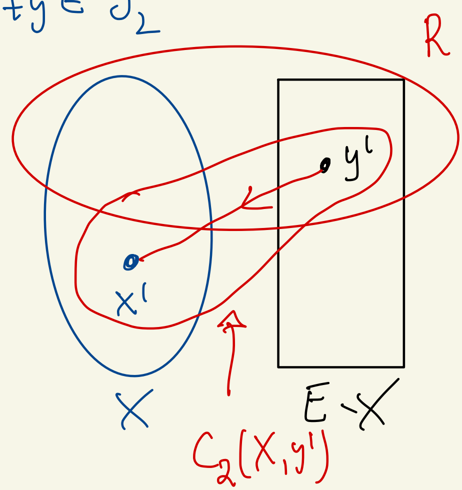
Now $C_1(X,y) \cap (X \cap R) \neq \emptyset$ as

$(X-R)+y \in \mathcal{J}_1$.

But then there is an arc $x \to y$ in A'_X from R to y implying that $y \in R$ contradiction (as $y \in (E-R)-X$)

- Suppose $r_2(R) > |X \cap R|$

Then $\exists\, y' \in R-X$ s.t. $X \cap R + y' \in \mathcal{J}_2$



But $X+y' \notin \mathcal{J}_2$ as $R \cap T_X = \emptyset$

Then $C_2(X,y') \cap X-R \neq \emptyset$

as $(X \cap R)+y' \in \mathcal{J}_2$

Now there is an arc $y' \to x'$ from R to E-R $\lightning$

We have shown that

$$r_1(E-R) = |X-R| \text{ and } r_2(R) = |X \cap R|$$