Schrijver application 1.7

$G=(V,E)$ $\quad s: E \to \mathbb{R}_+$ $\quad s(e) \Leftrightarrow$ strength of e

Reliability of a path P, $r(P) = \min\{s(e) \mid e \in E(P)\}$

$r_G(u,v) = \max\{r(P) \mid P \text{ is a } (u,v)\text{-path}\}$

Let T be a maximum weight spanning tree of G wrt the weight-function s.

Claim $\quad r_T(u,v) = r_G(u,v) \quad \forall u,v \in V(G)$

Proof: suppose $r_T(u,v) < r_G(u,v)$ for some pair $u,v \in V$

Let P be the unique (u,v)-path in T and let Q be a max. strength path from u to v

Denote P by $\quad u = u_1 u_2 \cdots u_r u_{r+1} = v$ and let X_i, $i = 1,2 \cdots r$, denote the connected component of $T - u_i u_{i+1}$

Then the edge $u_i u_{i+1}$ is the only edge from X_i to $V - X_i$ in T.

As $r(Q) > r(P)$ there exists an index $i \in \{1,2 \cdots r\}$ and an edge e of Q s.t $r(e) > r(u_i u_{i+1})$

Let $T' = T - u_i u_{i+1} + e$ and observe that

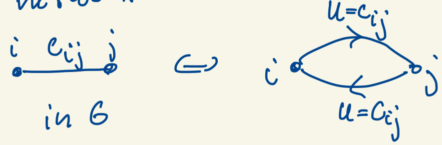
$r(T') = r(T) - r(u_i u_{i+1}) + r(e) > r(T)$ $\quad\lightning$

Exercise 3: Given $G=(V,E)$ and costs $c: E \to \mathbb{R}_+$

Goal find E' s.t $G-E'$ is disconnected and $c(E')$ is minimized for all E'' s.t $G-E''$ not connected.

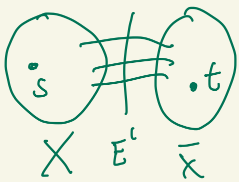
Solution via flows:

Given G, c we construct a network $N=(V, A, \ell\equiv 0, u)$ when $u_{ij} = c_{ij} = u_{ji}$



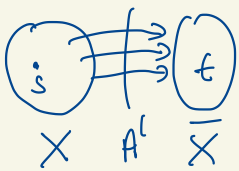
Also fix one vertex $s \in V$

Now if



is a cut in G where E' is the set of edges from X to $\bar{X}$ and $t \in \bar{X}$

Then



A' is the corresponding (s,t)-cut in N and $u(X,\bar{X}) = c(E')$

This shows that $\min\{c(E') \mid G-E' \text{ not connected}\}$

$$= \min\{u(X,\bar{X}) \mid (X,\bar{X}) \text{ is an } (s,t)\text{-cut for some } t \neq s\}$$

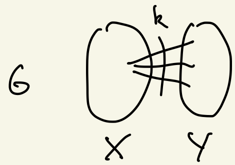
We can find a minimum (s,t)-cut using, say, Edmonds-Karp so by letting t run through all vertices in $V-s$ and taking the minimum of these (s,t)-cuts we can find E'. Running time $O(n)\cdot$ time for maxflow so polynomial.

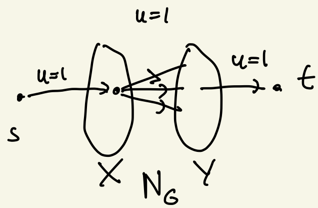
Exercise 4 = Schrijver 3.2

(i) Every regular bipartite graph has a perfect matching

proof 1 (via integrality thm for flows)

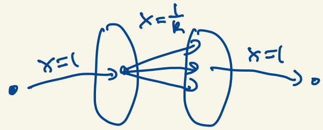
Given $G=(V,E)$ k-regular bipartite we make the corresponding flow network N_G:





$k \cdot |X| = k \cdot |Y| \Rightarrow |X| = |Y|$

In N_G every vertex in X has out-degree k and every vertex in Y has in-degree k so we obtain a maximum (s,t)-flow of value $|X|=|Y|$ as follows



By the integrality theorem, N_G has a maximum integer valued flow x and now $M^* = \{uv \mid u \in X, v \in Y \text{ and } x_{uv} = 1\}$ is a matching of size $|X|=|Y|$

proof 2 via Hall's theorem: suppose $\exists\, X' \subseteq X$ s.t. $|N(X')| < |X'|$ (*)

Then, using that all $k|X'|$ edges incident to X' go to $N(X')$ we have at least $k|X'|$ edges into $N(X')$. By (*) and the pigeonhole principle, some vertex in $N(X')$ has degree $> k$ ↯

(ii) Show that a k-regular bipartite G has k disjoint perfect matchings.

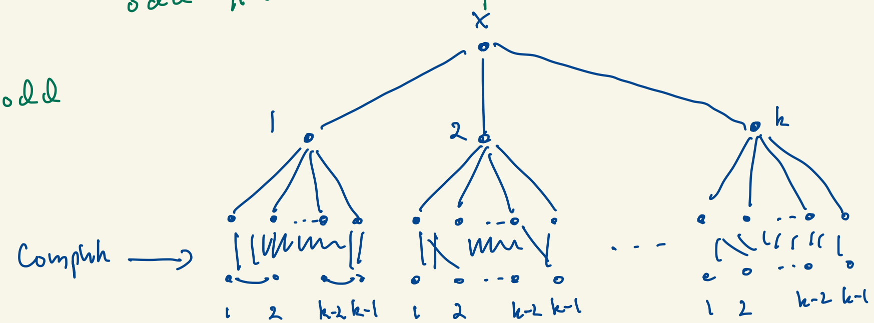
Proof: We saw that G has a perfect matching M. If $k=1$ we are done. Otherwise consider $G' = G - E(M)$. G' is $(k-1)$-regular and by induction it has $(k-1)$ disjoint perfect matchings.

(iii) Give for all $k>1$ an example of a k-regular graph with no perfect matchings.

We know from (i) that such a graph cannot be bipartite!

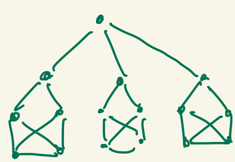
$k \geq 2$ even: Take the complete graph K_{k+1} it has an odd # vertices so no perfect matchings

$k \geq 3$ odd



G is k-regular and $G - x$ has $k \geq 3$ odd components so G has no perfect matchings

For $k=3$ G is

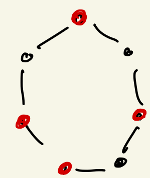


Exercise: Give an example of a graph G with $\nu(G) < \tau(G)$

Recall $\nu(G)$ = size of maximum matching

$\tau(G)$ = size of minimum vertex cover

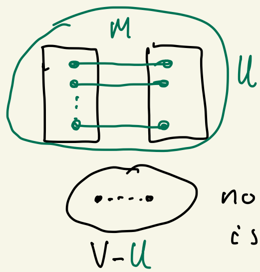
Odd cycles have $\tau = \nu + 1$:



clearly $\nu = 3$ and $\tau > 3$

$\tau = 4$

Let M be a matching of size $\nu(G)$:

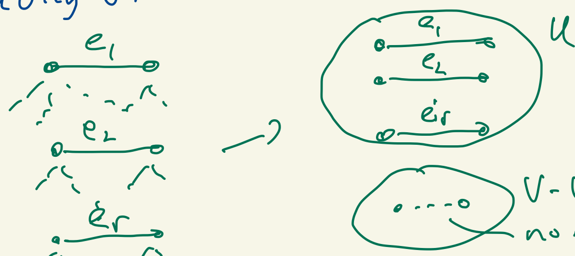


no edge in $V-U$ as M is maximum so U is a vertex cover and $|U| = 2|M|$

implying that $\tau \le 2\nu$

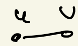
Same argument works if M is a maximal matching (still no edge in $V-U$).

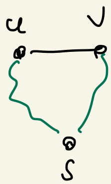
Easy to find a maximal matching by picking edges greedily until no edge remains:



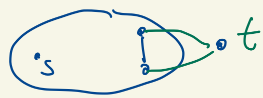
no edges, so U is a vertex cover

Exercise 6: Prove that if G is 2-connected ($= G-x$ connected $\forall x \in V(G)$)

Then $\forall s \in V(G)$ and $uv \in E(G)$ then exists a cycle C containing s and 

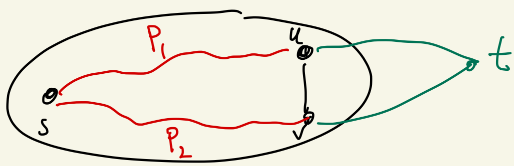


Proof: Let H be obtained from G by adding a new vertex t joined to u and v

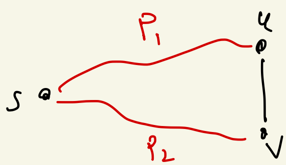


H is 2-connected since $H - v$ is connected $\forall v \in V(H)$

By Menger's theorem H has 2 internally disjoint (s,t)-paths must look like this



now



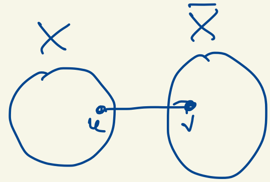
is the desired cycle.

Exercise 7 Prove: G has a strongly connected orientation

$\Updownarrow$

G is 2-edge connected

Proof: $\Downarrow$

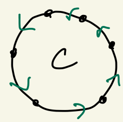


if uv is the unique edge between X and $\bar{X}$ then every orientation of G is non strong as we must orient uv as either $u \to v$ or $v \to u$.

$\Uparrow$: Assume G is 2-edge connected

Then G has a cycle C (as G is not a forest)

orient C as a directed cycle

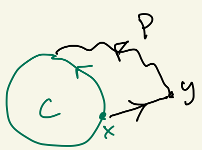


this is a strong digraph

If $V(C) = V(G)$ we are done

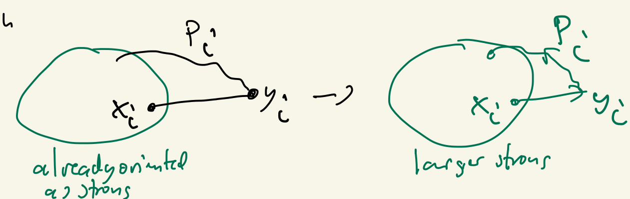
So assume $V(G) - V(C) \neq \emptyset$

As G is connected there is some edge xy with $x \in V(C)$ and $y \in V(G) - V(C)$. As G is 2-edge-connected there is a path P from y to V(C) in G−xy. Now orient as follows:



clearly the part which is oriented so far is strong.

Continue this way until we have all vertices included in the strong digraph



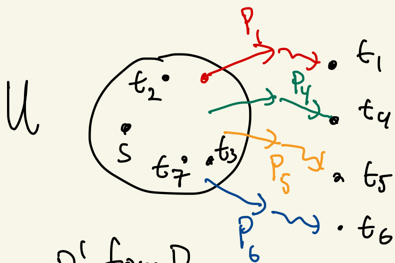
Edges of G not on C or one of the P_i's can be oriented arbitrarily

Exercise 8 = Schrijver 4.1

Let D be a digraph and $s, t_1, t_2 \dots t_k$ vertices of D (not necessarily distinct). Then there are arc disjoint paths $P_1, P_2 \dots P_k$ in D s.t $P_i = s \rightsquigarrow t_i$ if and only if

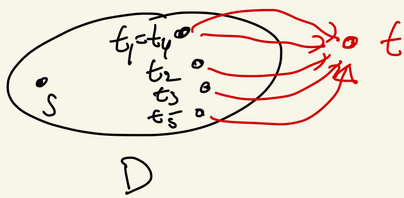
$$(*) \qquad d^+(U) \geq |\{i \mid t_i \notin U\}| \text{ for every } U \subseteq V \text{ with } s \in U$$

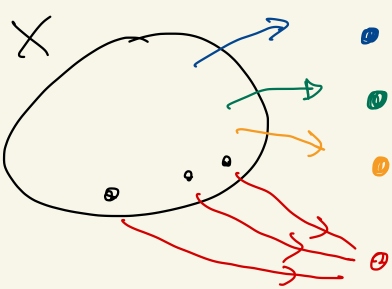
Proof: clearly $(*)$ holds if there are paths $P_1, P_2 \dots P_k$ as above since P_i must use at least one arc out of U if $t_i \in U$



Suppose $(*)$ holds and make D' from D by adding t and one arc from each t_i to t:

Now $(\square)$: $d^+(X) \geq k$ for all $X \subset V(D')$ with $s \in X$ and $t' \notin X$





Let $k' = |\{i \mid t_i \notin X\}|$

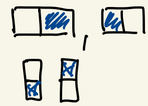
Then by $(*)$ and construction of D'

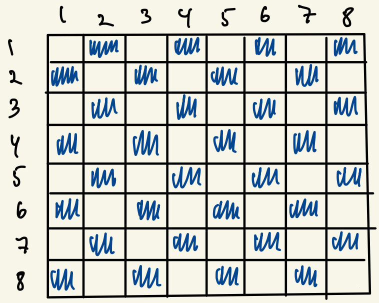
$$d^+(X) \geq k' + (k - k') = k$$

Now $(\square)$ and Menger's thm implies $\exists\, P'_1, P'_2 \dots P'_k$ arc-disj (s,t)-paths in D'. Deleting t from these we get the desired paths.

Exercise 10

8×8 chessboard + dominos like this

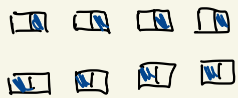




(a) We can cover the chessboard by 1×2 dominos:

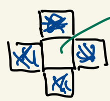
cover odd rows by

even rows by



(b) assume we delete cells $(1,1)$ and $(8,8)$. Show that now we can <u>not</u> cover the remaining cells by 1×2 and 2×1 dominos

Define G as the bipartite graph with vertex set □, ☒ so
V_1 = vertices that are '□', V_2 = vertices that are ☒
let uv be an edge in G if $u \in V_1$ and $v \in V_2$ and then are neighbours on the chessboard



Then each domino $\Leftrightarrow$ edge in G
so $\exists$ cover by dominos $\Leftrightarrow$ G has a perfect matching

But G has no perfect matching as $|V_2| = |V_1| + 2$
(we deleted $(1,1)$ and $(8,8)$ which are both white)