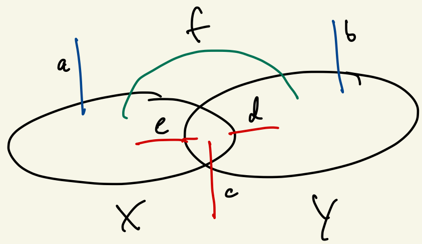
Solution of selected exercises on Note 4

Let $G=(V,E)$ be a graph. Then for every choice of $X,Y\subseteq V$ we have

(1) $d(X)+d(Y)\geq d(X\cap Y)+d(X\cup Y)$

(2) $d(X)+d(Y)\geq d(X-Y)+d(Y-X)$

Proof: just check the contribution of each edge to the two sides:



$$d(X)+d(Y)=(a+c+d+f)+(b+c+e+f)$$

$$=(c+d+e)+(a+b+c)+2c$$

$$=d(X\cap Y)+d(X\cup Y)+2d(X,Y)$$

$$\geq d(X\cap Y)+d(X\cup Y)$$

$$d(X)+d(Y)=(a+c+d+f)+(b+c+e+f)$$

$$=(a+e+f)+(b+d+f)+2c$$

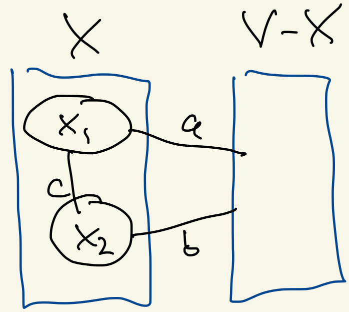
$$=d(X-Y)+d(Y-X)+2d(X\cap Y,V-(X\cup Y))$$

$$\geq d(X-Y)+d(Y-X)$$

Let $G=(V,E)$ be k-edge-connected $(\lambda(G) \geq k)$

Prove that if $d(X)=k$ for some $X \subset V$ then $\lambda(G[X]) \geq \lceil \frac{k}{2} \rceil$ when $G[X]$ is the subgraph induced by X.

Proof: consider a partition $X = X_1 \cup X_2$



Then we have

- $a+b = d(X) = k$
- $a+c \geq \lambda(G) = k$
- $b+c \geq \lambda(G) = k$

$$\Downarrow \quad (a+c)+(b+c)-(a+b) \geq k+k-k = k$$

$$\Updownarrow \quad 2c \geq k$$

$$\Updownarrow \quad c \geq \lceil \tfrac{k}{2} \rceil$$

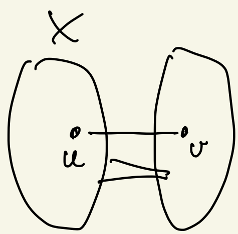
As $X_1 \cup X_2$ was an arbitrary partition of X we have shown that $d(Y) \geq \lceil \frac{k}{2} \rceil$ □.

$G=(V,E)$ is minimally k-edge connected if $\lambda(G)=k$ but $\lambda(G-e)=k-1 \quad \forall e\in E$

Problem: Given $G=(V,E)$ with $\lambda(G)=k$
Find $H=(V,E')$ s.t. H is minimally k-edge-connected

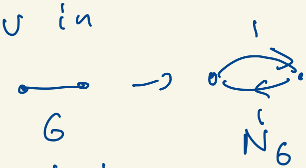
Subproblem: Given a graph $G=(V,E)$ with $\lambda(G)=k$ and a edge $uv\in E$. Decide if $\lambda(G-uv)=k$

Solution: removing uv can only lower the edge-connectivity if then $\exists\, X\subseteq V-v$ s.t $u\in X$



$d(X)=k$

This can be checked via flows: Find max flow from u to v in N_G when



each arc in N_G has capacity 1

This can be done in time $O(k\cdot m)$ via Ford-Fulkerson

Trivial solution:

- Order E as $e_1, e_2 \dots e_m$
- start with $E' = E$
- For $i \leftarrow 1$ to m

    if $H' = (V, E' - e_i)$ is k-edge-connected

    $E' \leftarrow E' - e_i$

By the remark on checking $\lambda(u,v)$ this takes $O(km)$ for each i

(more precisely $O(k|E'|)$)

So $O(km^2)$ in total

Better solution

Given G first find a small certificate $\hat{G}$ with $\lambda(\hat{G}) = \lambda(G) = k$ via max-back orderings

$\hat{G}$ = union of k first max back forests of G.

Then $\hat{m} = |E(\hat{G})| \leq k(n-1) = O(kn)$

Now run the previous algorithm on G' instead of G.

This takes time $O(k\hat{m}^2)$

as $\hat{m} = O(kn)$ we get total running time $O(k^3 n^2)$.

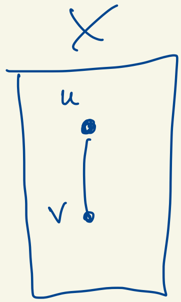
Problem show that if a graph $G=(V,E)$ is minimally k-edge-connected, then $\exists v \in V$ with $d(v) = k$.

Proof 1: via submodularity

Let $X \subset V$ have $d(X) = k$ and $|X|$ minimum among such sets. If $|X| = 1$ we are done so assume $|X| \geq 2$

Then $\lambda(G[X]) \geq \lceil \frac{k}{2} \rceil \geq 1$ as we have seen earlier

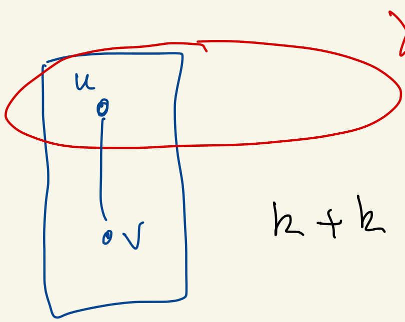
Let uv be an edge of G s.t $u, v \in X$:



as G is minimally k-edge connected $G' = G - uv$ has a set Y with $d_{G'}(Y) = k-1$

So $d_G(Y) = k$ and $|Y \cap \{u,v\}| = 1$

w.l.o.g $u \in Y$ $v \notin Y$



If $X \cup Y \neq V$ we get from [1]

$$\begin{aligned} k + k &= d(X) + d(Y) \\ &\geq d(X \cap Y) + d(X \cup Y) \\ &\geq k + k \end{aligned}$$

so $d(X \cap Y) = k$ contradicting the minimality of X

So $X \cup Y = V$ and $Y - X \neq \emptyset$ as $X \neq V$

Now using (2) we get

$$\begin{aligned} k + k = d(X) + d(Y) &\geq d(X - Y) + d(Y - X) \\ &\geq k + k \end{aligned}$$

So $d(X - Y) = k$ and this contradicts the minimality of X. $\square$.

Proof 2: via max-back orderings

Consider the algorithm $\mathcal{A}$ for finding $\lambda(G)$ via $n-2$ max back orderings.

This results in $\lambda(G) = k =$ degree of the last vertex $v_{n'}$ in one of the max-back orderings

$v_{n'}$

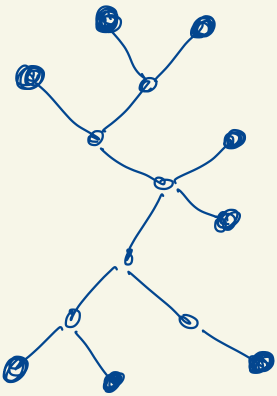
as $\lambda(G) = k$ we have $d(v_{n'}) \geq k$ in G so, since we keep all edges when identifying the two last vertices when running $\mathcal{A}$, we see that $v_{n'}$ has degree precisely k in G.

Now consider running $\mathcal{A}$ again starting the max-back orderings from $v_{n'}$ in each step. This will again find a new vertex $v_{n''} \neq v_{n'}$ with $d_G(v_{n''}) = k = d_G(v_{n'})$

So we have proved that every minimally k-edge-connected graph G has at least 2 vertices of degree k.

Problem Given a tree $T=(V,E)$

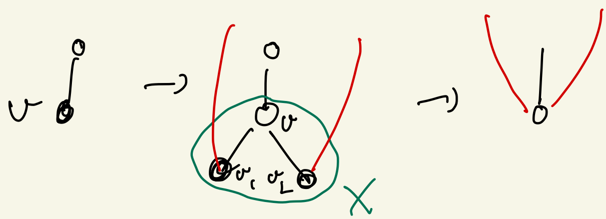
Find minimum cardinality set of edges F such that $G=(V, E\cup F)$ is 2-edge-connected



Solution Each leaf needs an edge from F so $|F| \geq \left\lceil \frac{\#\text{leaves}}{2} \right\rceil$

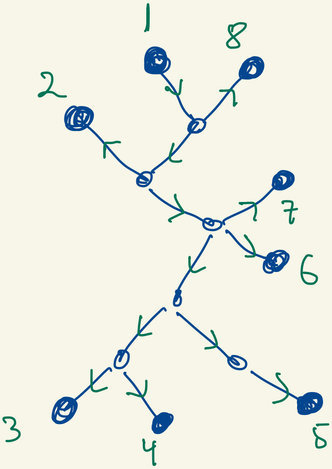
We prove that there is a solution with precisely $\left\lceil \frac{\#\text{leaves}}{2} \right\rceil$ edges

We can assume #leaves is even: If not replace one leaf by 2 by adding two new edges, solve the problem and contract the set X.

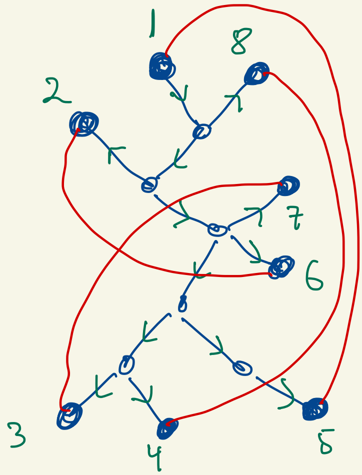


Contracting edges preserves edge connectivity.

Now run a depth-first-search starting in an arbitrary leaf and labelling the leaves in the order they are visited:

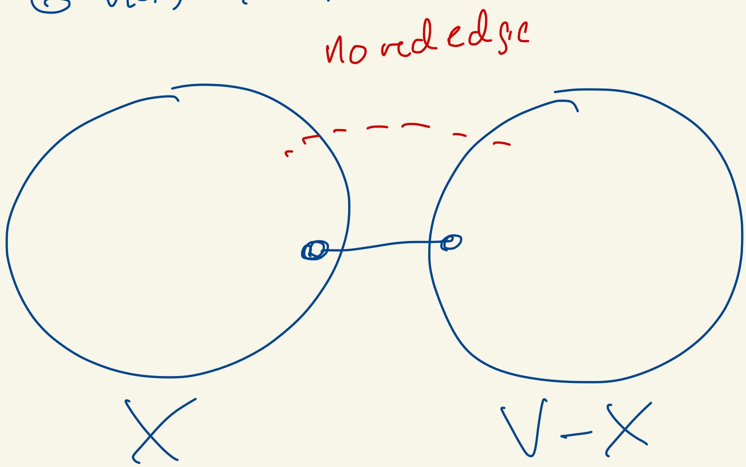


add the edges
$i - i+t$ for $i = 1, 2, \cdots t$
when #leaves $= 2t$

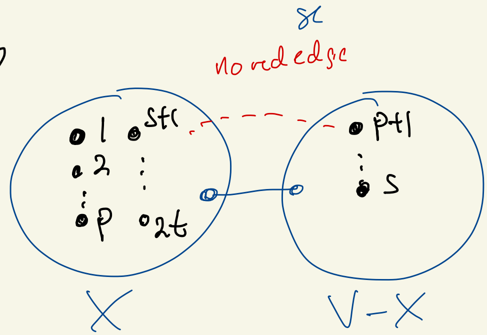


We claim that the resulting graph G is 2-edge connected.

Suppose G has a 1-edge-cut then



Without loss of generality leaf 1 is in X



Then, by the definition of DFS, there is a number $p < 2t$ such that

- leaves $1, 2, \dots, p$ are in X
- leaves $p+1, \dots, s$ are in $V-X$
- If $s < 2t$, then leaves $s+1, \dots, 2t$ are in X again

Case 1: $p \geq t$

Then each of the leaves $p+1, \dots, s$ have a red edge to X, contradiction

Case 2: $p < t$ and $s \leq t$

Then the leaf s has a red edge to $s+t \in X$, contradiction

Case 3 $p < t < s$:

Then the edge $1 - t+1$ goes from X to $V-X$, contradiction $\square$

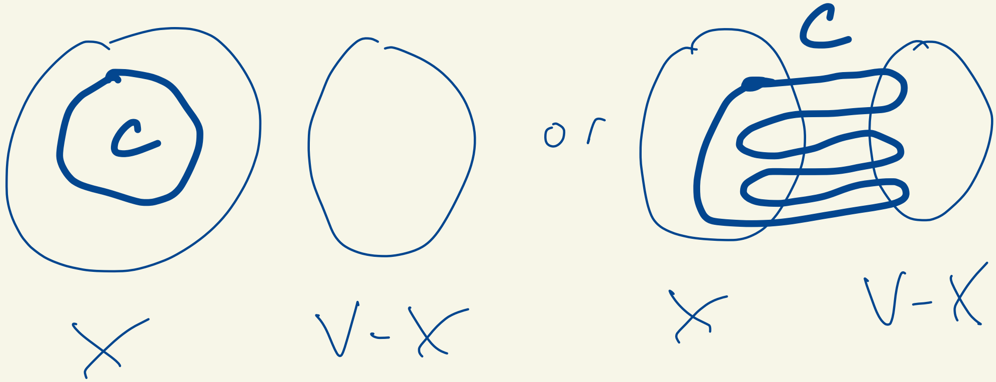
Problem

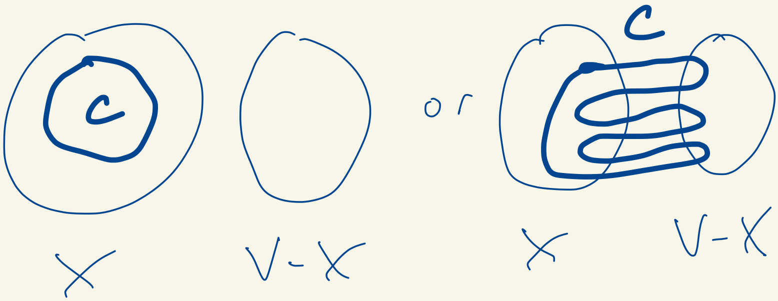
Let $G=(V,E)$ have $\lambda(G)=k$ and let E' be a minimal (w.r.t inclusion) set of edges s.t. $G'=(V,E\cup E')$ has $\lambda(G')=k+1$.

Show that there are no cycles in E' (so E' induces a forest)

Proof: suppose C

is a cycle in E' and consider a partition $(X, V-X)$ of V. Then we either have





- In the first case C has no edge between X and $V-X$ so we can delete any (all) edges of C without changing $d(X)$ which must already be at least $k+1$
- In the second case C has at least 2 edges between X and $V-X$ and using that $d_G(X) \geq k$ we see that we can delete an arbitrary edge of C without creating a cut of size k

Both cases show that E' is not minimal $\lightning$

<u>Problem</u> show that every minimally k-edge-connected graph has at most $k(n-1)$ edges.

<u>Proof</u>

Let $G=(V,E)$ be minimally k-edge-connected and let $H=(V,E')$ be the union of the first k max-back forests of G with respect to some max-back ordering $v_1, v_2, \dots, v_n$ of G.

Then $|E'| \le k(n-1)$ and $E' \subseteq E$

As G is minimally k-edge-connected, we have $E=E'$

so $|E| \le k(n-1)$ $\square$.