Tree-width (Niedermeier Chap 10)

Definition 10.1

A tree-decomposition of a graph $G=(V,E)$ is a pair $(\{X_i \mid i\in I\}, T)$ where T is a tree on vertices I, each $X_i \subseteq V$ and the following holds:

1. $\bigcup_{i\in I} X_i = V$
2. $\forall uv \in E \; \exists i \in I$ such that $u, v \in X_i$
3. $\forall i,j,k \in I$: if $i \sim j \sim k$ path in T

   then $X_i \cap X_k \subseteq X_j$

- The sets X_i, $i\in I$ are called the bags of the decomposition.
- The width of $(\{X_i \mid i \in I\}, T)$ is $\max_{i\in I} |X_i| - 1$
- The tree-width of G, denoted $tw(G)$ is the minimum width of a tree-decomposition of G.

Property 3. is called the consistency property
(very important in algorithmic applications of tree-decompositions)

Recall Theorem 4.8 (iii) on chordal graphs G

- Here $\{X_i \mid i \in I\}$ is the set of maximal cliques and the consistency property holds as $T[\mathcal{B}_v]$ is a subtree of T
- The width of this tree-decomposition for G is $\omega(G)-1$ and $tw(G) = \omega(G)-1$ since:

<u>Lemma</u> $tw(K_n) = n-1$

<u>proof</u>: suppose $tw(K_n) < n-1$ and let $(\{X_i \mid i \in I\}, T)$ be a tree-decomposition of G s.t. $\max_{i \in I} |X_i| < n$

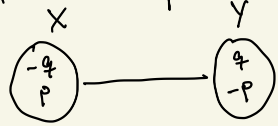
Let $X \in \{X_i \mid i \in I\}$ be a bag of maximum size

As $|X| < n$ there is some vertex q of K_n s.t. $q \notin X$

Let $Y \neq X$ be a bag containing q (by property 1.)

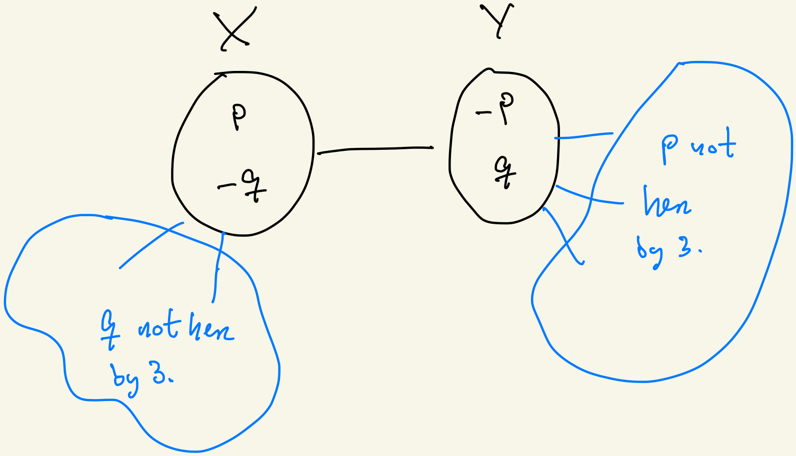
Let $p \in X$ be such that $p \notin Y$ (by maximality of $|X|$)

Now we have



note that we may assume X and Y are neighbours in T otherwise rename X:

$$\underbrace{X - X - \cdots - X}_{\text{equal bags}} - Y$$

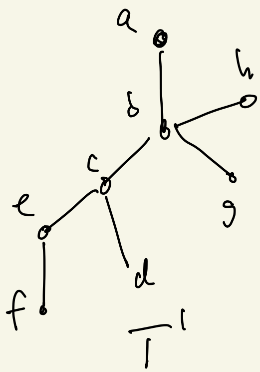


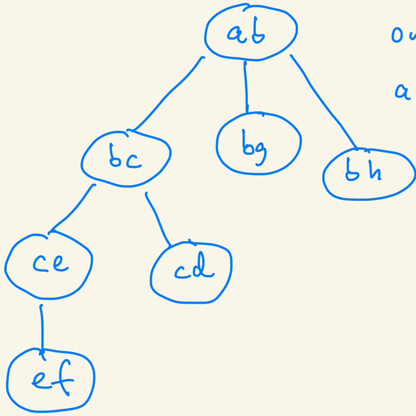
$\Rightarrow$ the edge pq violates 2. as no bag contains both p and q $\square$.

Note that $tw(G) \geq tw(H)$ for every induced subgraph H of G as we can obtain a tree-decomposition of H from a t.d for G letting $X_i' = X_i \cap V(H)$

Thus the tree-width of a chordal G is at least $\omega(G)-1$ and by Thm 4.8(iii) $tw(G) \leq \omega(G)-1$ so $tw(G) = \omega(G)-1$

Trees have tree-width 1:





one bag per edge of T' and T follows the structure of T'

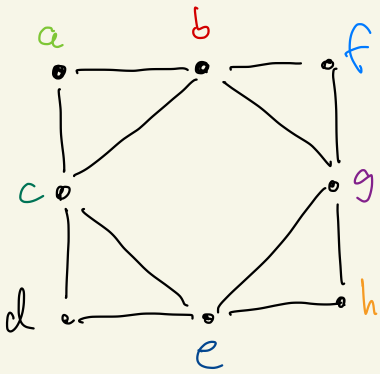
This is the reason for the definition of tree-width to be $\max_{i \in I} |X_i| - 1$ so that trees get $tw(T) = 1$.

Equivalent version of property 3.

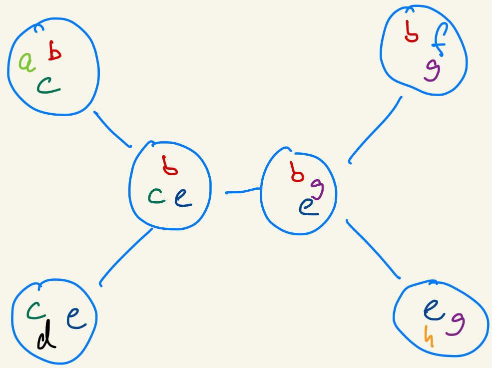
$3'$ $\forall v \in V$ the set of bags X_i with $v \in X_i$ form a subtree of T

G is called a partial k-tree if $tw(G) \leq k$

Example of a tree-decomposition



G



$tw(G) = 3-1=2$

Definition 10.2

A tree-decomposition $(\{X_i \mid i \in I\}, T)$ is nice if

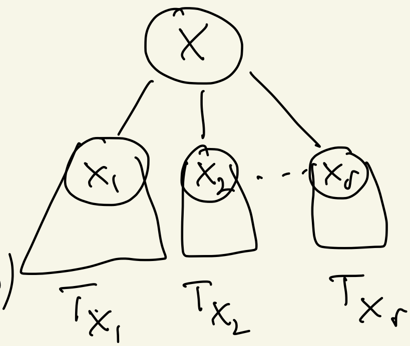
1. Every node has ≤ 2 children
2. If node i has 2 children j, k then $X_i = X_j = X_k$ and i is called a join node
3. If node i has only one child j then either
   (a) $|X_i| = |X_j|+1$ and $X_j < X_i$ (introduce node) or
   (b) $|X_i| = |X_j|-1$ and $X_i < X_j$ (forget node)

Lemma 10.3 Given a tree-decomposition of G with n nodes and width k, we can find a width k nice tree-decomposition with $O(n)$ nodes in time $O(n)$

proof omitted.

Lemma Every bag X in a t.d whose corresponding vertex in T has degree at least 2 is a vertex-separator.

proof:



consistency property

$\Downarrow$

Every $v \in V - X$ 'belongs' to a unique T_{X_i} (only in those bags)

Take $u \in X_1$ and $v \in X_2$ and suppose $G - X$ has a (u,v)-path P.

Let w be the first vertex on P not in $V(T_{X_1})$ (in G)
and let w' be its predecessor.

Then the edge $w'w$ is not in any bag of T_{X_1}

So by property 1 w' must belong to a bag of some T_{X_j}, $j \neq i$

but $w' \in V - X$ so this violates consistency property.

$\square$.

Lower bound for tree-width via cops and robbers

Game { on a graph $G=(V,E)$

- One robber is at a vertex $r \in V$ and can move to another vertex along edges in E at ∞ speed

- k cops want to catch the robber

- at each move cops can move to any vertex of V (they fly in helicopters and can land at any vertex)

- at each move $i+1$
  - (i) cops announce to which vertex set V^{i+1} $|V^{i+1}| \leq k$ they will move and before they move there
  - (ii) the robber moves to a vertex which is reachable from r in $G - V^i$
  - (iii) Now the cops move to V^{i+1} from V^i and the helicopters

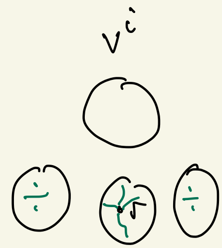
  at time $i=0$ all cops are in helicopters so robber can move anywhere

The game continues until the robber is caught

The cops win if they can put a cop on the vertex r occupied by the robber

The robber wins if (s)he can continue to escape forever.

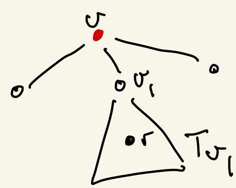
<u>Observation</u> in step $i+1$ (cops occupy $V^i \subseteq V$) the robber can only move within the connected component of $G-V^i$ which contains vertex r



<u>Lemma</u> 2 cops win on a tree

<u>Proof:</u> step 1: cops announce one vertex $v_0 \in V(T)$ $V^1=\{v_0\}$
and robber chooses $r \neq v_0$

Now let the edge vv_1 be the first edge on P_{vr} in T

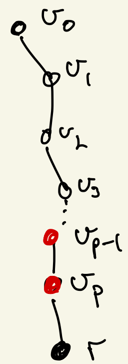


step 2 cops announce to put second cop at v_1 so $V^2=\{v_0,v_1\}$
robber must move to a vertex of $V(T_{v_1})-v_1$

Let v_1v_2 be first edge on P_{v_1r} in T

step 3 cops announce $V^3=\{v_1,v_2\}$ and robber must move to a vertex of $T_{v_2}-v_2$

Eventually we have robber at a leaf with a cop at its parent



Now cops announce to put the cop from v_{p-1} to the vertex r and the robber is caught †

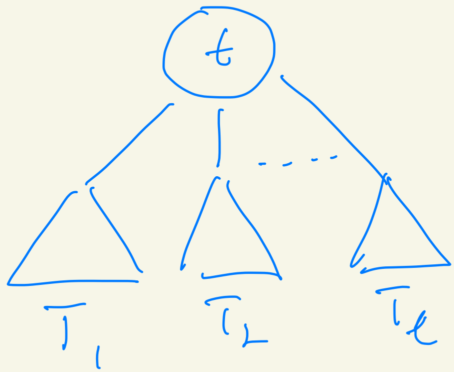
In fact we have:

<u>Lemma</u> if $tw(G) \le k$ then $k+1$ cops can win the game against a robber.

<u>Proof</u>

Let $(\{W_t \mid t \in I\}, T)$ be a t-d of width k

- pick a non-leaf t of T and announce to land $|W_t|$ cops at $V^0 = W_t$. (we have enough as $|W_t| \le k+1$

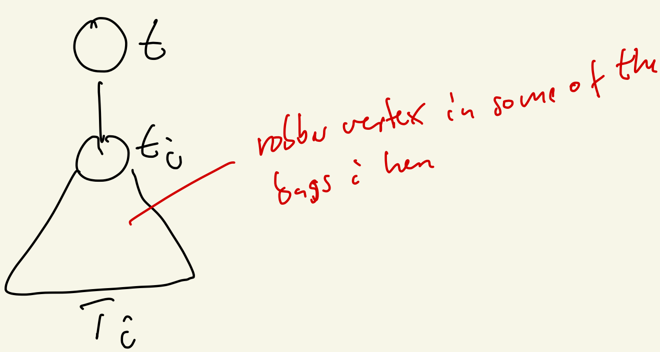


By Lemma W_t is a separator

Every connected component K of $G - W_t$ satisfies
$K \subseteq \bigcup_{t' \in V(T_i)} W_{t'}$ for some $i \in [\ell]$.

The robber moves to some connected component K_r of $G - W_t$

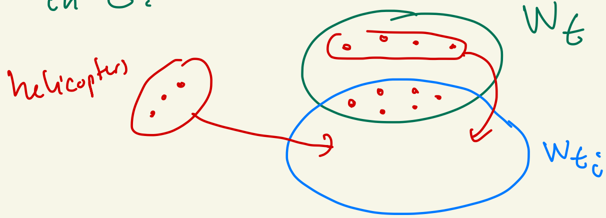
Let $i \in [\ell]$ be the unique index s.t $K_r \subseteq \bigcup_{t' \in V(T_i)} W_{t'}$



In the next step:

take all the cops in $W_t - W_{t_i}$ and annouce to place them and perhaps some cops not currently on W_t (they are in helicopters) on $W_{t_i} - W_t$ (ok since $|W_{t_i}| \leq k+1$)

in G:



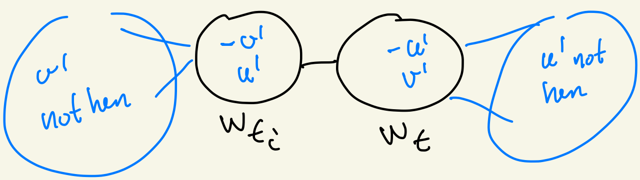
The robber cannot move out of K_r:

suppose there was a path from $u \in K_r - W_t$ to some $v \in V - W_t$ which does not intersect $W_t \cap W_{t_i}$.

Then there is an edge $u'v'$ in G with $u' \in W_{t_i} - W_t$ and $v' \in V - W_t$

However $u'v'$ is not in any bag by the consistency property:

$u' \notin W_t - W_{t_i}$ and $v' \notin W_{t_i} - W_t$



So the robber must move into a proper subtree of T_i and the cops are one step close to catching the robber.

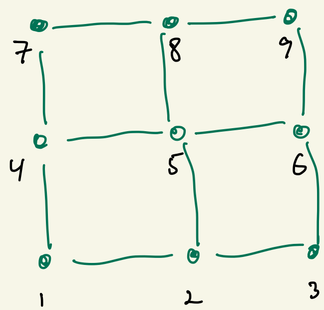
The $\ell\times\ell$-grid is the graph with vertices

$V = \{(i,j) \mid i,j \in [\ell]\}$ and edges

$$E = \{(i,j)(i,j+1) \mid 1 \le i \le \ell \wedge 1 \le j \le \ell-1\}$$
$$\cup\, \{(i,j)(i+1,j) \mid 1 \le i \le \ell-1 \wedge 1 \le j \le \ell\}$$

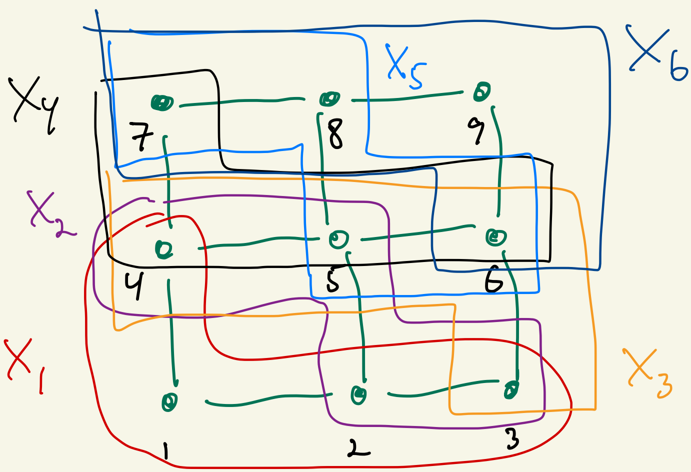
Example 3x3 grid

with different enumeration of V



Claim $tw(\ell\times\ell\text{-grid}) \le \ell$

Proof for 3x3-grid (generalizes easily)



$T =$ 1 — 2 — 3 — ... — 6 $(= \ell(\ell-1))$

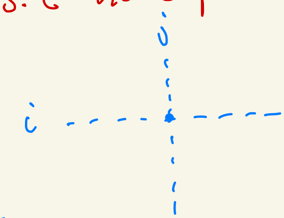
T is actually a path

lower bound via cops and robbers:

We prove that a robber can keep escaping $\ell-1$ cops on an $\ell\times\ell$-grid. This will imply that $tw(\ell\times\ell\text{-grid}) \geq \ell-1$.

In fact $tw(\ell\times\ell\text{-grid}) = \ell$ (more difficult)

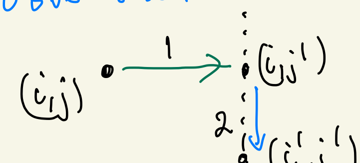
Observation: for every placement of $\ell-1$ cops on an $\ell\times\ell$-grid there exist $i,j \in [\ell]$ s.t. no cop in row i and no cop in column j



When cops announce their new positions the robber determines i and j and moves to vertex (i,j)

Now cops move and announce their next positions and robber find i', j' with no cop in row i' and no cop in column j'

The robber then moves to vertex (i',j') as follows



no cop in row i before move so $(i,j) \to (i,j')$ is free before cops move

No cop column j' after move so robber is free to move to (i',j')