Institut for Matematik og Datalogi
Syddansk Universitet

February 11, 2022
JBJ

DM867 - Spring 2022 – Weekly Note 3

Stuff covered in week 6

I gave an overview of Sections 3.1-3.5 in BJG. After this and watching the videos on flows, you should be ready to work with flows. I covered matching in bipartite graphs and showed how to find a maximum matching via flows. I then used the max flow min cut theorem to prove the theorems of Hall and König.

Lecture February 14, 2022:

- I will give a proof of Menger's Theorem based on flows and well as a proof using submodularity. See BJG 7.3.
- More on Matroids (circuits, rank function, dual matroid). PS 12.4 and SCH 10.1-10.2 and the notes below.
- Weighted bipartite matching. SCH 3.5 matching in a graph which is not bipartite.

Exercises February 17. 2022:

NB! There is a chance that this class will be replaced by a video lecture on max-back orderings. If that happens, we will discuss the exercises in week 8.

- SCH application 1.4.
- SCH application 1.7.
- Suppose you are given a connected undirected graph $G = (V, E)$ with costs on the edges and your task is to give an algorithm which finds a minimum cost set of $E' \subset E$ edges whose removal disconnects the graph (that is $G - E'$ is not connected). Explain how to do this in polynomial time (hint: use flows).
- SCH exercise 3.2. Hint for (i): you may either consider a maximal matching, apply Hall's theorem or use the integrality theorem for flows (as I did at the lecture in Week 6).
- Give an example of a graph G with $\nu(G) < \tau(G)$. Argue that for every graph G we have $\tau(G) \leq 2\nu(G)$. Suggest a polynomial algorithm for finding a vertex cover of size at most $2\tau(G)$ in a given graph G.

- Prove that if a graph is 2-connected (that is, there are at least two internally disjoint (s,t)-paths for every choice of distinct vertices $s, t \in V(G)$), then for every vertex s and edge uv of G there is a cycle C which contains s and the edge uv.
- Show that a graph G has a strongly connected orientation (we replace each edge uv by one of the arcs $u \to v, v \to u$) if and only if G is 2-edge-connected. Also describe an algorithm to find such an orientation or a bad cut.
- SCH Exercise 4.1.
- SCH application 4.1 be ready to discuss this in the class.
- Suppose you have a 8 by 8 chess board and dominos of size 1 by 2.
  (a) Show that you can cover the chess board by non-overlapping dominos.
  (b) Now suppose that we delete two diagonally opposite cornes of the chess board ($(1,1)$ and $(8,8)$). Show that the new chessboard cannot be covered by non-overlapping dominos. Hint: make a suitable bipartite graph and consider matchings in this.

Notes on matroids

Recall that a **base** of a matroid $M = (S, \mathcal{F})$ is a maximal independent set of $\mathcal{F}$.

Theorem 0.1 (Base axioms) *The set $\mathcal{B}$ bases of a matroid $M = (S, \mathcal{F})$ with $\mathcal{F} \neq \emptyset$ satisfy the following axioms:*

(B1) $\mathcal{B} \neq \emptyset$

(B2) $|B_1| = |B_2|$ *for all* $B_1, B_2 \in \mathcal{B}$.

(B3) If $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1$ *then there exists* $y \in B_2$ *such that* $B_1 - x + y \in \mathcal{B}$.

Proof: It is clear that the bases of M satisfy (B1) and (B2) and (B3) is a special case of the exchange axiom (consider $B_1 - x$ and B_2). $\diamond$
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The base axioms also define the set of all matroids of a set.

Proposition 0.2 *Let S be a set and let $\mathcal{B} \subseteq 2^S$ be a collection of subsets of S which satisfies (B1)-(B3). Define $\mathcal{F}_{\mathcal{B}} = \{X \subseteq S | \exists B \in \mathcal{B} : X \subseteq B\}$. Then $M_{\mathcal{B}} = (S, \mathcal{F}_{\mathcal{B}})$ is a matorid.*

Proof: Clearly $M_{\mathcal{B}}$ is s subset system so we just need to show that the exchange axiom holds for $\mathcal{F}_{\mathcal{B}}$. Let $X, Y \in \mathcal{F}_{\mathcal{B}}$ with $|Y| = |X| + 1$ and let B_X, B_Y be elements of $\mathcal{B}$ such that $X \subseteq B_X$ and $Y \subseteq B_Y$. Applying (B3) repeatedly we can delete the elements of $B_X - X$ one by one while adding a new element from $B_Y - B_X$ each time. Since $|B_X - X| = |B_Y - Y| + 1$ at some point in this process we have a base B'_X containing X such that the only element of $B_Y - B'_X$ that we can add to $B'_X - w$, $w \notin X$, is an element $y \in Y - X$. Now $B'_X - w + y$ contains $X + y$ so $X + y \in \mathcal{F}_{\mathcal{B}}$, showing that $Y - X$ contains an element y such that $X + y$ is independent. $\diamond$

Definition 0.3 (dual matroid) *Let $M = (S, \mathcal{F})$ be a matroid with base set $\mathcal{B}$ and rank $r(S) < |S|$. Define $\mathcal{F}^* = \{X | \exists B \in \mathcal{B} : X \cap B = \emptyset\}$. Then $M^* = (S, \mathcal{F}^*)$ is a matroid called the* **dual matroid** *of M.*

Proof: Let $\mathcal{B}^*$ be the set of bases of $\mathcal{F}^*$. We show that $\mathcal{B}^*$ satisfies the base axioms and then it follows from Proposition 0.2 that M^* is a matroid. By definition of $\mathcal{F}^*$, all maximal independent subsets of S have the same size and since $r(S) < |S|$ we have $\mathcal{B}^* \neq \emptyset$ so it only remains to prove that (B3) holds. Let $B_1^*, B_2^* \in \mathcal{B}^*$ and let $x \in B_1^* - B_2^*$ be arbitrary. Note that $(S - B_1^*) \cap (S - B_2^*) + x$ is a subset of $S - B_2^*$ and hence is independent in $\mathcal{F}$. Apply the exchange axiom (in M) to the independent sets $(S - B_1^*) \cap (S - B_2^*) + x$ and $S - B_1^*$ until we have a new base Z of M. This will satisfy $Z = (S - B_1^*) + x - z$ where $z \in (S - B_1^*) \cap B_2^* \subset B_2^*$ so we have shown that we can find $z \in B_2^*$ such that $B_1^* - x + z \in \mathcal{B}^*$. $\diamond$

Finding a negative cycle in a digraph

Theorem 0.4 *Let $D = (V, A)$ be a digraph with a special vertex s and let $w : A \to \mathbf{R}$ be a weightfunction. Let D_π be the successor digraph that we maintain while running the Bellmann-Ford algorithm. Then D_π will contain a cycle no later than iteration n of the algorithm if and only if D contains a negative cycle reachable from s.*

Proof: If D has a negative cycle C reachable from s, then it can be seen that D_π will contain a cycle no later than iteration k where k is the number of arcs on a shortest path from s to C plus the number of arcs in C. This is not a complete argument so you should try to make it more precise.

We prove the other direction below. Assume D_π is acyclic until iteration i and that a cycle C appears in iteration i. Consider the moment C appears and let $C = v_1 v_2 \dots v_k v_1$ where we have just added $v_k v_1$ to $A(D_\pi)$.

Note that at any time during the algorithm (and no matter whether D has a negative cycle or not) we always have $d(y) \geq d(x) + w(x, y)$ for every arc $xy \in A(D_\pi)$. This is because

$d(x)$ may have changed again but $d(y)$ has not.

Using that the arc $v_k v_1$ was just added we obtain

$$\begin{aligned}
d(v_1) &\geq d(v_k) + w(v_k, v_1) \\
&\geq d(v_{k-1}) + w(v_{k-1}, v_k) + w(v_k, v_1) \\
&\dots \\
&\geq d(v_1) + w(C),
\end{aligned}$$

implying that $w(C) < 0$. $\diamond$