Institut for Matematik og Datalogi — February 17, 2022
Syddansk Universitet — JBJ

DM867 – Spring 2022– Weekly Note 4

Stuff covered in Week 7

- Menger's Theorem (proofs via flows and submodularity)
- Schrijver section 3.5 on maximum weight bipartite matchings
- Basic stuff on matroids (circuits, dual matroid, base axioms)

Lectures in Week 8:

- Maxback orderings as described in the notes by Mette Eskesen (on the home page). You should watch the video lecture by midle of Week 8 so that you are ready to use max-back orderings in the exercises of this note.
- Non-bipartite Matching. SCH 5.1-5.2, see also PS Chapter 10.4-10.5. We will not cover this is full detail as almost all of you have seen it in MM856. It also means that the proof of the correctness of Edmonds blossom algorithm is not pensum.
- f-factors Here we want a spanning subgraph H where $d_H(v) = f(v)$ for all v where f is a prescribed function. So $f \equiv 1$ is the perfect matching problem. We will prove Tutte's 1-factor theorem from the structure obtained after running Edmonds's blossom algorithm.

Problems and applications to discuss Friday February 25, 2022:

- SCH application 4.1
- SCH exercise 3.18
- SCH exercise 3.23.
- Construct an example of a graph G which has two edge-disjoint spanning trees but no matter which max-back ordering you make of G you will not get two edge-disjoint spanning trees by taking the two first max-back forrests F_1, F_2. Hint: there is an example with 4 vertices.

- Show that for every graph $G = (V, E)$ and subsets $X, Y \subseteq V$ we have

$$d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \quad (1)$$
$$d(X) + d(Y) \geq d(X - Y) + d(Y - X) \quad (2)$$

- Show that if G is k-edge-connected and $d(X) = k$, then the subgraph $G[X] = (X, E(X))$ ($E(X)$ is the edges inside X) induced by X has $\lambda(G[X]) \geq \lfloor \frac{k}{2} \rfloor$.

- A graph $G = (V, E)$ is minimally k-edge-connected if $\lambda(G) = k$ but $\lambda(G - e) = k - 1$ for every $e \in E$.

  - Describe an efficient way to find, in a graph G with $\lambda(G) = k$ a spanning subgraph which is minimally k-edge-connected.
  - Show that every minimally k-edge-connected graph G has a vertex of degree k. Hint: consider a set X with $d(X) = k$ and $|X| \geq 2$, observe that it is connected if $k \geq 2$ and then use (1) on the graph obtained by deleting an edge inside X to obtain a smaller set with degree k is G.

- Let G be a tree. How many new edges must be added to G to make it 2-edge-connected? Try to construct an algorithm which adds as few edges as possible and try to formulate a min-max result.

- Let $G = (V, E)$ be a k-edge-connected graph and let H be a minimal set of new edges such that $G' = (V, E \cup H)$ is $(k+1)$-edge-connected. Prove that the edges of H form a forest.

- Use max-back orderings to show that every minimally k-edge-connected graph G contains a vertex of degree k (we proved this already but the goal now is to see that this follows from what you have learned about max-back orderings). Extend your argument to also show that in fact there are at least 2 vertices of degree 2. Give an example of a graph (multiple edges are allowed) which is minimaly k-edge-connected and has exactly 2 vertices of degree 2.

- Prove that every minimally k-edge-connected graph has at most $k(n-1)$ edges. Hint: recall what you learned about max-back forrests.