Institut for Matematik og Datalogi
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JBJ

DM867 – Fall 2022 – Weekly Note 6

Stuff covered in week 9

- Lovász's splitting theorem and augmenting the edge-connectivity of a graph. This is covered by notes at the end this weekly note.
- Arc-disjoint branchings. This is BJG Section 9.5. (Video lecture)

Classes in Week 10

- Nash-Williams orientation theorem. based on notes on this Note.
- Orientations with degree bounds. BJG Section 8.7 (we will cover pages 446 to 447 top as well as Theorem 8.7.3 and its proof).
- Finding subdigraphs with prescribed in- and out-degrees. BJG Section 3.11.3.
- Gomory-Hu trees. Based on Section 8.6 in Korte and Vygen, Combinatorial Optimization, Springer Verlag 2002.

Problems and applications to discuss on Thursday March 10

We discuss the remaining problems from Note 5 as well as the notes on Matroid connectivity below.

Notes on Nash-Williams Orientation Theorem

Recall the following consequence of Lovász's splitting theorem

Theorem 1 *Let $k \geq 2$ be an integer and let $G = (V + s, E)$ be a graph for which the degree of s is even, say $d(s) = 2p$ and we have*

$$\lambda(x, y) \geq k \text{ for every choice of } x, y \in V \tag{1}$$

Then we can find a pairing $(su_1, sv_1), \ldots (su_p, sv_p)$, $p = d(s)/2$ of the edges incident with s so that the graph $H = (V, E')$ that we obtain by deleting s and all its incident edges and adding the edges $u_1v_1, \ldots u_pv_p$ still satisfies (1).

A graph G is **minimally r-edge-connected** if it is r-edge-connected but $G - e$ is no longer r-edge-connected no matter which edge e we delete. In the exercises we have shown the following:

Lemma 1 *Every minimally r-edge-connected graph G has a vertex of degree r.*

Lemma 2 *Let $D = (V, A)$ be a p-arc-strong digraph and $u_1v_1, \ldots, u_pv_p$ any collection of p distinct arcs of D. Let $H = (V + w, A')$ be the digraph that we obtain from D by deleting the arcs $u_1v_1, \ldots, u_pv_p$, adding a new vertex w and the $2p$ arcs $u_1w, wv_1, \ldots, u_pw, v_pw$. Then H is p-arc-strong.*

Proof: We show that every set $\emptyset \neq X \neq V + w$ has out-degree at least p in H. For each arc u_iv_i that goes from X to $V - X$ in D, either the arc u_is or the arc s_iv_i will go from X to $V - X$ in H. Thus the out-degree of X is the same in D and H for all $X \subset V$ and for the set V, the out-degree is exactly p in H so H is p-arc-strong. $\square$

An **orientation** of a graph $G = (V, E)$ is a digraph $D = (V, A)$ that we can obtain from G by assigning each edge uv one of the two possible directions $u \to v$ or $v \to u$.

Now we are ready to prove Nash-Williams' orientation theorem

Theorem 2 (Nash-Williams) *Let $k \geq 1$ be an integer and $G = (V, E)$ a $2k$-edge-connected graph. Then G has an orientation $D = (V, A)$ which is k-arc-strong*

Proof: We prove the claim by induction on $n = |V|$. If $n = 2$ we have $V = \{u, v\}$ and there are at least $2k$ edges between u and v so we can just orient k of them from u to v and the remaining from v to u, which clearly gives a k-arc-strong orientation. Suppose now that the theorem holds for all $2k$-edge-connected graphs on at most $n-1$ vertices.

Delete a set $\tilde{E}$ of zero or more edges from G so that the resulting graph $G' = (V.E')$ is minimally $2k$-edge-connected. By Lemma 1 there is a vertex s of degree exactly $2k$ in H'. Now apply Theorem 1 to get a $2k$-edge-connected graph H'' on $n - 1$ vertices where we have replaced s and its $2k$ incident edges by k edges $x_1y_1, x_2, y_2, \ldots, x_ky_k$. By induction the graph H'' has a k-arc-strong orientation D''. By switching the names of x_i, y_i if necessary, we can assume that in D'' the edges $x_1y_1, \ldots, x_ky_k$ are all oriented as $x_1 \to y_1, x_2 \to y_2, \ldots, x_k \to y_k$. Next we let D' be obtained from D'' be replacing these arcs by the $2k$ arcs $x_1 \to s, s \to y_1, \ldots x_k \to s, s \to y_k$. By Lemma 2, D' is k-arc-strong. Finally we obtain the desired k-arc-strong orientation of G by orienting the edges in $\tilde{E}$ arbitrarily. $\square$

Corollary 1 *There exists a polynomial algorithm for finding a k-arc-strong orientation of a given $2k$-edge-connected graph.*

Proof: This follows from the proof above and the following remarks:

- We can test whether $G - uv$ is $2k$-edge-connected by checking (via flows) whether every set containing u but not v has degree at least $2k + 1$. Hence we can find $\tilde{E}$ by doing this check for each edge in G and deleting the currently considered edge uv if and only if every set containing u but not v has degree at least $2k + 1$. Note that after deleting some edges a remaining edge may become undeletable, even though it would have been OK to delete if we only wnated to delete that edge.

- We proved on Weekly note 5, that we have a polynomial algorithm for finding a pairing $(sx_1, sy_1), \ldots (sx_k, sy_k)$ of the $2k$ edges incident with s so that the graph H'' that we obtain by deleting s and all its incident edges and adding the edges $x_1y_1, \ldots, x_ky_k$ still satisfies (1).

Note that the algorithm will be recursive, since we repeat the steps above until we reach a graph on just 2 vertices which we orient and then expand the orientation as we go back in the recursion, while always lifting k arcs from the previous graph back to the vertex s from which we performed the last splitting and then adding zero or more edges arbitrarily oriented (they were the ones we called $\tilde{E}$).

1 Notes on matroids

Recall that a **circuit** of a matroid $M = (S, \mathcal{I})$ is a minimal **dependent set** $C \subseteq S$, that is, $C \notin \mathcal{I}$ but $C - x \in \mathcal{I}$ for every $x \in C$.

Theorem 3 *Let $M = (S, \mathcal{I})$ be a matroid. The set $\mathcal{C}$ of circuits of M satisfies the following properties*

(C1) If $X, Y \in \mathcal{C}$ and $X \subseteq Y$ then $X = Y$ (no proper subset of a circuit is a circuit).

(C2) If $X, Y \in \mathcal{C}$, $X \neq Y$ and $u \in X \cap Y$, then there exist a circuit $Z \in C$ such that $Z \subseteq X \cup Y - u$.

(C3) If $X, Y \in \mathcal{C}$, $X \neq Y$, $x \in X - Y$ and $u \in X \cap Y$, then there exist a circuit $Z \in C$ such that $Z \subseteq X \cup Y - u$ and $x \in Z$.

(C4) If $X, Y \in \mathcal{C}$, $X \cap Y \neq \emptyset$, $X \neq Y$, $x \in X - Y$ and $y \in Y - X$, then there exist a circuit $Z \in C$ such that $Z \subseteq X \cup Y$ and $x, y \in Z$.

Proof: The first claim is immediate from the definition of a circuit.

To Prove (C2) suppose that there is no circuit inside $X \cap Y - u$ which means that this is an independent set. Then, using that M satisfies the exchange axiom, starting from

$W = \{u\}$ we can add elements from $X \cap Y - u$ to the current set W until we end with an independent set W' of the same size as $X \cup Y - u$, but this means that either X or Y is fully contain in W', contradicting that M is a subset system (every subset of an independent set is independent).

Suppose that (C3) does not hold and let X, Y a counterexample such that $|X \cup Y|$ is minimum among all counterexamples. Let $x \in X$ and $u \in X \cap Y$ be such that there is no circuit containing x in $X \cup Y - u$. By (C2) there exists a circuit $Z_1 \subseteq X \cup Y - u$. By the assumption above we have $|Z_1 \cup Y| < |X \cup Y|$, since $x \notin Z_1$ so Z_1, Y is not a counterexample. This means that there exists a circuit $Z_2 \subset Z_1 \cup Y - y$ containing u, where y is in $Z_1 \cap Y - X$ (there must be such an element as Z_1 cannot be properly contained in X. Now we have $|Z_2 \cup X| < |X \cup Y|$ so Z_2, X is not a counterexample. Consequently there exists a circuit $Z_3 \subseteq X \cup Z_2 - u$ with $x \in Z_3$, contradicting that X, Y is a counterexample since $Z_3 \subseteq X \cup Y$. This proves (C3).

Finally suppose that (C4) does not hold and let X, Y be a counterexample such that $|X \cup Y|$ is minimum among all counterexamples and let $x \in X - Y, y \in Y - X$ be such that there is no circuit in $X \cup Y$ that contains both x and y. Let $z \in X \cap Y$, then by (C3) there is are circuits $Z_1, Z_2 \subset X \cap Y - z$ with $y \in Z_1$ and $x \in Z_2$. Note that $Y - z \subset Z_1$ and $X - z \subset Z_2$ must hold, since otherwise we get a contradiction by considering X, Z_1 or Y, Z_2. This implies that $Z_1 \cap Z_2 \neq \emptyset$. However, we have $|Z_1 \cup Z_2| < |X \cup Y|$ (since $z \notin Z_1 \cup Z_2$) so by the minimality of $X \cup Y$ there exists a circuit $Z \subseteq Z_1 \cup Z_2 \subset X \cup Y$ which contains both x and y, contradicting that X, Y is a counterexample. □

A matroid $M = (S, \mathcal{I})$ is **connected** if the following holds for every partition $U, S - U$ into two non-empty sets: there exists a circuit $C \in$ such that $C \cap U$ and $C \cap S - U$ are both non-empty.

Theorem 4 *A matroid $M = (S, \mathcal{I})$ is connected if and only if every pair of elements $x, y \in S$ lie in some circuit $C \in \mathcal{C}$ of M.*

Proof: If every pair of elements of S is in a circuit, then clearly M is connected (just take $x \in U$ and $y \in S - U$ arbitrary and let C be a circuits containing x, y. Suppose now that M is connected but there is some pair of elements $x, y \in S$ so that they are in no circuit together. Let X be the union of all circuits containing x. Then $X, S - X$ is a partition of S with $y \in S - X$. Since M is connected there is a circuit C with intersects both X and $S - X$. Let $a \in C \cap X$, $b \in C \cap S - X$ and let C_a be a circuit that contains x and a. Then $a \in C_a \cap C$, $x \in C_a - C$ and $b \in C - C_a$ so by (C4) there is a circuit C' containing x and b, contradicting the definition of X. □