Institut for Matematik og Datalogi
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JBJ

DM867 – Spring 2022 – Weekly Note 10

Second set of exam problems
These are now posted on itslearning The solutions must be handed in by May 2nd at 9.00 a.m.

Stuff covered in Week 13, 2022

- The Steiner tree problem. We use Chapter 42 in the notes by Khuller on the home page
- The k-path problem for directed graphs. BJG Sections 9.1-9.2. In particular I proved that for fixed k, the k-path problem can be solved in polynomial time for acyclic digraphs
- The proof that the 2-path problem is NP-complete for digraphs. BJG Section 9.2.
- The k-path problem for undirected graphs. I said some words about this. The most important thing is that the k-path problem is polynomially solvable for undirected graphs for any fixed k. Robertson and Seymour proved in a series of papers that there is an algorithm for the k-path problem with a running time which is $O(f(k)n^3)$. Here $f(k)$ is a very fast growing function of k, but when k is fixed f(k) is a constant so the running time is $O(n^3)$. This implies that we can solve problems where we look for a subdivision of a given graph H in another graph G in polynomial time. I also illustrated how to use the polynomial algorithm for the k-path problem in acyclic digraphs to solve a similar problem for acyclic digraphs. There are notes about both problems at the bottom of this note.

Classes in Weeks 14-16
There are no classes in weeks 14 and 15.

- The intersection problem for 3 or more matroids is NP-complete. PS 12.6.3
- Weighted Matroid intersection. PS 12.6.1. We will just mention that this problem can be solved in polynomial time for two matroids and give an application to minimum cost out-branchings.
- Chordal graphs (originally called triangulated graphs). These are graphs with no induced cycle of length more than 3. They will play an important role in the study of tree-width. The presentation is based on Chapter 4 in the book 'Algorithmic Graph Theory and Perfect Graphs by M.C. Golumbic. This chapter is available from the home page.

- We will define tree-width and tree-decompositions of graphs. These constitute a very important tool to obtain efficient algorithms for classes of graphs with low tree-width. This is based on Chapter 10 in the book:"Invitation to fixed parameter algorithms" by R. Niedermeier, Oxford 2006. This is available from the home page. We will continue on tree-width in Week 17.

1 Notes on finding subdivisions for (di)graphs in (acyclic di)graphs

Theorem 1 (Robertson and Seymour, 1995) *For every fixed natural number k there is an algorithm of complexity*[1] $O(n^3)$ *for deciding for a given input graph G and distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G whether G has vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is a (s_i, t_i)-path.*

A **subdivision** of a graph $H = (V_H, E_H)$ in a graph $G = (V, E)$ is a subgraph $G' = (V', E')$ of G and a mapping of H to G' with the property that its is 1-1 on the vertices of H and every edge $e = uv \in E_H$ is mapped to a path P_{uv} from $f(u)$ to $f(v)$ such that every vertex of $P_{uv} - \{f(u), f(v)\}$ has degree 2 in G' (we replace the edge uv by a path in G' and no two paths corresponding to different edges of H intersect except possibly at their ends). This definition also makes sense if H has loops as such a loop at u corresponds to a cycle through $f(u)$ in G'. A subdivision of a digraph is defined analogously.

Corollary 1 *For every graph $H = (V_H, E_H)$ there exists a polynomial algorithm $\mathcal{A}_H$ which for a given input graph $G = (V, E)$ decides whether G contains a subdivision of H.*

Proof: Let $H = (V_H, E_H)$ be given and assume first that we have fixed a 1-1 mapping $f : V_H \to V$. If there is an edge $uv \in E_H$ such that $f(u)f(v)$ is an edge in G (possibly $u = v$ and then $f(u)f(u)$ is a loop in G), then we can use this edge to realize the path corresponding to the edge uv and consider H minus this edge and G minus the edge $f(u)f(v)$. Hence we can first trim off (select) such pairs and then assume that $f(V_H)$ (the set of images of V_H) is an independent set in G.

Fix an ordering of the edges around each vertex in H: if u has k neighbours then we label these $v_{u,1}, v_{u,2}, \ldots, v_{u,k}$ (notice that the same vertex gets many different labels, one for each of its neighbours in V_H). Clearly for a given edge $e = uv \in E_H$ this gives two labels l_{uv} and l_{vu} (the number it has in u's labelling and in v's labelling). Now consider the graph G_H that we obtain from G by replacing each vertex $f(u)$ by $d_H(u)$ copies, that is, replace $f(u)$ by an independet set $F(u) = \{f(u)^1, f(u)^2, \ldots f(u)^{d_H(u)}\}$ on $d_H(u)$ vertices and join each of these to all neighbours of $f(u)$ in G.

[1] The constant here depends heavily on k: the complixity is $O(f(k)n^3)$ where $f(k)$ grows VERY fast in k.

We claim that now G contains a H-subdivision G' where the vertices of H are $\{f(u)|u \in V_H\}$ if and only if G_H contains a collection of disjoint paths $\{P_{uv}|uv \in E_H\}$ where P_{uv} starts in $f(u)^{l_{uv}}$ and ends in $f(v)^{l_{vu}}$. This is easy to see: if the paths exist in G_H then we obtain G' by contracting (identifying) each set $F(u)$ to the single vertex $f(u)$. Conversely, if we are given a subdivision G' of H then we obtain the paths by splitting up each $f(u)$ into $d_H(u)$ distinct vertices. Thus it follows from Theorem 1 that for a fixed 1-1 mapping of $V(H)$ to $V(G)$ we can decide in time $O(n^3)$ whether this mapping extends to a subdivision of H in G. Thus, in polynomial time, we can check for all the $\binom{|V(G)|}{|V(H)|}$ 1-1 mappings of $V(H)$ to $V(G)$ to see whether at least one extends to a homeomorphism of H to G in polynomial time (H is fixed so its size is a constant). □

Theorem 2 (Fortune, Hopcroft and Wyllie, 1980) *For any fixed natural number k there exists a polynomial algorithm for deciding whether a given acyclic digraph $D = (V, A)$ with specified vertices $s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k$ has vertex-disjoint paths $P_1, P_2, \dots, P_k$ such that P_i is a (s_i, t_i)-path.*

Corollary 2 *For every acyclic digraph $H = (V_H, A_H)$ there exists a polynomial algorithm for deciding whether a given acyclic digraph $D = (V, A)$ contains a subdivision of H.*

Proof: As above it is sufficient to show that we can decide in polynomial time whether a fixed 1-1 mapping of $V(H)$ to $V(D)$ extends to a homoemorphism of H to D so we assume below that a 1-1 mapping of $V(H)$ to $V(G)$ is given.

As above we may assume that the vertices of H are mapped to an independent set in D (if $f(u)f(v)$ is an arc and $uv \in A_H$ then use $f(u)f(v)$ to realize that path and delete the arc uv from A_H. If $f(u)f(v)$ is an arc of D and uv is not and arc of A_H, then we can never use the arc $f(u)f(v)$ in a homeomorphism (because paths must be internally disjoint) and hence we can delete the arc $f(u)f(v)$ from D without changing the problem. Finally if uv is an arc of H and $f(v)f(u)$ is an arc of D, then there cannot exist a solution for the given mapping f as this would imply that D contained a cycle.).

For each vertex $u \in V_H$ fix and ordering of the arcs entering u and an ordering of the arcs leaving u: We label the $d^-_H(u)$ in-neighbours of u $v^-_{u,1}, v^-_{u,2}, \dots, v^-_{u,d^-_H(u)}$ and we label the $d^+_H(u)$ out-neighbours of u by $v^+_{u,1}, v^+_{u,2}, \dots, v^+_{u,d^+_H(u)}$. As in the proof above, for a given arc $e = uv \in A_H$ this gives two labels l^+_{uv} and l^-_{uv} (the number it has in u's out-labelling and in v's in-labelling). Given the 1-1 mapping $f : V_H \to V(G)$ we make a new acyclic digraph G_H by replacing each vertex $f(u)$, $u \in V_H$ by two sets $I_{f(u)} = \{v^-_{u,1}, v^-_{u,2}, \dots, v^-_{u,d^-_H(u)}\}$ and $O_{f(u)} = \{v^+_{u,1}, v^+_{u,2}, \dots, v^+_{u,d^+_H(u)}\}$ and joing every in-neighbour x of $f(v)$ in G to every vertex y in $I_{f(v)}$ by an arc $x \to y$ and every vertex p of $O_{f(v)}$ to every out-neighbour q of $f(v)$ in G (it is possible that one of the sets $I_{f(v)}, O_{f(v)}$ is empty in which case we add no arcs corresponding to that set).

Now it is easy to show that D contains a subdivision of H if and only if D_H contains vertex disjoint paths $\{P_{uv}|uv \in A_H\}$ where P_{uv} starts in l^+_{uv} and ends in l^-_{uv}. □