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April 22, 2022
JBJ

DM867 – Spring 2022 – Weekly Note 11

Stuff covered in week 16, 2022
I showed how to solve the first set of exam problems. See the note below for an argument that I could not produce on the fly Thursday.

Classes in Week 17, 2022:

- Chordal graphs (originally called triangulated graphs). These are graph with no induced cycle of length more than 3. They will play an important role in the last part of the course. The presentation is based on Chapter 4 in the book 'Algorithmic Graph Theory and Perfect Graphs by M.C. Golumbic. This chapter is available from the home page.
- We show how to recognize chordal graphs using Lexicographic BFS and discuss nice algorithmic properties of chordal graphs as well as prove that a graph is chordal if and only if it is the interesection graph of subtrees of some tree.
- We will introduce tree-width and tree-decompositions of graphs. These constitute a very important tool to obtain efficient algorithms for classes of graphs with low tree-width. This is based on Chapter 10 in the book:"Invitation to fixed paramerter algorithms" by R. Niedermeier, Oxford 2006. This is available from the home page.

Comment on Problem 6B question b)

Let us recall the situation from Problem 6B b): we have a balanced k-arc-strong orientation D' want to orient the previously deleted edge st and add it to D' so that the new digraph is still balanced. The only difficult situation is if $d^+_{D'}(s) - d^-_{D'}(s) = d^+(t) - d^-(t)$ and without loss of generality (by reversing all arcs if necessary) we can assume that $d^+_{D'}(s) - d^-_{D'}(s) = d^+(t) - d^-(t) = 1$. Let

$$X^+ = \{v \in V(D')|d^+(v) - d^-(v) = 1\} \text{ and } X^- = \{v \in V(D')|d^-(v) - d^+(v) = 1\}$$

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Since we count the number of arcs by summing the out-degrees and also by summing the in-degrees, we see that $|X^-| = |X^+| = r$ for some $r \geq 2$ $(s, t \in X^+)$. Let $X^- = \{z_1, z_2, \ldots, z_r\}$. Then our goal is to find a path P starting at s and ending in some vertex z_i such that we can reverse this path without destroying k-arc-connectivity. If we have P we can reverse it and orient the edge st from s to t.

By our previous observation in question a) we know that if there is no good path P from s to X^-, then for each $i \in [r]$ there exists a set $Y_i \subseteq V - s$ with $z_i \in Y_i$ and $d^-(Y_i) = k$. Let us call a set $X \subseteq V - s$ with $d^-(X) = k$ **critical**. By submodularity, if two critical sets X, X' intersect then each of $X \cap X'$ and $X \cup X'$ are critical. Now let $\mathcal{F}$ be a family of critical sets covering X^- such that $|\mathcal{F}|$ is minimum. Then the sets in $\mathcal{F}$ are disjoint (we just showed that the union of two intersecting critical sets is critical). We claim that $|\mathcal{F}| = 1$ in which case the argument I gave in class works.

Consider an arbitrary critical set X. We claim that $|X \cap X^+| \geq |X \cap X^-|$. Suppose not, then with $A(X)$ denoting the set of arcs inside X we get (all degrees are in D':

$$\begin{aligned} d^+(X) &= [\sum_{v \in X} d^+(v)] - |A(X)| \\ &= [\sum_{v \in X} d^+(v)] - ([\sum_{v \in X} d^-(v)] - d^-(X)) \\ &= [\sum_{v \in X} (d^+(v) - d^-(v))] + d^-(X) \\ &= [\sum_{v \in X} (d^+(v) - d^-(v))] + k \\ &< k \end{aligned}$$

This contradicts that D' is k-arc-strong.

Applying this to $\mathcal{F}$ we conclude that $|\mathcal{F}| = 1$: If there are 2 or more sets in $\mathcal{F}$ then there must be an intersecting pair as the sets in $\mathcal{F}$ cover all r vertices in X^- but at most $r - 1$ vertices of X^+. Hence if the sets in $\mathcal{F}$ were all pairwise disjoint, at least one of the sets $X \in \mathcal{F}$ would have $|X \cap X^+| < |X \cap X^-|$, contradicting what we found above.

So $\mathcal{F}$ consists of only one critical set Y and now we get a contradiction again as $|Y \cap X^+| < |Y \cap X^-|$.

Preliminary(!) list of exam questions

In the parantheses I give examples of things you can talk about, but there are many other possible choices.

1. **Edge-connectivity of graphs** (Menger's theorem, max back orderings and determining the edge-connectivity via these, Gomory Hu trees, edge-connectivity augmentation (both the general case via splitting off methods and augmention a tree to a 2-edge-connected graph)).

2. **Matchings** (Tutte's 1-factor theorem, f-factors, theorems by König and Hall and application of these e.g. in kernelization, weighted bipartite matching).

3. **Lovasz's splitting theorem with applications** (edge-connectivity augmentation, using the theorem to prove Nash-Williams orientation theorem).

4. **Arc-disjoint branchings** (Edmonds' branching theorem and the algorithmic proof of this, relation to edge-disjoint spanning trees (problem 6A in first set of exam problems), proving that every k-arc-strong D is weakly-k-linked).

5. **(Arc-)Disjoint paths in digraphs** (Proof that the 2-path problem is NPC, k-path problem for acyclic digraphs).

6. **Matroid intersection and partition** (Edmonds' algorithm for finding a maximum size common independent set of two matroids, correctness of the algorithm, matroid partition, applications of matroid intersection and partition, such as finding edge-disjoint spanning trees).

7. **Chordal graphs** (recognition via LexBFS, perfect elimination orderings, representation as intersection graphs, polynomial algorithms for clique, chromatic number etc).

8. **Tree-width** (definitions and properties of tree-decomposotions, cops and robber game and tree-width, relation to chordal graphs, using tree-decompositions to solve vertex cover and chromatic number).

9. **Fixed parameter tractability and kernels** (what is fixed parameter tractability?, kernel for feedback vertex set problem, kernel for the MSSS problem (if we cover it in the course)).