Institut for Matematik og Datalogi
Syddansk Universitet

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JBJ

DM867 – Spring 2022– Weekly Note 13

Stuff covered in week 18

- I introduced tree-width and tree-decompositions of graphs. These constitute a very important tool to obtain efficient algorithms for classes of graphs with low tree-width. This is based on Chapter 10 in the book:"Invitation to fixed paramerter algorithms" by R. Niedermeier, Oxford 2006. This is available from the home page. We covered (parts of) Sections 10.1, 10.2 and 10.4. Only the things discussed on my slides are pensum.
- I also showed how to relate the game of cops and robbers to tree-width of graphs. There is no litterature for this, except my lecture notes.
- I showed how to use dynamic programming via a tree-decomposition $(\{X_i|i \in I\}, T)$ of a graph G to find a minimum vertex cover of G in time $O(2^\omega \omega |I|)$, where ω is the size of the largest bag in the tree-decomposition. Another example of the use of tree-decompositions is given in the notes below.

Plan for Week 19 (last teaching)

- I will lecture on the color-coding method which allows one to check for a cycle of length $\log n$ in a graph on n vertices in polynomial time. This is based on the paper by Alon et all that you can access via the top link under supplementary notes.
- We will talk about the exam.

Finding the chromatic number of a graph G by dynamic programming based on a tree-decomposition of G:

- I gave an argument that for every graph G we have $\chi(G) \leq tw(G)+1$, where $\chi(G)$ is the chromatic number of G and $tw(G)$ is the tree-width of G, that is, $\beta-1$ where β is the maximum bag size of some tree decomposition of G. The proof uses that we have $\chi(G) = tw(G)+1$ for chordal graphs: Given a tree-decomposition $(\{X_i : i \in I\}, I)$ of G we add new edges E' to G so that in the resulting graph G' each of the subgraphs $G'[X_i]$, $i \in I$ are cliques. Clearly $\chi(G) \leq \chi(G')$ and the claim now follows from the fact that G' is a chordal graph whose maximal cliques are exactly those induced by the X_i's.

- Here is a sketch of a dynamic programming algorithm for finding $\chi(G)$ when we are given a tree-decomposition $(\{X_i : i \in I\}, T)$ of G: Let ω denote the size of a largest bag (X_i) and consider all possible colourings of the X_i's by colours $1, 2, \ldots, \omega$ (there are $|X_i|^\omega$ of these). For each such colouring $C_i : X_i \to \{1, 2, \ldots, \omega\}$ we initialize $m(C_i)$ as ∞ if C_i is not a legal colouring (some edge in $G[X_i]$ received the same colour in both ends) and otherwise $m(C_i)$ is $\beta(C_i)$ which is the number of different colours used. Furthermore, we also keep a bit-vector $\gamma(C_i)$ which codes which of the ω colours are used in the colouring C_i (so $\beta(C_i)$ equals the number of 1's in $\gamma(C_i)$).

  After this initialization we are ready to start updating the value $\gamma(C_i)$ and hence $\beta(C_i)$ and $m(C_i)$ using dynamic programming guided by the tree T: When we update the info for X_i from the info for a child X_j we first indentify the set $Z = X_i \cap X_j$ and then, for all of the $|Z|^\omega$ different colourings of $|Z|$ in turn: if C is such a colouring then for every proper colouring C_i of X_i which agrees (that is, uses exactly the same colours on Z as C) we update as follows: For every colouring C_j of X_j which agrees with C and which is a legal colouring of X_j consider the number of 1's in the OR-sum of the bitvectors $\gamma(C_i)$ and $\gamma(C_j)$ and make the new $\gamma(C_i) := \gamma(C_i)\ OR\ \gamma(C_{j'})$, where j' is chosen such that the total number of used colours (bits that are 1) in C_i and $C_{j'}$ is minimum.

  We preform this updating for all children of X_i and continue around the tree in an in-order traversal of T. It can be shown that this will result in the root bag X_r containing a colouring $C_r(X_r)$ whoose value $m(C_r) = \chi(G)$.

  Note that the process above considers the "same" colouring MANY times because a lot of the colourings in the X_i's are identical up to a renumbering of the colours.