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A POLYNOMIAL ALGORITHM FOR THE 2-PATH PROBLEM FOR SEMICOMPLETE DIGRAPHS*

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Abstract. This paper presents polynomially bounded algorithms for finding a cycle through any two prescribed arcs in a semicomplete digraph and for finding a cycle through any two prescribed vertices in a complete k-partite oriented graph. It is also shown that the problem of finding a maximum transitive subtournament of a tournament and the problem of finding a cycle through a prescribed arc set in a tournament are both NP-complete.

Key words. tournaments, semicomplete digraphs, k-partite tournaments, polynomial algorithms, NP-completeness, feedback vertex set, feedback arcset, 2-path problem

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1. Introduction. For general digraphs, it is easy to see, using standard transformations, that the following three problems are equivalent from an algorithmic point of view: (i) Given four distinct vertices u_1, u_2, v_1, v_2 in a digraph D, decide whether D has disjoint paths connecting u_1 to v_1 and u_2 to v_2; (ii) Given two arcs e_1, e_2 in a digraph D, decide whether D has a cycle through e_1 and e_2; and (iii) given two vertices u and v in a digraph D, decide whether D has a cycle through u and v.

The problem of finding a cycle through two prescribed vertices or arcs in a digraph is NP-complete, as shown by Fortune, Hopcroft, and Wyllie [5]. Hence all three of the above problems are NP-complete for general digraphs. However, if we restrict ourselves to a certain class of digraphs, then the complexity (as a measure of difficulty) of these problems can vary considerably. For example, the third problem is trivial for tournaments, while the other two, as we see below, are not quite trivial, even though they prove to be polynomially decidable. Note also that, for semicomplete digraphs (that is, digraphs with no two nonadjacent vertices) the first two problems are equivalent from an algorithmic point of view.

If $e_1 = y_1x_2$ and $e_2 = y_2x_1$ are given arcs in a digraph D, then a necessary condition for D to have a cycle through e_1 and e_2 is that for each vertex z of D, $D - z$ has a path from x_i to y_i for $i = 1$ or 2. This condition is also sufficient when D has no two disjoint cycles, as shown in [11]. In general, however, it is far from sufficient even for tournaments. Indeed, there are examples of 2-connected tournaments and of 4-connected semicomplete digraphs that contain two independent arcs not contained in a cycle [3]. Thus there seems to be no simple structural characterization of the semicomplete digraphs with no cycle through two given arcs. However, we conjecture that there exists a polynomially bounded algorithm for the k-path problem in semicomplete digraphs. The *k-path problem* is the following: Given distinct vertices $u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_k$ in a digraph D, decide whether D has k disjoint paths $P_1, P_2, \cdots, P_k$ such that P_i is a (u_i, v_i)-path for $i = 1, \cdots, k$. We verify this here for $k = 2$. In the last section, we show that, if k is not fixed, then the problem is NP-complete.

In [5] it is shown that the k-path problem is in P for acyclic digraphs. In [10] the 2-path problem is completely solved in the case of acyclic digraphs. We also present a polynomially bounded algorithm for finding a cycle through two given vertices in

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a digraph that have the same neighbours. This shows that the third of the above problems has an easy solution in terms of a polynomial algorithm, for complete k-partite digraphs.

2. Terminology and preliminaries. Most of the notation is the same as in [2], [9], but, for completeness, we repeat most of it here, except for the very standard notation, for which we refer to [4].

A *digraph* D consists of a pair $V(D), E(D)$, where $V(D)$ is a finite set of *vertices* and $E(D)$ is a set of ordered pairs xy of vertices called *arcs.* In our definition of a digraph, we do not allow multiple arcs in the same direction between two vertices. An *oriented* graph is a digraph with no cycle of length 2. A *semicomplete* digraph is a digraph with no nonadjacent vertices. A *tournament* is an oriented graph with no nonadjacent vertices. Thus tournaments are a special subclass of the semicomplete digraphs.

If there is an arc from x to y in the digraph D, then we say that x *dominates* y, and we use the notation $x \rightarrow y$ to denote this. We also sometimes denote the arc xy by the symbol $x \rightarrow y$. For any subset A of $V(D) \cup E(D)$, $D - A$ denotes the subgraph obtained by deleting all vertices of A and their incident arcs and then deleting the arcs of A still present. We write $D - x$ instead of $D - \{x\}$ when $x \in V(D) \cup E(D)$. For a given vertex x of a digraph D, $d^+(x)$ (respectively, $d^-(x)$) denotes the number of vertices dominated by x in D (respectively, dominating x in D). We also call $d^+(x)$ (respectively, $d^-(x)$) the *outdegree* (respectively, the *indegree*) of x.

The subgraph *induced* by a vertex set A of D is defined as $D - (V(D) \setminus A)$ and is denoted by $D\langle A\rangle$. We often write $x \in D$ instead of $x \in V(D)$ or $x \in E(D)$, but the meaning is always clear. A *path* is a digraph with vertex set $x_1, x_2, \cdots, x_n$ and arc set $x_1 \rightarrow x_2, x_2 \rightarrow x_3, \cdots, x_{n-1} \rightarrow x_n$ such that all the vertices and arcs shown are distinct. We call such a path an (x_1, x_n)-path and denote it by $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n$. If P is a path containing a subpath from x to y, then we let $P[x, y]$ denote the part of P from x to y. A *cycle* is defined analogously. A *k-cycle* is a cycle of length k.

A *component* D' of a digraph D is a maximal subdigraph, such that, for any two vertices $x, y \in D'$, D' contains an (x, y)-path and a (y, x)-path. A digraph D is *strong* if it has only one component. D is *k-connected* if, for any set A of at most $k - 1$ vertices, $D - A$ is strong.

The *local connectivity* from x to y in a digraph D is the maximum number of internally disjoint paths in D from x to y.

Let x and y be vertices of a digraph D such that there is no arc from x to y. An *(x, y)-separator of size k* is a set S of k vertices of $D - \{x, y\}$ that separates x from y in D; that is, there is no (x, y)-path in $D - S$. We also sometimes call S a *k-separator* of x and y. A k-separator S of x and y is called *trivial* if either x has outdegree zero or y has indegree zero in $D - S$.

The following theorem gives a sufficient condition, in terms of local connectivities, for the existence of disjoint (x_1, y_1)-, (x_2, y_2)-paths in a semicomplete digraph T.

THEOREM 2.1 (see [3]). *Let T be a semicomplete digraph and let x_1, x_2, y_1, y_2 be distinct vertices of T. If $T - \{x_i, y_i\}$ has three internally disjoint (x_{3-i}, y_{3-i})-paths and $T - \{x_{3-i}, y_{3-i}\}$ has two internally disjoint (x_i, y_i)-paths, for $i = 1$ or 2, then T has a pair of disjoint (x_1, y_1)-, (x_2, y_2)-paths.*

In [3] this was shown to be the best possible in the sense that "three" cannot be replaced by "two," and "two" cannot be replaced by "one."

3. Cycles through two vertices with the same neighbours. Before we describe the main algorithm, we point out that in some special cases it is easy to decide if a digraph D has a cycle through two prescribed vertices x, y. If $x \rightarrow y$, then we just check if D has a (y, x)-path. The next result deals with the case where x and y are nonadjacent, but have the same neighbours.

THEOREM 3.1. *There exists a polynomially bounded algorithm for the following problem: Given two nonadjacent vertices x, y in a digraph D such that a vertex z in D is adjacent to x if and only if z is adjacent to y, find a cycle through x and y or show that such a cycle does not exist.*

Proof. We first assume that there are two internally disjoint (x, y)-paths P_1, P_2 and two internally disjoint (y, x)-paths Q_1, Q_2 in D. We claim that D then has a cycle through x and y. If one of P_1 and P_2 has length 2, then the union of that path and one of Q_1, Q_2 is a cycle through x and y. So assume that P_1, P_2, Q_1, Q_2 all have length at least 3. Let x' (respectively, y') be the successor of x (respectively, predecessor of y) on P_1. Then, by the assumption of the theorem, x is adjacent to y', and y is adjacent to x'. If $x \rightarrow y'$, or $x' \rightarrow y$, then we obtain a cycle through x and y as above. On the other hand, if $y' \rightarrow x$ and $y \rightarrow x'$ are arcs, then they are contained in a path P_1' from y to x such that $V(P_1') = V(P_1)$. Then $P_1' \cup P_2$ is a cycle through x and y. If the paths P_1, P_2, Q_1, Q_2 do not exist, then we can assume that P_1, P_2 do not exist, and so there is a vertex z such that $D - z$ has no path from x to y. Let $V(D) - z = A \cup B$ such that $x \in B, y \in A, A \cap B = \emptyset$ and there is no arc from B to A. Now D has a cycle through x and y if and only the following conditions are satisfied:

(i) There is a vertex $a \in A \setminus \{y\}$ such that $D - a$ has a path from z to y and $y \rightarrow a$;

(ii) There is a vertex $b \in B \setminus \{x\}$ such that $D - b$ has a path from x to z and $b \rightarrow x$.

Note that if a exists in (i), then a dominates x, and if b exists in (ii), then y dominates b. Since all the steps in the argument can be checked in polynomial time, the proof is complete. □

Note that Theorem 3.1 and the remark preceeding it imply the following result.

COROLLARY 3.2. *There exists a polynomial algorithm for deciding if two vertices in a complete k-partite oriented graph are on a common cycle (regardless of the value of k).*

The restriction of this result to the case where $k = 2$ was found by Manoussakis and Tuza [7].

4. Two technical results.

We now turn to the main algorithm, which is based on Theorem 2.1. In this section, we prove a theorem that deals with a special case of the 2-path problem for semicomplete digraphs that do not satisfy the condition of Theorem 2.1. We also prove a lemma that allows us to reduce the problem to a smaller one in certain cases.

THEOREM 4.1. *Let T be a semicomplete digraph, and let x_1, x_2, y_1, y_2 be distinct vertices of T such that, for each $i = 1, 2$, there are two, but not three, internally disjoint (x_i, y_i)- paths in $T - \{x_{3-i}, y_{3-i}\}$. Suppose that all (x_i, y_i)-separators of size 2 in $T - \{x_{3-i}, y_{3-i}\}$ are trivial, for $i = 1, 2$. Then T has a pair of disjoint $(x_1, y_1)-, (x_2, y_2)$-paths.*

Proof. When we refer to indegrees and outdegrees, below, it is to be understood that, for instance, $d^-(y_1) = 2$ means that y_1 has indegree 2 in $T - \{x_2, y_2\}$. Also, when we refer to a 2-separator of x_i and y_i, it is to be understood that this is in $T - \{x_{3-i}, y_{3-i}\}$, for $i = 1, 2$.

We may assume without loss of generality that every (x_i, y_i)-path has length at least 3, since otherwise it follows easily, from the assumption of the theorem, that T

has the desired paths. Also we may assume, without loss of generality, that there is no arc $x_i \to y_{3-i}$ for $i = 1,2$.

We can assume that all (x_i, y_i)-separators of size 2 induce a 2-cycle, for $i = 1, 2$. For if, without loss of generality, $\{x, y\}$ is an (x_1, y_1)-separator of size 2, such that there is only one arc between x and y, say $x \to y$, then we add the arc $y \to x$. Now, if P_1 and P_2 are disjoint (x_1, y_1)-, (x_2, y_2)-paths in T with the arc $y \to x$ added, then P_2 does not use the arc $y \to x$, and, if P_1 uses that arc, then we can replace part of P_1 by either the arc $x_1 \to x$ or $y \to y_1$, one of which exists by the assumption that all 2-separators of x_1, y_1 are trivial. We now distinguish between four cases, depending on the half degrees of $x_i, y_i, i = 1, 2$.

Case 1. We have that $d^-(y_1), d^-(y_2) \geq 3$.

Then the assumption of the theorem implies that $d^+(x_1) = d^+(x_2) = 2$. Let r, r', a, a' be chosen such that $x_1 \to r, r'$ and $x_2 \to a, a'$. By the assumption that all (x_2, y_2)-paths have length at least 3, $y_2 \to a, a'$. Hence it follows from the fact that all the 2-separators are trivial and induce directed 2-cycles that there are three internally disjoint (a, y_2)-paths and three internally disjoint (a', y_2)-paths in $T - \{x_1, y_1, x_2\}$. Now Theorem 2.1 implies the existence of disjoint (x_1, y_1)-, (a, y_2)-paths (respectively, disjoint (x_1, y_1)-, (a', y_2)-paths) in $T - \{x_2\}$, ensuring the existence of the desired paths in T, unless a (respectively, a') is contained in an (x_1, y_1)-separator of size 2. Thus we may assume that $\{a, a'\} = \{r, r'\}$, and now the existence of the desired paths follows from Menger's theorem. Now we assume that $\min\{d^-(y_1), d^-(y_2)\} = 2$ and, by directional symmetry, $\min\{d^+(x_1), d^+(x_2)\} = 2$.

Case 2. We have that $d^-(y_i), d^+(x_{3-i}) \geq 3, i = 1$ or 2

We may assume, without loss of generality, that i = 2. Hence $d^+(x_2) = d^-(y_1) = 2$. Let a, a', s, s' be chosen such that $x_2 \to a, a'$ and $s, s' \to y_1$. As in Case 1, we can assume that $\{a, a'\} = \{s, s'\}$. $T - \{x_2, y_2, y_1\}$ has a pair of disjoint (x_1, s)-, (x_1, s')-paths P_1, P_2, by the assumption in the theorem. Let P_1 and be P_2 be chosen such that they are minimal; i.e., no proper subset of the vertices of P_i induces a semicomplete digraph containing a path with the same endvertices, $i = 1, 2$. Let P be any (x_2, y_2)-path in $T - \{x_1, y_1\}$. Go backward on P from y_2, and let u be the first vertex in $P_1 \cup P_2$ that we encounter. If $u \in \{s, s'\}$, then it is easy to see that T has the desired paths. Suppose, without loss of generality, that u is on P_1. If P_1 is the path $x_1 \to u \to s$, then T has the desired paths, since $\{u, s\}$ is not a 2-separator of x_2 and y_2. Thus we may assume that P_1 has length at least 3. Now it follows from the minimality of P_1 that $P_1 \cup \{x_2\} \cup P[u, y_2] - \{x_1\}$ contains an (x_2, y_2)-path that is disjoint from the (x_1, y_1)-path $P_2 \cup \{s' \to y_1\}$ (s dominates the successor of x_1 on P_1 and $x_2 \to s$). Now we assume that $\min\{d^+(x_1), d^-(y_2)\} = \min\{d^+(x_2), d^-(y_1)\} = 2$

Case 3. We have that $d^-(y_i) \geq 3, d^-(y_{3-i}) = d^+(x_{3-i}) = 2, i = 1$ or 2.

We may assume, without loss of generality that i = 2. By the assumption of the theorem $d^+(x_2) = 2$.Let a, a', r, r', s, s' be chosen such that $x_1 \to r, r'$, and $s, s' \to y_1$ and $x_2 \to a, a'$. As in Case 1, we can assume that $\{a, a'\} \subset \{r, r', s, s'\}$. If there is an arc from $u \in \{r, r'\}$ to $v \in \{s, s'\}$, then either the desired paths exist or $\{a, a'\} = \{u, v\}$. Hence we may assume that there is at most one such arc. Thus we may assume, without loss of generality that there are no such arcs ending in s. Suppose first that $s \notin \{a, a'\}$. Then, by the assumption of the theorem, $T - \{x_1, y_1, s\}$ contains two internally disjoint (x_2, y_2)-paths and $T - \{x_1, y_1, x_2, y_2\}$ contains three internally disjoint (r, s)-paths. Thus we conclude, by Theorem 2.1, that the desired paths exist, unless $r \in \{a, a'\}$. Similarly, $r' \in \{a, a'\}$. If $\{r, r'\} = \{a, a'\}$, however, then any two disjoint paths from $\{x_1, x_2\}$ to $\{y_1, y_2\}$ (which exist by Menger's theorem) can be modified into the desired paths. Hence we may assume that $s \in \{a, a'\}$. Now either the desired paths exist, or there can be no arc $\alpha \to \beta$ from $\{r, r'\}$ to $\{s, s'\}$ ending in s' either, since that would imply a 2-separator of x_2, y_2 (namely, $\{\alpha, \beta\}$) that is different from $\{a, a'\}$, a contradiction. Thus, by the same argument as above, $s' \in \{a, a'\}$; i.e., $\{a, a'\} = \{s, s'\}$. Now we can argue as in Case 2 to show that the desired paths exist in this case. Now, by directional symmetry, there only remains one case.

Case 4. We have that $d^+(x_1) = d^-(y_1) = d^+(x_2) = d^-(y_2) = 2$

Let $a, a', b, b', r, r', s, s'$ be chosen such that $x_1 \to r, r'$, and $s, s' \to y_1$ and $x_2 \to a, a'$, and $b, b' \to y_2$.

Suppose first that $\{a, a', b, b'\} = \{r, r', s, s'\}$. If $\{a, a'\}$ equals $\{r, r'\}$ (respectively, $\{s, s'\}$), then we conclude that T has the desired paths using the same arguments as we did in in Case 1 (respectively, Case 2). Otherwise, we obtain an (x_i, y_i)-path of length 3 for $i = 1$ or 2, and it follows from the assumption of the theorem that this path does not separate x_{3-i} from y_{3-i}.

Thus we may assume that $\{a, a', b, b'\} \neq \{r, r', s, s'\}$. By symmetry and directional symmetry, we may assume that $s \notin \{a, a', b, b'\}$.

Then the assumption of the theorem implies that there are two internally disjoint (x_2, y_2)-paths in $T - \{x_1, y_1, s\}$, since s is not in any 2-separator of x_2, y_2. Also, we may assume that $s {\to} r, r'$, since otherwise the existence of the desired paths follows from the fact that s is not in a 2 separator of x_2, y_2. By assumption of the theorem and the fact that all 2-separators induce a 2-cycle, there are three internally disjoint (r, s)-paths (respectively, (r', s)-paths) in $T - \{x_1, x_2, y_1, y_2\}$. We now distinguish between two further subcases.

Case A. We have that $\{r, r'\} \not\subset \{a, a', b, b'\}$

Say, without loss of generality, that $r \notin \{a, a', b, b'\}$. Let $T' = T - \{x_1, y_1\}$. As we have seen, $T' - \{x_2, y_2\}$ has three internally disjoint (r, s)-paths. If $T' - \{r, s\}$ has two internally disjoint (x_2, y_2)-paths, then Theorem 2.1 implies the existence of disjoint (r, s)-, (x_2, y_2)-paths in T', and hence T has the desired paths. Otherwise, there exists a vertex z such that $T' - \{r, s, z\}$ has no (x_2, y_2)-path. Now $T' - r$ has an (x_2, s)-path L_1 and an (x_2, z)-path L_2 such that $L_1 \cap L_2 = \{x_2\}$ by Menger's theorem and the fact that r is not in any 2-separator of x_2, y_2. Similarly, since s is not in any 2-separator of x_2, y_2, $T' - s$ has an (r, y_2)-path L_3 and a (z, y_2)-path L_4 such that $L_3 \cap L_4 = \{y_2\}$. Since the last vertex of $L_3 - y_2$ dominates the first vertex of $L_1 - x_2$, T has an (x_1, y_1)-path disjoint from $L_2 \cup L_4$, showing that T has the desired paths (L_3 is not just an arc, since r does not dominate y_2. Similarly, L_1 is not just an arc, since x_2 does not dominate s).

Case B. We have that $\{r, r'\} \subset \{a, a', b, b'\}$.

If $\{r, r'\} = \{a, a'\}$ (respectively, $\{r, r'\} = \{b, b'\}$) then we conclude as in Case 1 (respectively, Case 2) that T has the desired paths. Hence we may assume, without loss of generality, that $\{r, r'\} = \{a, b\}$. Since $s \notin \{a, a', b, b'\}$, there is one of a, a', b, b' not in $\{r, r', s, s'\}$. If $b' \notin \{r, r', s, s'\}$, then we conclude, as above, that $\{a, a'\} = \{r, s'\}$ (or $\{a, a'\} = \{r', s'\}$, in which case the proof is analogous to the proof given below). Now, however, the arc between $r' = b$ and $s' = a'$ implies an (x_i, y_i)-path of length 3 for $i = 1$ or 2, implying the existence of the desired paths, by the assumption of the theorem (r', s' is not a 2-separator of any of the pairs x_i, y_i, $i = 1, 2$). Thus we may assume that $b' \in \{r, r', s, s'\}$, which implies that $b' = s'$. (We choose the notation such that $r = a, r' = b$.)

If $b \to s$, then T has the desired paths, since $r' = b$ and $\{r', s\}$ is not a 2-seperator of x_2, y_2. So we may assume that $s \to b$.

Suppose first that $a' \to s$. Then T has the desired paths with $x_2 \to a' \to s \to b \to y_2$ being one of them, unless $\{a', s, b\}$ is an (x_1, y_1)-separator. Suppose that $\{a', s, b\}$ is an (x_1, y_1)-separator. Then x_1 and a (respectively, b' and y_2) are not separated by $\{a', s, b\}$. Let R_1 be an (a, s)-path in $T - \{x_2, y_2, a', b\}$, and let R_2' be an (x_1, y_1)-path in $T - \{x_2, y_2, b, s\}$. Then R_2' contains a' and an (a', b')-path R_2. Since $R_1 \cap R_2 = \emptyset$, they can be extended to the desired paths in T (since $x_1 \to a, s \to y_1, x_2 \to a', b' \to y_2$).

Suppose now that $s \to a'$. Let $T'' = T - \{x_1, y_1, x_2, y_2\}$. By Menger's theorem and the assumption of the theorem, $T'' - b$ has two internally disjoint (a, b')-, (a', b')-paths and two internally disjoint (r, s)-, (r, s')-paths. Also, $b \to s'$, since $\{b, s'\}$ is a 2-separator of x_2, y_2, and we have assumed that all these form 2-cycles. Now it is easy to see that T'' has three paths from $\{a, b, a'\}$ to $\{s, s'\}$ that are disjoint, except that two of them contain s'. Two of these can be extended to the desired paths. (If the path that ends in s starts in a', then T'' has disjoint (a', b)-, (r, s')-paths, because $s \to b$.) Hence, if $s \notin \{a, a', b, b'\}$, then T has the desired paths. This completes the proof of the theorem. □

LEMMA 4.2. *Let T be a semicomplete digraph, and let x_1, x_2, y_1, y_2 be distinct vertices such that there are two internally disjoint (x_2, y_2)-paths in $T - \{x_1, y_1\}$. Suppose that there exists a nontrivial 2-separator $\{x, y\}$ of x_2 and y_2 in $T - \{x_1, y_1\}$ such that there is no arc from $B - x_2$ to y_1, where A and B form any patition of $T - \{x_1, y_1, x, y\}$ such that $x_2 \in B$, $y_2 \in A$, and all arcs between A and B go from A to B.*

Transform T into a new semicomplete digraph T' as follows:

1. *If x_1 dominates some vertex in $A - y_2$ then*
   - *If there exists a vertex $b \in B - x_2$ such that $b \to x$ and there is an (x_2, y)-path in $T(B \cup \{y\} \setminus \{b\})$, then add all arcs from $A - y_2$ to x that are not present already.*
   - *If there exists a vertex $b \in B - x_2$ such that $b \to y$ and there is an (x_2, x)-path in $T(B \cup \{x\} \setminus \{b\})$, then add all arcs from $A - y_2$ to y that are not present already.*
2. *Add the arcs $x_2 \to x, x_2 \to y$ if they are not present already;*
3. *Add the arc $x_1 \to z$ for $\{z, w\} = \{x, y\}$ if $T(B \cup \{z, w, x_1\})$ has a pair of disjoint $(x_1, z)-, (x_2, w)$-paths, and the arc $x_1 \to z$ is not present already;*
4. *Delete the vertices of $B - x_2$.*

Call the added *arcs* special arcs. *Then the resulting semicomplete digraph T' has disjoint (x_1, y_1)-, (x_2, y_2)-paths if and only if T also has them.*

Proof. Suppose that P and Q are disjoint $(x_1, y_1)-, (x_2, y_2)$-paths in T chosen such that they are minimal; i.e., no proper subset of P or Q is a path from x_i to y_i for $i = 1$ or 2. If $x, y \in V(Q)$, then P is entirely in $T(A \cup \{x_1, y_1\})$ since there is no arc from B to y_1. Let z be that of x, y that is closest to y_2 on Q. Then P is in T', and $Q' = \{x_2 \to z\} \cup Q[z, y_2]$ is in T'. Now suppose that Q contains only one of x, y, and let z denote that one, and w the other. Then $Q[z, y_2] \cap B = \emptyset$. If P does not intersect B, then P and $Q' = \{x_2 \to z\} \cup Q[z, y_2]$ are in T'. Thus we can assume that P intersects B. Then $w \in V(P)$ since there are no arcs from B to y_1. If $P[x_1, w]$ intersects A, then the minimality of P implies that P contains the sequence $a \to b \to w$ for some $a \in A, b \in B$. Let $P' = P[x_1, a] \cup \{a \to w\} \cup P[w, y_1]$ and $Q' = \{x_2 \to z\} \cup Q[z, y_2]$. Then P', Q' are the desired paths in T'. If $P[x_1, w]$ does not intersect A, then we let $P' = \{x_1 \to w\} \cup P[w, y_1]$ and take Q' as above. Thus if T has the desired paths, then so has T'.

Conversely, suppose that P', Q' are disjoint (x_1, y_1)-, (x_2, y_2)-paths in T', chosen such that they are minimal. We may assume, without loss of generality, that $P' \cup Q'$ contains at least one special arc. By the minimality of Q', exactly one of x, y belongs to Q'. Let z be that one, and w the other. If P' contains the special arc $x_1 \rightarrow w$, then, by the definition of the special arcs, this is the only special arc in P', and $T(B \cup \{x_1, w, z\})$ has a pair of disjoint (x_1, w)-, (x_2, z)-paths P^*, Q^*. Then $Q^* \cup Q'[z, y_2]$ and $P^* \cup P'[w, y_1]$ are the desired paths in T. If P' contains a special arc of the form $a \rightarrow w$ for some $a \in A$, then we know, from the definition of the special arcs, that there exists a vertex $b \in B$, dominating w such that $T((B \cup \{z\}) \setminus \{b\})$ has an (x_2, z)-path Q^*. Let $Q = Q^* \cup Q'[z, y_2]$ and $P = P'[x_1, a] \cup \{a \rightarrow b \rightarrow w\} \cup P'[w, y_1]$. Then P, Q are the desired paths in T. Finally, if P' contains no special arcs, then P' and $Q'[z, y_2]$, together with some (x_2, z)-path in $T(B \cup \{z\})$, are the desired paths. This proves the lemma. □

5. A polynomial algorithm for the 2-path problem for semicomplete digraphs.

The idea in the algorithm is either to settle the problem, or else to reduce the problem to a smaller one (i.e., construct a semicomplete digraph S such that the desired paths exist in T if and only if there exist some corresponding paths in S), and then let the algorithm call itself recursively. The crucial step in the algorithm is when the local connectivity from x_i to y_i is precisely 2 for $i = 1, 2$. If all 2-separators of x_i, y_i are trivial for $i = 1, 2$, then no reduction is possible. Fortunately, this is precisely the situation in Theorem 4.1, and we can conclude that the desired paths exist. In the case when a nontrivial 2-separator exists, say, for x_2, y_2, we show how to use the structure of the nontrivial 2-separators of x_2, y_2 to decide the existence of the desired paths, or to reduce the problem to a smaller one.

THEOREM 5.1. *There exists a polynomial algorithm for the following problem for semicomplete digraphs: Let T be a semicomplete digraph and x_1, x_2, y_1, y_2 be four different vertices of T. Decide whether T has a pair of disjoint (x_1, y_1)-, (x_2, y_2)-paths.*

Proof. Clearly, if there is no (x_i, y_i)-path in $T - \{x_{3-i}, y_{3-i}\}$, $i = 1$ or 2, then the desired paths do not exist. This is easy to check, and we may thus assume that there is an (x_i, y_i)-path in $T - \{x_{3-i}, y_{3-i}\}$ for $i = 1, 2$. If T is not strong, then it is easy to see that the desired paths exist, except possibly when x_1, x_2, y_1, y_2 all belong to the same strong component of T. Then we can reduce the problem to that component. Since it is easy to find the strong components of T, we can thus assume that T is strong. We can also assume that T contains none of the arcs $x_1 \rightarrow y_1, x_2 \rightarrow y_2$, since otherwise the desired paths exist.

From Theorem 2.1, we know that, if $T - \{x_i, y_i\}$ has three internally disjoint (x_{3-i}, y_{3-i})-paths and $T - \{x_{3-i}, y_{3-i}\}$ has two internally disjoint (x_i, y_i)-paths, for i=1 or 2, then T has the desired paths. Thus the algorithm first checks this and stops if this condition is met. We now distinguish between two cases.

Case 1. The local connectivity from x_i to y_i in $T - \{x_{3-i}, y_{3-i}\}$ is 1 for i=1 or 2.

Without loss of generality, we may assume that i=2. By Menger's theorem, there exists an $x \in T - \{x_1, y_1\}$ such that there is no (x_2, y_2)-path in $T - \{x, x_1, y_1\}$. We can assume that there is an (x_1, y_1)-path in $T - \{x, x_2, y_2\}$, since otherwise the desired paths do not exist in T. For a given x, which separates x_2 from y_2 in $T - \{x_1, y_1\}$, we define A and B as follows:

$$A = \{v \in T - \{x\} \mid v \text{ can reach } y_2 \text{ by a path in } T - \{x, x_1, y_1\}\},$$

$$B = V(T) - A - \{x, x_1, y_1\}.$$

Then all the arcs between A and B are leaving A.

If A contains a vertex a dominated by x_1 such that $T - \{x_1, y_1, a\}$ has an (x, y_2)-path P_1, and B contains a vertex b dominating y_1 such that $T - \{x_1, y_1, b\}$ has an (x_2, x)-path P_2, then $P_1 \cup P_2$ and $x_1 \rightarrow a \rightarrow b \rightarrow y_1$ are the desired paths. Suppose that a does not exist. Now the desired paths exist if and only if $T(B \cup \{x, x_1, y_1\})$ contains a pair of disjoint $(x_1, y_1)-, (x_2, x)$-paths. Thus we have reduced the problem to a smaller one. The case where b does not exist is analogous.

Case 2. There exist two internally disjoint (x_i, y_i)-paths in $T - \{x_{3-i}, y_{3-i}\}$ for i=1, 2.

If all 2-separators of x_1 and y_1 in $T - \{x_2, y_2\}$ and all 2-separators of x_2 and y_2 in $T - \{x_1, y_1\}$ are trivial, then it follows from Theorem 4.1 that T has the desired paths. Hence we may assume, by renaming if necessary, that there exists a nontrivial 2-separator $\{x, y\}$ of x_2 and y_2 in $T - \{x_1, y_1\}$. Define A, B as follows:

$$A = \{v \in T - \{x_1, y_1\} \mid \text{v can reach } y_2 \text{ by a path in } T - \{x, y, x_1, y_1\}\},$$

$$B = T - A - \{x_1, y_1, x, y\}.$$

Then all arcs between A and B go from A to B, and neither x_1 nor y_1 belong to $A \cup B$. Since $\{x, y\}$ is a nontrivial 2-separator, we have that $\mid A \mid, \mid B \mid \geq 2$. We can assume, without loss of generality, that $x \rightarrow y$. There are two subcases to consider.

Subcase 2.1. There are no arcs from x_1 to $A - y_2$, or there are no arcs from $B - x_2$ to y_1.

We may assume that the latter case holds, since the former case can be treated similarly (using an analogous version of Lemma 4.2 to contract A into one vertex, just as shown below for B in the latter case). Our next step is to contract B into one vertex x_2. First, we check, using the flow version of Menger's theorem (see, e.g., [4]), whether there exist disjoint (x_1, z)-, (x_2, w)-paths in $T(B \cup \{x_1, x, y\})$, where $\{z, w\} = \{x, y\}$ (using the flow version, we can actually find these paths if they exist). If T has such paths, then we apply our algorithm to $T(B \cup \{x_1, x, y\})$ to decide whether $T(B \cup \{x_1, x, y\})$ also has disjoint (x_1, w)-, (x_2, z)-paths. If $T(B \cup \{x_1, x, y\})$ also has these paths, then we conclude that the desired paths exist. This follows, since there is no arc from $B - x_2$ to y_1 and we know that there are two internally disjoint paths P_1, P_2 from $\{x, y\}$ to y_2 in $T(A \cup \{x, y\})$. Also, $T - y_2$ has a path P from x_1 to y_1. Going backward on P until we meet a vertex in one of the previously mentioned paths, P_1, P_2, or x_1 gives a configuration that contains the desired paths: Suppose that the first vertex of $P_1 \cup P_2$ that we encounter when going backward from y_1 on P is the vertex u on P_1. Let Q_1, Q_2 be disjoint (x_1, x)-, (x_2, y)-paths in $T(B \cup \{x_1, x, y\})$. Then $Q_1 \cup P_1[x, u] \cup P[u, y_1]$ and $Q_2 \cup P_2$ are the desired paths. Thus we may assume that $T(B \cup \{x_1, x, y\})$ does not have both pairs of paths.

Now we contract B to $\{x_2\}$ giving a new semicomplete digraph T', as described in Lemma 4.2. By Lemma 4.2, the resulting semicomplete digraph T' has disjoint (x_1, y_1)-, (x_2, y_2)-paths if and only T also has them. Thus we have reduced the problem to a smaller one.

Subcase 2.2. $x_1 \rightarrow r$ for some $r \in A - y_2$, and $s \rightarrow y_1$ for some $s \in B - x_2$.

Then T contains the path $x_1 \rightarrow r \rightarrow s \rightarrow y_1$. If $T - \{x_1, r, s, y_1\}$ has an (x_2, y_2)-path, then T has the desired paths. Hence we may assume that this is not the case. Then $\{r, s\}$ must separate x_2 from y_2 in $T - \{x_1, y_1\}$. Since the local connectivity from x_2 to y_2 in $T - \{x_1, y_1\}$ is 2, r does not destroy all paths from $\{x, y\}$ to y_2 in $T(A \cup \{x, y\})$, and s does not destroy all paths from x_2 to $\{x, y\}$ in $T(\{x, y\} \cup B)$. Thus, since $x \rightarrow y$

and $T - \{x_1, r, s, y_1\}$ has no (x_2, y_2)-path, we conclude that in $T(A \cup \{x, y\} - \{r\})$ there is an (x, y_2)-path but no (y, y_2)-path, and in $T(B \cup \{x, y\} - \{s\})$ there is an (x_2, y)-path, but no (x_2, x)-path. Thus $\{r, s\}$ is also a nontrivial 2-separator of x_2 and y_2 in $T - \{x_1, y_1\}$ (x is included in the new A and y in the new B).

Now look at the sets A' and B' that correspond to $\{r, s\}$ as A and B correspond to $\{x, y\}$ (the remainder of the discussion deals only with A', B'; hence the reader may disregard that x, y, A, B). If there is no arc from x_1 to $A' - y_2$ or there is no arc from $B' - x_2$ to y_1, then we are in Case 2.1, where we can settle the problem or reduce. Hence we may assume that we are in Case 2.2 with A' and B' instead of A, B. Thus we have some $r' \in A' - y_2$ and some $s' \in B' - x_2$ such that $x_1 \rightarrow r' \rightarrow s' \rightarrow y_1$ is a path in T. Again, we may assume that there is no (x_2, y_2)-path in $T - \{x_1, r', s', y_1\}$. As above, $T(A' \cup \{r, s\} - \{r'\})$ has an (r, y_2)-path but no (s, y_2)-path, and in $T(B' \cup \{r, s\} - \{s'\})$ there is an (x_2, s)-path, but no (x_2, r)-path. If $r \rightarrow s'$ or $r' \rightarrow s$, then it is easy to see that T contains the desired paths (for example, if $r \rightarrow s'$, then there is an (x_2, y_2)-path in $T - \{x_1, r, s', y_1\}$ since s' does not separate x_2 from s in $T(B' \cup \{r, s\})$, and r does not separate s from y_2 in $T(A' \cup \{r, s\})$). So assume that these arcs do not exist. Also, we must have that $x_2 \rightarrow s'$ and $r' \rightarrow y_2$: Suppose that there is no arc from x_2 to s'. Let P_1 and P_2 be internally disjoint (x_2, r)- , (x_2, s)-paths in $T(B' \cup \{r, s\})$. By the above argument, P_1 contains s', and, by the assumption, $P_1[x_2, s']$ has length at least 2. Let u be the first vertex after x_2 on $P_1[x_2, s']$. We cannot have that $u \rightarrow r$, since then s' would not separate x_2 from r in $T(B' \cup \{r, s\})$ as assumed above. Thus $r \rightarrow u$, and now we easily find an (x_2, y_2)-path that avoids the (x_1, y_1)-path $x_1 \rightarrow r \rightarrow u \cup P_1[u, s'] \cup \{s' \rightarrow y_1\}$ (when the vertices of this path are removed, there is still a path from x_2 to s and then to y_2, namely, P_2 and any (s, y_2)-path in $T(A' \cup \{r, s\})$). Similarly, we get that $r' \rightarrow y_2$. Now if we look at the nontrivial 2-separator $\{r', s'\}$, we can argue similarly that either T has the desired paths, or $x_2 \rightarrow s$ and $r \rightarrow y_2$. Hence we may assume that we have the following paths in T:

$$x_1 \rightarrow r \rightarrow s \rightarrow y_1, x_1 \rightarrow r' \rightarrow s' \rightarrow y_1, x_2 \rightarrow s \rightarrow r' \rightarrow y_2, x_2 \rightarrow s' \rightarrow r \rightarrow y_2.$$

Now any (x_1, y_1)-path in $T - \{x_2, y_2\}$ must contain at least two of the vertices r, r', s, s', to destroy both of the above (x_2, y_2)-paths. On the other hand, no minimal (x_1, y_1)-path that does not start with $x_1 \rightarrow r$ or $x_1 \rightarrow r'$ contains more than one of these four vertices, since $\{s, s'\} \rightarrow y_1$. Thus, if $\{r, r'\}$ is not a 2-separator of x_1 and y_1 in $T - \{x_2, y_2\}$, then T contains the desired paths. Hence we may assume that $\{r, r'\}$, and, similarly, $\{s, s'\}$ are 2-separators of x_1 and y_1 in $T - \{x_2, y_2\}$. That is, all minimal (x_1, y_1)-paths start with one of the arcs $x_1 \rightarrow r$, $x_1 \rightarrow r'$ and end with one of the arcs $s \rightarrow y_1, s' \rightarrow y_1$. Now it is easy to see that T contains the desired paths if and only if at least one of $T - \{x_2, s, r', y_2\}$, $T - \{x_2, s', r, y_2\}$ contains an (x_1, y_1)-path. (Remember that we have assumed that there is no (x_2, y_2)-path in $T - \{x_1, r, s, y_1\}$ or $T - \{x_1, r', s', y_1\}$).

This completes the proof of the theorem, since it is easy to see that all our actions can be done in a polynomially bounded number of steps, and, for each application of the algorithm, the number of vertices decreases before the next time the algorithm is applied. In fact, the complexity of the algorithm is of order $O(n^5)$: The analysis of all separating sets of size 2 takes $O(n^4)$ time, the flow calculation and all other actions at most $O(n^3)$ time. Hence, between two consecutive calls of the algorithm we spend at most $O(n^4)$ time. Thus, if $T(n)$ denotes the time complexity of our algorithm, then $T(n) \leq T(n - x + 1) + T(x + 3) + O(n^4)$, where x denotes the size of B in Subcase 2.1. This implies that $T(n) = O(n^5)$. Above n denotes the number of vertices in the semicomplete digraph. $\square$

We make no claim as to the optimality of this algorithm. Our sole objective has been to demonstrate that the problem is polynomially solvable for semicomplete digraphs. However, we feel that an algorithm with a significantly lower time complexity is considerably more complicated to describe. This is supported by the fact that there exist highly nontrivial families of 4-connected semicomplete digraphs that are not 2-linked (i.e., in such a digraph, there exist four vertices u, v, x, y such that there do not exist disjoint $(u, v)-, (x, y)-$paths). Hence it is very unlikely that a simple algorithm exists, since this would imply a simple characterization of those semicomplete digraphs that are not 2-linked (see also §1, p. 366).

Note that it is easy to use this algorithm to obtain a polynomial algorithm that finds the desired paths, given that they exist: Successively, reverse all arcs out of x_1 and, each time, call the algorithm to check for the existence of the desired paths. If an arc is identified, whose reversal destroys the property of having the desired paths, then this arc must be part of such a pair of paths in the current semicomplete digraph. Suppose this is the arc $x_1 \rightarrow u$. Now remove x_1, rename u by x_1, and repeat the above step for the new x_1. Eventually, we get that the current x_1 dominates y_1 and that the current semicomplete digraph T contains an (x_2, y_2)-path that does not contain any of x_1 and y_1. Hence the (x_1, y_1)-path given by the sequence of successively identified vertices and any (x_2, y_2)-path in the final semicomplete digraph with x_1, y_1 removed is a pair of disjoint paths as desired.

COROLLARY 5.2. *There exists a polynomal algorithm for the following problem for semicomplete digraphs. Let T be a semicomplete digraph and let x, y, z be distinct vertices of T. Decide whether T has an (x, z)-path through y.*

Proof. This correponds to the situation $y_1 = x_2$ in the two path problem. In the above proof, we required that x_1, x_2, y_1, y_2 are distinct vertices, but it is easy to see that this special case of the 2-path problem reduces to the general case in polynomial time within the class of semicomplete digraphs. Hence the result follows. Details are given in [1]. □

6. Two NP-complete problems on tournaments.

There are many NP-complete problems on graphs. One example is the problem of finding a Hamiltonian cycle. That problem has an easy solution for tournaments, and not many "natural" NP-complete problems on tournaments are known. (Since every oriented graph is a subgraph of a tournament, it is, of course, possible to formulate a number of artificial NP-complete problems involving (subgraphs of) tournaments.) We mention here two basic NP-complete tournament problems.

THEOREM 6.1. *The problem q of finding a cycle through a prescribed arc set in a tournament is* NP*-complete.*

Proof. To see this, we reduce the problem of finding a Hamiltonian cycle in a directed graph to q. Let D be a directed graph. First, we split every vertex v of D into two vertices v_1 and v_2 such that all arcs entering v (respectively, leaving v) now enter v_1 (respectively, leave v_2). We also add the "distinguished" arc $v_1 v_2$. This transforms D into a bipartite digraph D'. We add arcs from vertices of index 1 to vertices of index 2 in D (whenever the arcs in the opposite direction are not already present). We add all arcs between vertices of the same index and orient them at random. Then the resulting tournament has a cycle through the distinguished arcs if and only if D has a Hamiltonian cycle. □

This proves that, if k is not fixed, then the k-path problem NP-complete for tournaments.

THEOREM 6.2. *The problem p_1 of finding a largest transitive subtournament* *(or, equivalently, the problem of finding a smallest vertex set meeting all cycles) is* NP*-complete.*

Proof. It is well known that the problem p_2 of finding a largest set of independent (i.e., pairwise nonadjacent) vertices in an undirected graph is NP-complete. Now we reduce p_2 to p_1 by a polynomial time reduction. Let G be an undirected graph with vertex set $v_{1,0}, v_{2,0}, \cdots, v_{n,0}$. We form a tournament T as follows : We add, for each $i = 1, 2, \cdots, n$, a set of n+1 new vertices $v_{i,1}, v_{i,2}, \cdots, v_{i,n+1}$. Now T contains the directed arcs $v_{i,k}v_{j,m}$ whenever $i > j$ or $i = j$ and $k > m$, unless $k = m = 0$ and $v_{i,0}, v_{j,0}$ are adjacent in G. In that case, T contains the arc $v_{j,0}v_{i,0}$. Now a vertex set S in G is a largest independent set if and only if $T - (V(G) \setminus S)$ is a largest transitive subtournament of T. □

Note that problem of finding a feedback vertex set, i.e., a minimum set of vertices such that every directed cycle is incident with this set, corresponds to p_1 for tournaments; hence Feedback vertex set is NP-complete for tournaments.

CONJECTURE 6.3. *The problem of finding a minimum set of arcs in a tournament whose reversal results in a transitive tournament is* NP*-complete.*

Such a set of arcs is also called a feedback arc set. This problem is known to be NP-complete for general digraphs [6, p. 192].

Note added in proof. After the submission of this manuscript, the second author [12] has shown that for general digraphs there is no degree of connectivity that will ensure that a digraph is 2-linked, i.e., has disjoint (u, v)-, (x, y)-paths for any choice of distinct vertices u, v, x, y.

We also want to point out that the result of Theorem 6.2 was found independently by Speckenmeyer; see [8].

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