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ON THE PROBLEM OF FINDING DISJOINT CYCLES AND DICYCLES IN A DIGRAPH

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We study the following problem: Given a digraph D, decide if there is a cycle B in D and a cycle C in its underlying undirected graph $UG(D)$ such that $V(B)\cap V(C)=\emptyset$.

Whereas the problem is $\mathcal{NP}$-complete if, as additional part of the input, a vertex x is prescribed to be contained in C, we prove that one can decide the existence of B,C in polynomial time under the (mild) additional assumption that D is strongly connected. Our methods actually find B,C in polynomial time if they exist. The behaviour of the problem as well as our solution depend on the *cycle transversal number* $\tau(D)$ of D, i.e. the smallest cardinality of a set T of vertices in D such that $D-T$ is acyclic: If $\tau(D)\geq 3$ then we employ McCuaig's framework on intercyclic digraphs to (always) find these cycles. If $\tau(D)=2$ then we can characterize the digraphs for which the answer is "yes" by using topological methods relying on Thomassen's theorem on 2-linkages in acyclic digraphs. For the case $\tau(D)\leq 1$ we provide an algorithm independent from any earlier work.

1. Introduction

All graphs and digraphs are supposed to be finite, and they may contain loops or multiple arcs or arcs. Notation follows [1], and we recall the most relevant concepts here. A digraph D is *acyclic* if it does not contain a cycle, and it is *intercyclic* if it does not contain two disjoint cycles. A *cycle transversal* is a set S of vertices of D such that $D-S$ is acyclic, and the *cycle transversal number* $\tau(D)$ is defined to be the size of a smallest cycle transversal. McCuaig characterized the intercyclic digraphs of minimal in- and out-degree at least 2 in terms of their cycle transversal number and de-

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Mathematics Subject Classification (2000): 05C38, 05C20, 05C85 signed a polynomial time algorithm that, for any digraph D, either finds two disjoint cycles or a structural certificate for being intercyclic [6].

The problem of deciding whether for a given digraph D and $b,c \in V(D)$ there exist disjoint cycles B,C in D with $b \in V(B)$ and $c \in V(C)$ is known to be $\mathcal{NP}$-complete by the classic dichotomy of Fortune, Hopcroft, and Wyllie on the *fixed directed subgraph homeomorphism problem* [4]: For some pattern digraph H, not part of the input, we want to decide for an input digraph D and an injection f from $V(H)$ to $V(D)$ if we can extend f on $V(H) \cup A(H)$ such that every loop xx maps to a cycle of D containing $f(x)$ and every arc xy with $x \neq y$ maps to an $(f(x), f(y))$-path, and the resulting paths and cycles are *internally disjoint*, i.e. no internal vertex of either object is a vertex of another one [1]. The dichotomy then states that the problem is solvable in polynomial time if the arcs of H have the same initial vertex or if they have the same terminal vertex, and is $\mathcal{NP}$-complete in all other cases [4].

We recently studied an extension of this where H might be a mixed graph (having both arcs and edges) and the edges of H are asked to be mapped to cycles and paths of $UG(D)$, the *underlying undirected graph* of D [2]. [2] We found it surprising that, as a consequence of the resulting dichotomy, the problem is already $\mathcal{NP}$-complete when there is both an arc and an edge in the pattern graph. In particular, the problem of deciding whether for a digraph D and $b,c \in V(D)$ there exists a cycle B in D and a cycle C in $UG(D)$ with $b \in V(B)$, $c \in V(C)$, and $V(B) \cap V(C) = \emptyset$, is $\mathcal{NP}$-complete. Our proof, however, shows that even the weaker problem to decide whether for a digraph D and $c \in V(D)$ there exists a cycle B in D and a cycle $C \in UG(D)$ with $c \in V(C)$ and $V(B) \cap V(C) = \emptyset$ is $\mathcal{NP}$-complete, even if we are assuming that, in addition, D is strongly connected.

So the question arised what happens if we prescribe no vertices at all:

Problem 1.1. Given a strongly connected digraph D, decide if there is a cycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$.

We will show that there is a polynomial time algorithm for this problem. It is quite likely that this is extendable to the case of not necessarily strongly connected digraphs, but we will not take a closer look at this here. Our methods rely heavily on $\tau(D)$: Using McCuaig's result mentioned above we prove that the answer is "yes" if $\tau(D) \geq 3$. If $\tau(D) = 2$ then we can characterize the strongly connected digraphs for which the answer is "yes" by using

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[1] Where, in the case of a cycle C of D assigned to a loop of H at x, we consider its internal vertices to be all but $f(x)$.

[2] We are always assuming that D and $UG(D)$ have the same set of vertices and arcs, respectively, i.e. they differ only by means of incidence relations.

topological methods relying on Thomassen's theorem on 2-linkages in acyclic digraphs [10]. For the case $\tau(D) \leq 1$ we provide an algorithm independent from any earlier work. We concentrate on the decision problem, Problem 1.1, but our methods and proofs can be transformed to actually find B, C as there if the answer is "yes".

It is not too likely that there is a much simpler algorithm than the one presented along the lines of this paper, because such an algorithm must be able to work on a difficult class of digraphs of cycle transversal number 2 which we define in Section 4. Many of these *vaults* are "yes"-instances, but to locate the *niches* which actually bring cycles B, C as in Problem 1.1 to live requires, apparently, some insight into their structure.

The undirected graphs without two disjoint cycles have been characterized by Lovász [5], generalizing earlier statements of Dirac for the 3-connected case [3]. The digraphs without two disjoint cycles have been characterized by McCuaig [6], as discussed above. One might estimate that the difficulty of Problem 1.1 is somewhere in between the (easier) purely undirected version and the framework for the purely directed version provided in McCuaig's paper. In contrast to this initial intuition it seems to us that *our mixed version needs more effort* [3] *than even the directed one*. We believe this not because our methods actually rely on the theorems of McCuaig and Thomassen, but because in the mixed version we can not, like in the directed case, employ three important concepts:

(i) Symmetry of the objects we are looking for,
(ii) strongly connectedness – the general directed version reduces immediately to this case, whereas here we needed to add it as a condition to the input digraph —, and, finally,
(iii) reduction by contracting an arc which is either the unique out-arc at its tail or the unique in-arc at its head – in the directed case, this does not change the answer, so that we can immediately assume that all vertices have in- and out-degree at least 2.

We certainly cannot overcome (i). As we show in the next section, strongly connectedness or reduction as in (iii) is not an issue if the transversal number of the input digraph is at least 3. In Section 4 we show how to overcome the difficulties related to (iii) in the case that the cycle transversal number is 2.

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[3] In a sufficiently untechnical sense – all three problems are in $\mathcal{P}$, and we omit any refined time analysis.

2. Cycle transversal number ≥ 3

Let us recall some digraph terminology. An arc e from x to y in a digraph D is usually denoted by xy, where $t(e) := x$ is its *tail* and $h(e) := y$ is its *head*. e is also called an *out-arc* at x and an *in-arc* at y, and its presence causes y to be an *out-neighbor* of x and x an *in-neighbor* of y. If $t(e)=h(e)$ then e is a *loop*. The *multiplicity* of e is the number of arcs *parallel* to e, i.e. arcs f with $t(f)=t(e)$ and $h(f)=h(e)$. The *out-degree* $d_D^+(x)$ of a vertex x is the number of arcs whose tail is x, the *in-degree* $d_D^-(x)$ is the number of arcs whose head is x. A vertex x is a *source* if $d_D^-(x)=0$, and a *sink* of D if $d_D^+(x)=0$. If D is nonempty we set $\delta^+(D) := \min\{d_D^+(x)\colon x \in V(D)\}$, $\delta^-(D) := \min\{d_D^-(x)\colon x \in V(D)\}$, and $\delta(D) := \min\{\delta^+(D), \delta^-(D)\}$. Given a path P formed by vertices $x_0,\ldots,x_\ell$ in that order, we write $\ell(P) := \ell$ to express its length, that is, the number of its arcs. Moreover, for $0 \leq i \leq j \leq \ell$, we define $P[x_i,x_j]$ to be the subpath induced by the vertices x_k with $i \leq k \leq j$ in P [4], and also write $P[x_i,x_j) := P[x_i,x_j]-\{x_j\}$, $P(x_i,x_j] := P[x_i,x_j]-\{x_i\}$, and $P(x_i,x_j) := P[x_i,x_j]-\{x_i,x_j\}$ for brevity. We call x_0 the *initial vertex* and x_ℓ the *terminal vertex* of P, and these are the *end-vertices* of P. If $\ell(P) > 0$ then the arc in P from x_0 to x_1 is the *initial arc*, whereas the one from $x_{\ell-1}$ to x_ℓ is the *terminal arc* of P. We appeal to the readers intuition for other notions or aspects of path algebra, such as concatenation etc. D is *strongly connected* if there is an (a,b)-path for all vertices a,b in D.

In order to handle the case that the cycle transversal number is at least 3 we need to rephrase McCuaig's Structure Theorem on intercyclic digraphs. D is *in reduced form* if it is simple, loopless, strongly connected, and $\delta(D) \geq 2$. Let $\mathcal{P}(x_1,\ldots,x_s;y_1,\ldots,y_t)$ be the class of acyclic simple digraphs D such that $x_1,\ldots,x_s$, $s \geq 2$, are the sources of D, $y_1,\ldots,y_t$, $t \geq 2$, are the sinks of D, every vertex which is neither a source nor a sink has in- and out-degree at least 2, and, for $1 \leq i < j \leq s$ and $1 \leq k < \ell \leq t$, every (x_i,y_ℓ)-path intersects every (x_j,y_k)-path. By a Theorem of Metzlar [7], such a digraph can be embedded in a disk such that $x_1,x_2,\ldots,x_s,y_t,y_{t-1},\ldots,y_1$ occur, in this cyclic order, on its boundary. Let $\mathcal{T}$ be the class of simple, loopless digraphs with minimum in- and out-degree at least 2 which can be obtained from a digraph in $\mathcal{P}(x^+,y^+;x^-,y^-)$ by identifying $x^+=x^-$ and $y^+=y^-$. Let D_7 be the digraph from Figure 1. Let $\mathcal{K}$ be the class of digraphs D with $\tau(D) \geq 3$ and $\delta(D) \geq 2$ which can be obtained from a digraph K_H from $\mathcal{P}(w_0,z_0;z_1,w_1)$ by adding at most one arc connecting w_0,z_0, adding at most one arc connecting w_1,z_1, adding a 4-cycle $x_0x_1x_2x_3x_0$ disjoint from

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[4] This will never be mistakable for the notion $D[X]$ for the subdigraph induced by a set X of vertices in D.

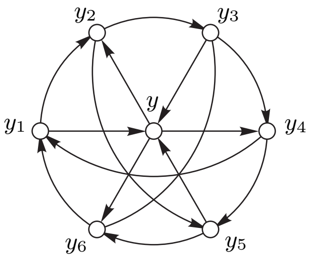

Figure 1. The digraph D_7 [6].

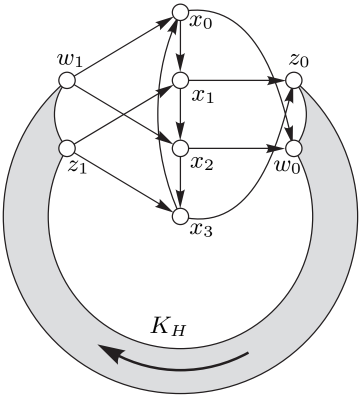

Figure 2. The digraphs from $\mathcal{K}$ [6]. The arrow in the grey area symbolizing the acyclic (plane) digraph K_H indicates that z_0, w_0 are its sources and z_1, w_1 are its sinks.

K_H and adding eight single arcs $w_1x_0, w_1x_2, z_1x_1, z_1x_3, x_0w_0, x_2w_0, x_1z_0, x_3z_0$ (see Figure 2).

Let $\mathcal{H}$ be the class of digraphs D with $\tau(D) \geq 3$ and $\delta(D) \geq 2$ such that D is the union of three arc-disjoint digraphs $H_\alpha \in \mathcal{P}(y_4, y_3, y_1; y_5, y_2)$, $H_\beta \in \mathcal{P}(y_4, y_5; y_3, y_1, y_2)$, and $H_\gamma \in \mathcal{P}(y_1, y_2; y_3, y_4)$, where y_1, y_2, y_3, y_4, y_5 are the only vertices in D occuring in more than one of $H_\alpha, H_\beta, H_\gamma$ (see Figure 3).

Theorem 2.1. [6, Theorem 3.1] *The class of intercyclic digraphs in reduced form is* $\mathcal{T} \cup \{D_7\} \cup \mathcal{K} \cup \mathcal{H}$.

Lemma 2.2. [6, Lemma 4.1] *A digraph* D *is* $\mathcal{T}$ *if and only if it is an intercyclic digraph in reduced form with* $\tau(D) \leq 2$.

We are now able to prove the following.

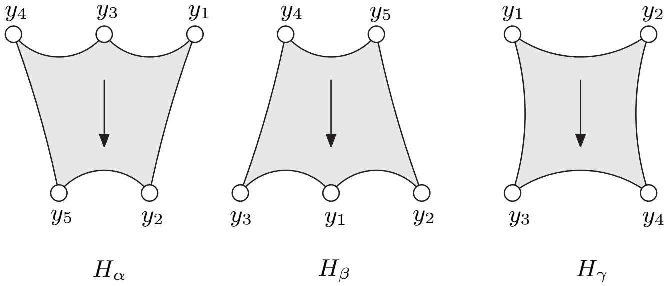

Figure 3. The digraphs from $\mathcal{H}$ [6].

Theorem 2.3. *Every strongly connected digraph with $\tau(D)\geq 3$ admits a cycle B in D and a cycle C in $UG(D)$ with $V(B)\cap V(C)=\emptyset$, and these cycles can be found in polynomial time.*

Proof. We first reduce D by subsequently deleting arcs which have a parallel and contracting arcs which are the unique out-arc at their initial vertex or the unique in-arc at their terminal vertex, as long as we can. Let D' be the resulting digraph. It has been shown in [6, P. 209] that $\tau(D')=\tau(D)\geq 3$ and either we find a loop in D', causing disjoint cycles B,C in D, or D' is in reduced form. By Theorem 2.1 and Lemma 2.2 we therefore know that there are disjoint cycles B,C in D unless D is from $\{D_7\}\cup\mathcal{K}\cup\mathcal{H}$. In fact, there is a polynomial time algorithm outlined in [6, Section 9] which either finds B and C, or a certificate that D is contained in $\{D_7\}$, $\mathcal{K}$, $\mathcal{H}$ (i.e. objects as in the definition of these classes).

In the digraph D_7 as in Figure 1, the cycle formed by $y_1y_2y_3y_4$ is disjoint from the cycle of $UG(D_7)$ formed by y,y_5,y_6. So D_7 is a "yes"-instance. Now let H be a digraph from $\mathcal{K}$ and consider $K_H,w_0,z_0,z_1,w_1,x_0,x_1,x_2,x_3$ as in the definition. Since $C=x_0x_1x_2x_3x_0$ is a cycle in D we may assume that $UG(K_H)$ is acyclic. Since H has out-degree at least 2, at least one of w_0,z_0 must have out-degree at least 2 in K_H. By symmetry, we may assume that w_0 has two distinct out-arcs in K_H. Since $UG(K_H)$ is acyclic, there are paths starting with these arcs and terminating in w_1,z_1, and they have only w_0 in common. Therefore, K_H contains a (w_0,w_1)-path P. Together with x_0, P forms a cycle in H disjoint from the cycle of $UG(H)$ formed by z_0,x_1,x_2,x_3. So the digraphs in $\mathcal{K}$ are "yes"-instances, too.

Finally, let us consider a digraph H from $\mathcal{H}$ and let y_1, y_2, y_3, y_4, y_5, H_α, H_β, H_γ be as in the definition. Observe that if P is an (y_i,y_j)-path in some H_δ then $\delta\in\{\alpha,\beta,\gamma\}$ is determined by i,j unless $(i,j)=(4,2)$. Since $\{y_1,y_2\}$ is not a cycle transversal, there must be cycle avoiding all vertices and arcs of H_γ except possibly y_3,y_4; this cycle must contain subpaths P

from H_α and Q from H_β. Furthermore, as every such P must terminate in y_5 and every such Q must terminate in y_3, the cycle contains only one such P,Q, respectively, and $P =: P_{35}$ is a (y_3,y_5)-path in H_α whereas $Q =: P_{53}$ is a (y_5,y_3)-path in H_β. Similarly, since neither of $\{y_3,y_4\},\{y_4,y_5\},\{y_5,y_2\}$ are cycle transversals, we find paths $P_{51},P_{15},P_{23},P_{32},P_{14},P_{41}$, where P_{ij} is a (y_i,y_j)-path in the respective $H_{\delta(i,j)}$. Since y_2 has out-degree at least 2 and H_γ is acyclic, there is a path Q from y_2 to one of y_3,y_4 in H_γ which avoids the initial arc of P_{23}. If Q terminates in y_3 then it must intersect P_{14} by the properties of H_γ. Thus we find, in either case, two (y_2,y_4)-paths Q',Q'' with distinct initial arcs. It follows that $UG(H_\gamma)$ contains a cycle avoiding y_1,y_3, and this cycle is disjoint from the cycle in D formed by $P_{35} \cup P_{53}$ (or by $P_{51} \cup P_{15}$). ∎

The results above obviously extend to the case of arbitrary digraphs: Suppose D is any digraph with $\tau(D) \geq 3$. Then it has at least a cycle B and hence at least one nontrivial strong component D', containing B. If it had another nontrivial strong component then we would take a cycle C from this. Otherwise $\tau(D') \geq 3$, and we apply Theorem 2.3 to D'.

3. Cycle transversal number 2

We start this section with the case of simple, 2-regular digraphs, i. e. simple digraphs where every vertex has in- and out-degree 2. Recall that the *square* of a digraph D is obtained from D by adding a single arc from a to b for all pairs $(a,b) \in V(D) \times V(D)$ such that there is an (a,b)-path of length 2 but no shorter (a,b)-path.

Theorem 3.1. *A simple 2-regular digraph D admits a cycle B in D and a cycle C in $UG(D)$ such that $V(B) \cap V(C)$ are disjoint if and only if D is not the square of an odd cycle.*

Proof. It is easy to see that the square of an odd cycle does not admit B,C as in the statement.

Conversely, suppose that B,C as in the statement do not exist. Then D must be strongly connected, and hence contains a cycle. Let us consider a shortest cycle $B = x_0x_1 \dots x_{\ell-1}x_0$ in D, and set $H := D - V(B)$. Then $UG(H)$ is a forest on $|V(D)| - \ell$ vertices. Let us denote by $c \geq 1$ the number of components of $UG(H)$. As D is 2-regular, every vertex of B is incident with 2 arcs not in B, and every component X of $UG(H)$ is incident with $2|X| + 2$ arcs not in H. Consequently, the number of arcs connecting some vertex from $V(B)$ to some vertex from $V(H)$ is equal to $2|V(B)|$ and, at the same time, equal to $2|V(H)| + 2c$, implying that $\ell = |V(D)|/2 + c/2$.

Let r be a vertex of degree at most 1 in $UG(H)$. By symmetry, we may assume that there are two arcs $e_1 \neq e_2$ from $V(B)$ to r and at least one arc, say, f, from r to some vertex in $V(B)$ (otherwise we reverse all arcs). If the endvertices of e_1, e_2 in $V(B)$ were not consecutive on B then $\ell \geq 4$ and we may assume without loss of generality that they are x_0 and x_i for some i with $2 \leq i \leq \ell-2$ and that x_j with $0 < i < j$ is the endvertex of f; but then $x_0 r x_j x_{j+1} \ldots x_{\ell-1} x_0$ is a cycle in D shorter than B, a contradiction. Hence we may assume after relabelling that there is an arc from each of x_0 and x_1 to r.

Let X denote the set of all vertices v in H such that there is an (r,v)-path in H. Since $D[X]$ is a tree rooted at r, we deduce that there are exactly $|X|+1$ arcs from some vertex in X to some vertex in $V(B)$. Observe that

$$\left.\begin{array}{l}\text{if } x_s \in V(B) \text{ is an out-neighbor of some vertex } y \text{ at}\\ \text{distance } k \text{ from } r \text{ in } D[X], \text{ then } s \leq k+2,\end{array}\right\} \tag{1}$$

because if we denote the (r,y)-path in $D[X]$ by P then $B' := x_0 P x_s x_{s+1} \ldots x_{\ell-1} x_0$ is a cycle of length $(\ell - s) + k + 2 = |V(B')| \geq |V(B)| = \ell$. Moreover,

$$\left.\begin{array}{l}\text{if, for some } \varepsilon \in \{0,1\},\ x_\varepsilon \text{ is an out-neighbor of a vertex}\\ y \text{ in } X \text{ at distance } k \text{ from } r \text{ then } k = |X|-1 \text{ and}\\ |X| = |V(H)| = |V(B)|-1,\end{array}\right\} \tag{2}$$

because, again, if we denote the (r,y)-path in $D[X]$ by P then $B' = x_\varepsilon P x_\varepsilon$ is a cycle of length $k+2$ so that $|X|+1 \geq k+2 \geq |V(B)| > |V(H)| \geq |X|$.

If $D[X]$ was not a path then all vertices in X have distance at most $|X|-2$ from r, so that their $|X|+1$ out-neighbors in $V(B)$ were vertices x_s with $s \leq |X|$ by (1); since any such x_s has only one in-neighbor in $V(H)$, we deduce that the $|X|+1$ out-neighbors of X in $V(B)$ are exactly the vertices $x_0, x_1, \ldots, x_{|X|}$, which contradicts (2).

So $D[X]$ is a path $r = r_0 r_1 \ldots r_{|X|-1}$. Again by (1), the $|X|+1$ out-neighbors of the r_i in $V(B)$ are vertices x_s with $s \leq |X|+1$. Since any such x_s has only one in-neighbor in $V(H)$, we deduce that x_ε is among them for some $\varepsilon \in \{0,1\}$. By (2) we know that $H = D[X]$ and $|X| = \ell - 1$. Hence *each* of x_0, x_1 has an in-neighbor y in X, and by (2) y must have distance $k = |X|-1$ from r, so $y = r_{|X|-1}$. For the remaining proof, we set $k := |X|-1$, hence

$$\text{there is an arc from } r_k \text{ to each of } x_0, x_1. \tag{3}$$

We prove by induction that

$$x_{i+2} \text{ is an out-neighbor of } r_i \text{ for all nonnegative } i < k. \tag{4}$$

Suppose (4) is true for all nonnegative i smaller than some nonnegative $i_0 < k$. As $D[X]$ is a path, r_{i_0} has an out-neighbor x_{j_0} in $V(B)$. As every x_j has only one in-neighbor in X and since, for $j < i_0+2$, this in-neighbor is either r_k by (3) or r_i for some $i < i_0$ by induction hypothesis, we see that $j_0 \geq i_0+2$. Since $x_0 r_0 r_1 \dots r_{i_0} x_{j_0} x_{j_0+1} \dots x_{\ell-1} x_0$ is a cycle of length $\ell - j_0 + i_0 + 2$ we deduce $j_0 \leq i_0 + 2$, proving (4).

Similarly, we prove by induction that

$$(5) \qquad r_i \text{ is an out-neighbor of } x_{i+1} \text{ for all positive } i \leq k.$$

Suppose (5) is true for all positive i smaller than some positive $i_0 \leq k$. x_{i_0+1} must have an out-neighbor r_{j_0} in X. As the in-neighborhood of r_0 is $\{x_0, x_1\}$ and the in-neighborhood of r_j for positive $j < i_0$ is $\{r_{j-1}, x_{j+1}\}$ by induction hypothesis, we see that $j_0 \geq i_0$. Since $x_1 x_2 \dots x_{i_0+1} r_{j_0} r_{j_0+1} \dots r_k x_1$ is a cycle of length $k - j_0 + i_0 + 2 = \ell - j_0 + i_0$, we deduce that $j_0 \leq i_0$, proving (5).

The definitions of $x_0 x_1 \dots x_{\ell-1} x_0$ and $r_0 r_1 \dots r_k$ together with (3), (4), and (5) determine D to be the square of the odd cycle $x_0 x_1 r_0 x_2 r_1 x_3 r_2 \dots x_{\ell-2} r_k x_{\ell-1} x_0$. ∎

To handle the case that the cycle transversal number equals 2, we employ Theorem 3.1 and the following theorem by Thomassen [10] (see also [9]).

Theorem 3.2. [10, cf. pp. 73/74] *Let D be a simple acyclic digraph with two sources x^+, y^+ and two sinks x^-, y^- where every vertex distinct from x^+, x^-, y^+, y^- have in- and out-degree at least 2 such that every (x^+, x^-)-path meets every (y^+, y^-)-path in D. Then D can be embedded in the 2-dimensional unit disc S such that x^+, y^+, x^-, y^- are mapped to $(-1,0)$, $(0,-1)$, $(+1,0)$, $(0,+1)$, respectively.*

From this we will get the main statement of this section, Theorem 3.3. To *split a vertex a into a^+, a^-* in some digraph D means to add two new vertices a^+, a^- to $D - a$ and maintaining all arcs, where an arc which previously terminated in a now terminates in a^- and an arc which previously started in a now starts in a^+.

Theorem 3.3. *Let D_0 be an intercyclic digraph with $\tau(D_0) = 2$ and $\delta(D_0) \geq 2$. Then there is a cycle B in D_0 and a cycle C in $UG(D_0)$ with $V(B) \cap V(C) = \emptyset$ if and only if D_0 is not among the following digraphs.*

(i) A complete digraph on 3 vertices (with arbitrary multiplicities).

(ii) A digraph obtained from a cycle Z on at least 3 vertices by adding a new vertex a and at least one arc from a to every $b \in V(Z)$ and at least one arc from every $b \in V(Z)$ to a.

(iii) A digraph obtained from a cycle Z of odd length ≥ 5 by taking its square and adding an arbitrary collection of arcs parallel to those of Z.

Proof. It is easy to see that the graphs in (i),(ii),(iii) do not admit B,C as in the statement.

Conversely, suppose that there are no cycles B, C as in the statement. Observe that an intercyclic digraph with cycle transversal number 2 must be loopless. If some vertex x in D_0 had only one out-neighbor, say, y, then every cycle of D_0 which contains x must contain y, too. Since $\tau(D_0) \geq 2$, we thus knew that there must be a cycle B in $D_0 - \{x,y\}$, and, since x has out-degree at least 2, $\{x,y\}$ would constitute a cycle in $UG(D_0)$ disjoint from B, contradiction. Hence we may assume that the underlying simple digraph D_1 of D_0 still has minimum out-degree at least 2, and, analogously, has minimum in-degree at least 2.

If D_1 is 2-regular then, by Theorem 3.1, it is the square of an odd cycle $C_0 = c_0c_1c_2\dots c_{\ell-1}$. Which arcs might have parallels in D_0? If $\ell = 3$ then all arcs might, so let us look at the case that $\ell \geq 5$. If the arc c_0c_2 would have multiplicity at least 2 then c_0c_2 constitutes a 2-cycle in $UG(D_0)$, disjoint from any cycle [5] formed by $c_1c_3c_4\dots,c_{\ell-1}$. Hence the arcs c_0,c_2 and, by symmetry, all arcs c_i,c_{i+2} (indices modulo ℓ) are simple. This proves the statement of the Theorem in the case that D_1 is 2-regular.

So we may assume that D_1 is not 2-regular. It suffices to prove that D_1 is a digraph obtained from a cycle on at least 3 vertices by adding a new vertex a and one arc from a to every b and one from b to a for every $b \in V(Z)$, as, for such a digraph, adding any set of arcs parallel to arcs incident with a will never generate B,C as in the statement, whereas adding at least one arc parallel to an arc not incident with a will do.

Obviously, $\tau(D_1) = \tau(D_0) = 2$. Let $\{x,y\}$ be a cycle transversal of D_1. Take D_1, split x into x^+ and x^- and y into y^+ and y^-, and call the resulting simple acyclic digraph D. By Theorem 3.2, D can be embedded in the 2-dimensional unit disc S such that x^+,y^+,x^-,y^- are mapped to $(-1,0),(0,-1),(+1,0),(0,+1)$, respectively.

A *cross through* z is a quadruple (P,Q,R,S), where P is an (x^+,z)-path, Q is a (y^+,z)-path, R is a (z,x^-)-path, S is a (z,y^-)-path, and P,Q,R,S have pairwise only z in common. Let us restate Lemma 3.1 of [10].

Claim 1. For every $z \in V(D) - \{x^+,x^-,y^+,y^-\}$, there is a cross through z.

For completeness we add a proof. Every $X \subseteq V(D) - \{x^+,y^+,x^-,y^-\}$ containing z must have more than one out-neighbor in D, as otherwise $D[X]$

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[5] Here it is important that $\ell \geq 5$.

had minimum out-degree at least 1 and, thus, contained a cycle – but D is acyclic. Thus, by Menger's theorem, there exist two $(z, \{x^+, y^+, x^-, y^-\})$-paths which have only z in common. Clearly, one of them is a (z, x^-)-path and the other one is a (z, y^-)-path, and we call them R, S, respectively. Similarly, there is an (x^+, z)-path P and a (y^+, z)-path Q which have only z in common, and since D is acyclic, each of P, Q has only z in common with each of R, S. Thus, (P, Q, R, S) is a cross through z, proving Claim 1. ∎

As we are aiming for a polynomial time algorithm that actually finds B, C as in the statement of the theorem provided that they exist, and as our arguments will be partly topological, it is relevant to find an embedding as above in polynomial time. There are various ways to achieve this. First, we may add 4 arcs to D forming a 4-cycle Z traversing x^+, y^+, x^-, y^- and add another arc e connecting x^+, x^- and then embed the result D'' in the plane in polynomial time. By Claim 1, the two triangles in $Z+e$ are non-separating and, thus, faces of the embedding of D''. Hence Z bounds a face in $D''-e$ which easily yields an embedding of D as wanted. An alternative, second way is to obtain D'' by adding Z to D as above and adding a new vertex w and a single arc connecting w to each vertex from z. The eight newly introduced arcs thus form a 4-wheel. We again embed D'' in the plane in polynomial time and modify this embedding in such that the four triangles become faces. This is possible, because if $\varDelta := uvw$ is such a triangle and there is a component F of $D'' - \varDelta$ not containing the arc in $Z - \{u, v\}$ then the neighbors of $V(F)$ are among u, v, so that we can reembed $V(F)$ and the arcs connecting it to u, v in a sufficiently small stripe around uv in the region incident with uv and *not* bounded by $\varDelta$. This again yields easily an embedding as required. The advantage of this alternative is that it easily generalizes and does not rely on the degrees in the given graph. A third way to actually obtain the embedding is described by Metzlar in [8].

Now let us fix such an embedding.

Claim 2. If (P, Q, R, S) is a cross through some z then $V(D) = V(P \cup Q \cup R \cup S)$.

Set $X := P \cup Q \cup R \cup S$ and suppose, to the contrary, that there is a vertex $z' \in V(D) - V(X)$. Then there is a cross (P', Q', R', S') through z' by Claim 1. Let r, s be the first vertex in $V(X)$ on R', S', respectively, and let p, q be the last vertex in $V(X)$ on P', Q', respectively. Then $P'' := P'[p, z']$, $Q'' := Q'[q, z']$, $R'' := R'[z', r]$, $S'' := S'[z', s]$ have length at least one, have pairwise only z' in common, and their end-vertices p, q, r, s different from z' are their respective unique vertices from X. By planarity, $\{p, q, r, s\}$ is contained in one of $P \cup Q$, $Q \cup R$, $R \cup S$, $S \cup P$. Therefore, there exists two distinct

x,y in $\{p,q,r,s\}-\{z\}$ such that $\{x,y\}$ is a subset of some $Z\in\{P,Q,R,S\}$. Let X'',Y'' be the unique objects from $\{P'',Q'',R'',S''\}$ which contain x,y, respectively. Then $(Z-\{z\})\cup X''\cup Y''$ contains a cycle in $UG(D)$ either disjoint from the (x^+,x^-)-path $P\cup R$ or disjoint from the (y^+,y^-)-path $Q\cup S$ – a contradiction. This proves Claim 2. ▮

Claim 3. If (P,Q,R,S) is a cross through some $z\in V(D)-\{x^+,x^-,y^+,y^-\}$ then P,Q,R,S are induced paths, and z has in- and out-degree 2.

Suppose that P is not an induced path. Since D is acyclic, there are no backward chords, so that there is an (x^+,z)-path P' such that $V(P')$ is a proper subset of $V(P)$. But then (P',Q,R,S) is a cross through z violating Claim 2. Similarly, Q,R,S are induced paths. As every in- or out-neighbor of z is in one of P,Q,R,S by Claim 2, the last part of the statement of Claim 3 follows ▮

Since we are assuming that D_1 is not 2-regular, one of x^+,y^+ must have more than two out-neighbors in D. Without loss of generality, we may assume that x^+ has. Let z be an out-neighbor of x^+ distinct from y^-. Since D_1 has no loops, $z\neq x^-$. By Claim 1, there exists a cross (P,Q,R,S) through z, and by Claim 3, P has length 1. We set $Z:=Q\cup S$.

Let t be the out-neighbor of z on R, and let z' be *any* out-neighbor of x^+ distinct from y^-. By planarity and Claim 2, $z'\in Z$, and by Claim 3, the out-neighbor $t_{z'}$ of z' distinct from its out-neighbor on Z is in $R(z,x^-)$. If it was distinct from t then $(x^+z',Z[y^+,z'],z'R[t_{z'},x^-],Z[z',y^-])$ would be a cross at z' not covering t, contradicting Claim 3. Hence $t_{z'}=t$ for all out-neigbors $z'\neq y^-$ of x^+.

Suppose, to the contrary, that $t\neq x^-$. Then by Claim 3, t has in-degree 2, so that there are at most two out-neighbors of x^+ distinct from y^-. As x^+ has more than two out-neighbors, y^- must be an out-neighbor of x^+, and there exist exactly two out-neighbors $z\neq z'$ distinct from y^-. Let v be an out-neighbor of y^+ distinct from the out-neighbor of y^+ on Z. By Claim 3 and planarity, $v\in R[t,x^-]$, but $v\neq t$ as z,z' are the only in-neighbors of t in D. It follows that the subdigraph formed by $y^+vR[v,x^-]$ and x^+y^- in D corresponds to a cycle in D_1 disjoint from the cycle of $UG(D)$ formed by $zt,z't$, and the subpath of Z connecting z,z', contradiction.

It follows that $D-\{x^+,x^-\}=Z$ and that every out-neighbor of x^+ in $D-y^-$ is an in-neighbor of x^-. As, by Claim 3, every vertex of $Z-y^+$ must have an in-neighbor in D not on Z, every such vertex is an out-neighbor of x^+. Using symmetry we see that every vertex of $Z-y^-$ is an in-neighbor of x^-, which yields the desired structure. ▮

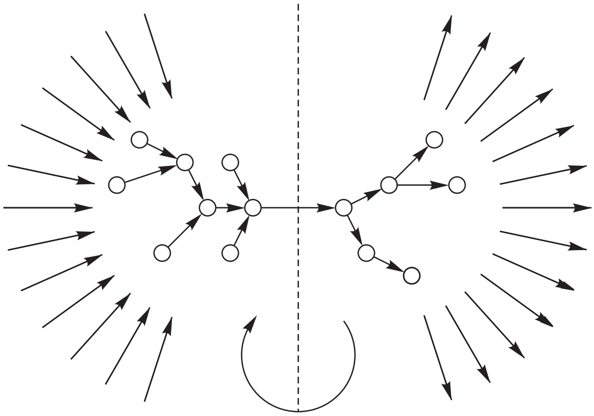

Figure 4. A typical display member P_v. Arcs not in $A(P_v)$ but incident with some vertex from $V(P_v)$ will start in its out-tree or terminate in its in-tree (or both). The in- and out-trees are displayed on the left or right hand side of the drawing, respectively. Instead of being adjacent as indicated, their respective roots might be the same ("thought of being on the dashed line").

4. Vaults and Wheels

In this section, we use Theorem 3.3 to characterize the strongly connected digraphs D of cycle transversal number 2 such that every cycle of D meets every cycle of $UG(D)$ (i. e. the "no"-instances of Problem 1.1). Our proofs can be transformed in polynomial time algorithms which decide if, for a given strongly connected input digraph D of cycle transversal number 2 there exists a cycle B in D and a cycle C in $UG(D)$ such that $V(B) \cap V(C) = \emptyset$, and finds them if the answer is "yes". We leave the details to the reader.

A *reduction* D' of a digraph D is obtained from D by contracting arcs e which are the unique out-arc at its initial vertex or the unique in-arc at its terminal vertex as long as it is possible. It is clear that every vertex v of the reduction D' either corresponds to a nonempty set of arcs which form a subdigraph P_v of D where P_v is connected in $UG(D)$, or is a vertex of D, forming the arcless digraph P_v; we call the family $(P_v)_{v \in V(D')}$ the *display* of the reduction. [6]

Lemma 4.1. *Let D be a strongly connected digraph without vertices of both in- and out-degree* 1. *Then*

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[6] We took the symbol P_v for the display members, as they turn out to be paths in many cases.

(i) there is only one reduction D', up to the labelling of the newly introduced vertices in the contraction process,

(ii) for its display $(P_v)_{v\in V(D')}$, each P_v is either the union of an in-tree L_0 and an out-tree R_0 which have only their root in common, or the union of an in-tree L_0 and an out-tree R_0 disjoint from R_0 plus an additional arc from the root of L_0 to the root of R_0, such that, in both cases,

(iii) every arc in $A(D)-A(P_v)$ starting in P_v starts in R_0 and every arc in $A(D)-A(P_v)$ terminating in P_v terminates in L_0. (See Figure 4.)

Proof. For a digraph D let $L(D)$ be the set of arcs which are the unique out-arc at its initial vertex and $R(D)$ be the set of arcs which are the unique in-arc at their terminal vertex. Let D/e denote the digraph obtained from D by contracting $e\in A(D)$ to a vertex v_e. To be more precise, D/e is the digraph with vertices $(V(D)-V(e))\cup\{v_e\}$, where v_e is a new vertex and with arcs $A(D)-\{e\}$, where the initial vertex of $f\in A(D)-\{e\}$ is the initial vertex of f in D if that one is not in $V(e)$ and v_e otherwise, and the terminal vertex of f is the terminal vertex of f in D if that one is not in $V(e)$ and v_e otherwise. Now let D be as in the statement, and let $e\in L(D)\cup R(D)$. It is easy to see that D/e is as in the statement again, and that $L(D/e)=L(D)-\{e\}$ and $R(D/e)=R(D)-\{e\}$, so that, by definition, there is only one reduction D' up to the labelling of the newly introduced vertices, and $\bigcup_{v\in V(D')}A(P_v)=R(D)\cup L(D)$.

The subdigraph L formed by the arcs of $L(D)$ cannot contain an induced cycle C, because then there is no arc from any vertex in $V(C)$ to any vertex $V(D)–V(C)$ – but there must be such an arc because D is strongly connected and $V(D)-V(C)$ is nonempty since D is not a cycle. Hence L is the disjoint union of in-trees. Similarly, the subdigraph R formed by the arcs of $R(D)$ is the disjoint union of out-trees. Let L_0 be a maximal in-tree in L, and let x_0 be its root. Suppose that f is an arc not in L_0 terminating in $y\in V(L_0)$. If y is a non-leaf of L_0 then f is not the unique in-arc at y, and the same holds if y is a leaf of L_0 because y then has out-degree 1 and, thus, in-degree at least 2. Hence f is not in R, and f is not in L by maximality of L_0. If f is an arc not in L_0 starting in $y\in V(L_0)$ then clearly $y=x_0$, and $f\notin L$. In particular, L_0 is an induced subdigraph of D. If f is an arc from y to x in $A(L_0)\cap A(R)$ then it is the unique in-arc at x so that no out-arc at x is unique, implying that f is the unique in-arc at x_0 in D. These considerations imply that L_0 cannot share vertices with more than one maximal out-tree R_0 in R and if there is such an R_0 then either L_0,R_0 have only x_0 in common or they share the unique in-arc f at x_0 in L_0, which is, at the same time, the unique out-arc at the root y_0 of R_0. Symmetrically, R_0 does not share vertices with maximal in-trees of L distinct from L_0. Since every P_v is either arcless or the union of maximal in-trees of in L and maximal out-trees in R and $UG(P_v)$ is connected, we see that P_v is either arcless or a maximal in-tree in L or a maximal out-tree in R or the union of a maximal in-tree L_0 in L and a maximal out-tree in R_0 in R which have either only their roots in common unless there is only one in-arc f at the root x_0 of L_0 which is at the same time the unique out-arc of the y_0 root of R_0 and $A(L_0)\cap A(R_0)=\{f\}$ and $V(L_0)\cap V(R_0)=\{x_0,y_0\}$. This implies the statement of the Lemma, where, in the latter case, we take P_v to be the union of L_0 and the out-tree $R_0-\{y_0\}$ rooted at x_0. ∎

From Lemma 4.1 one easily deduces that any two reductions of some (strongly connected) digraph D are equal up to the labelling of the newly introduced vertices and up to the choice of a surviving arc from (a) each maximal path whose interior vertices have in- and out-degree 1 in D, and from (b) each cycle containing at most one vertex not having in- and out-degree 1 in D. We feel encouraged enough to call any such reduction *the reduction* of D.

Let us define vaults and niches. Let $\ell \geq 5$ be odd, let $P_0,\dots,P_{\ell-1}$ be disjoint nonempty paths, and, for each $i \in \{0,\dots,\ell-1\}$, let a_i the initial vertex, d_i be the terminal vertex, and b_i,c_i be vertices of P_i such that either b_ic_i is an arc on P_i or $b_i=c_i\in\{a_i,d_i\}$. Suppose that D is obtained from the disjoint union of the P_i by

(i) adding at least one arc from some vertex in $P_i[c_i,d_i]$ to some vertex from $P_{i+1}[a_{i+1},b_{i+1}]$ (multiarcs may occur), and
(ii) adding a single arc from d_i to a_{i+2}, for all $i\in\{0,\dots,\ell-1\}$,

where the indices are taken modulo ℓ. Any digraph of such a form is called a *vault*. We say that the vault D has a *niche*, if there exist arcs pq,rs from P_i to P_{i+1} such that p occurs before r on P_i and q occurs after s on P_{i+1}. In that case, $a_iP_ip\ qP_{i+1}d_{i+1}\ a_{i+3}P_{i+3}d_{i+3}\ \dots\ a_{i-2}P_{i-2}d_{i-2}$ is a cycle of D, disjoint from the cycle of $UG(D)$ constituted by rs and the path $rP_id_i\ a_{i+2}P_{i+2}d_{i+2}\ a_{i+4}P_{i+4}d_{i+4}\ \dots\ a_{i-1}P_{i-1}d_{i-1}\ a_{i+1}P_{i+1}s$. Figure 5 shows a vault with $\ell=5$, where all paths $a_iP_ib_i$ or $c_iP_id_i$ have seven vertices; the grey areas indicate the set of arcs connecting $c_iP_id_i$ to $a_{i+1}P_{i+1}b_{i+1}$, a niche would correspond to a pair of arcs which can be drawn without crossing in such an area. Vaults are strongly connected. They may contain vertices of in- and out-degree 1, but, as they occur only as internal vertices of the P_i, we deduce that every vault D is a subdivision of a vault D' without vertices of in- and out-degree 1, where D' has a niche if and only if D has.

Theorem 4.2. *Let D be a strongly connected digraph with cycle transversal number* 2 *whose reduction is as in (iii) of Theorem 3.3. Then there is a cycle*

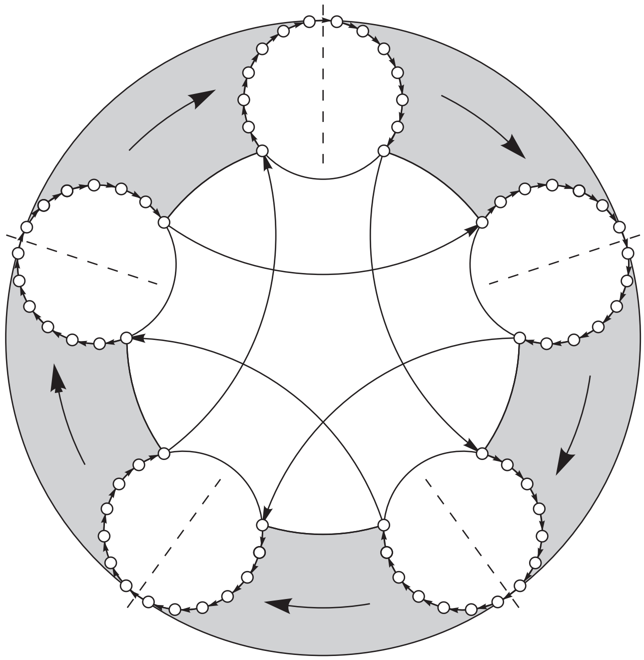

Figure 5. A typical vault. The five central arcs must have multiplicity 1 and are the only arcs from P_i to P_{i+2}.

B in D and a cycle C in $UG(D)$ with $V(B)\cap V(C)=\emptyset$ if and only if D is not a subdivision of a vault without a niche.

Proof. Without loss of generality, we may assume that there is no vertex of in- and out-degree equal to 1. Let $P_0,\dots,P_{\ell-1}$ be the display of the reduction (in cyclic order, according to the cycle Z in (iii) of Theorem 3.3). By Lemma 4.1, P_i is the union of an in-tree L_i rooted at b_i and an out-tree R_i rooted at c_i as described there.

Suppose first that there are no objects B,C as in the statement. We have to show that D is a vault without niches. Let x be any leaf of R_i. If the unique arc e from P_i to P_{i+2} does not start in x then there are two arcs from x to P_{i+1}, so that $UG(D[\{x\}\cup V(P_{i+1})])$ has a cycle disjoint from any directed cycle of D using e and "going around only once". Hence R_i is a path from, say, c_i to d_i, and there is a unique arc from d_i to P_{i+2}. Analogously, L_i is a path from, say, a_i to b_i, and there is a unique arc from P_{i-2} to a_i, so that D is a vault as described in the definition. We have seen that if there was a niche then there are B,C as in the statement.

Conversely, suppose that D is a vault without niches, certified as in the definition, and suppose, to the contrary, that there are B,C as in the statement. As C does not contain all vertices of D, it contains an arc rs from P_i to P_{i+1} for some i. B cannot contain the arc d_ia_{i+2} because then it would also contain b_id_i as a subpath and, hence, r. Similarly, B cannot contain $d_{i-1}a_{i+1}$. Hence B traverses both $b_iP_ic_i$ and $b_{i+1}P_{i+1}c_{i+1}$ and contains an arc pq from P_i to P_{i+1}. Consequently, p occurs before r on P_i, and q occurs after s on P_{i+1}. But this certifies a niche, contradiction. ∎

A *multiwheel* is a digraph obtained from a cycle Z of length at least 3 by adding a new vertex a and, for each $x\in V(Z)$, k arcs from a to x and ℓ arcs from x to a, where $k+\ell>0$. A *split multiwheel* is obtained from such a wheel by splitting a into a^+ and a^- and add a single arc from a^- to a^+.

Theorem 4.3. *Let D be a strongly connected digraph with cycle transversal number 2 whose reduction is as in (ii) of Theorem 3.3. Then there is a cycle B in D and a cycle C in $UG(D)$ with $V(B)\cap V(C)=\emptyset$ if and only if D is not a subdivision of either a multiwheel or a split multiwheel.*

Proof. Without loss of generality we may assume that D has no vertices of in- and out-degree 1. It is easy to check that multiwheels or split multiwheels D do not admit B,C as in the statement. Conversely, suppose that B,C as there do not exist. Let $P_0,\dots,P_{\ell-1},P_\ell$ be the display of the reduction, where P_ℓ corresponds to the center of the wheel. By Lemma 4.1, P_i can be described by in- and out-trees L_i,R_i as there. First look at P_i for some $i\in\{0,\dots,\ell-1\}$, and let x be a leaf of R_i. If the unique arc connecting P_i to P_{i+1} (indices modulo ℓ) would not start in x then there would be a cycle in $D-(\{x\}\cup V(P_\ell))$ and a cycle in $UG(D[\{x\}\cup V(P_\ell)])$ (since x has out-degree at least 2 in D). Hence R_i is a path, whose terminal vertex is the inital vertex of the arc from P_i to P_{i+1}. Analogously, L_i is a path whose inital vertex is the terminal vertex of the arc from P_{i-1} to P_i (indices modulo ℓ). Therefore, $D-V(P_\ell)$ is a cycle. Consider a leaf x of R_ℓ, and let $i\in\{0,\dots,\ell-1\}$. If there is an arc from some $y\in V(P_\ell)-\{x\}$ to P_i and there are two arcs from x to $D-(V(P_\ell)\cup V(P_i))$ then $D[V(P_\ell)\cup V(P_i)]$ contains a cycle B and $UG(D-V(P_\ell)\cup V(P_i))$ contains a cycle C, but we supposed that such B,C do not exist. This implies that every leaf of R_ℓ must be adjacent to every P_i, $i\in\{0,\dots,\ell-1\}$. Using the argument above with ends x,y of R_ℓ, we see that R_ℓ cannot have two leaves, and that R_ℓ is a path such that all arcs from R_ℓ to $V(D)-V(P_\ell)$ start at its terminal vertex. Analogously, L_ℓ is a path such that all arcs from $V(D)-V(P_\ell)$ to L_ℓ terminate at its initial vertex. From this it follows that D is a multiwheel or a split multiwheel, as desired. ∎

A *trivault* is obtained by six disjoint digraphs R_i, L_i, $i \in \{0,1,2\}$, where each R_i is either a nontrivial out-star with root b_i or a (b_i, x_i)-path and each L_i is either a nontrivial in-star with root c_i or a (y_i, c_i)-path, as follows:

(i) for each $i \in \{0,1,2\}$ either add a single arc from c_i to b_i or identify b_i, c_i,

(ii) for distinct $i, j \in \{0,1,2\}$, if R_i is a nontrivial out-star and L_j is a nontrivial in-star, add a single arc from each leaf of R_i to c_j and from b_i to every leaf of L_j and an arbitrary number of arcs (possibly 0) from b_i to c_j,

(iii) for distinct $i, j \in \{0,1,2\}$, if R_i is a nontrivial out-star and L_j is a path, select $v \in L_j$ and add a single arc from each leaf of R_i to v, at least one arc from b_i to y_j, and an arbitrary number of arcs (possibly 0) from b_i to each $z \in L_j[y_j, v]$,

(iv) similarly, for distinct $i, j \in \{0,1,2\}$, if R_i is a path and L_j is a nontrivial in-star, select $v \in R_i$ and add a single arc from v to each leaf of L_j, at least one arc from x_i to c_j, and an arbitrary number of arcs (possibly 0) from each $z \in R_i[v, x_i]$ to c_j, and

(v) if, for distinct $i, j \in \{0,1,2\}$, R_i, L_j are paths, then add at least one arc from x_i to some vertex of L_j, and at least one arc from some vertex of R_i to y_j, and add an arbitrary number of arcs (possibly 0) from each $z \in R_i$ to each $w \in L_j$.

We say that such a trivault has a *niche* if there are distinct $i, j, k \in \{0,1,2\}$ such that either

(a) R_i, L_j are paths and there are arcs pq, rs such that p occurs before r on R_i and q occurs after s on L_j, or

(b) R_i is a path, containing an in-neighbor x of L_k such that there are at least two arcs from $R_i(x, x_i]$ to L_j, or

(c) L_i is a path containing an out-neighbor y of R_k such that there are at least two arcs from R_j to $L_i[y_i, y)$.

Figure 6 shows a typical trivault. Allowing $\ell = 3$ in the definition of vaults on page 653 will produce other trivaults. Observe that every trivault is strongly connected. It might contain a vertex of in- and out-degree 1; however, this is either in some path $R_i - x_i$ or in some path $L_i - y_i$, and contracting any arc (on that path) incident with it produces, consequently, a smaller trivault, and it will have a niche only if the original one had a niche. Hence we can consider every trivault as a subdivision of a trivault without vertices of in- and out-degree 1, which has a niche if and only if the original one had.

Theorem 4.4. *Let D be a strongly connected digraph with cycle transversal number 2 whose reduction is as in (i) of Theorem 3.3. Then there is a cycle*

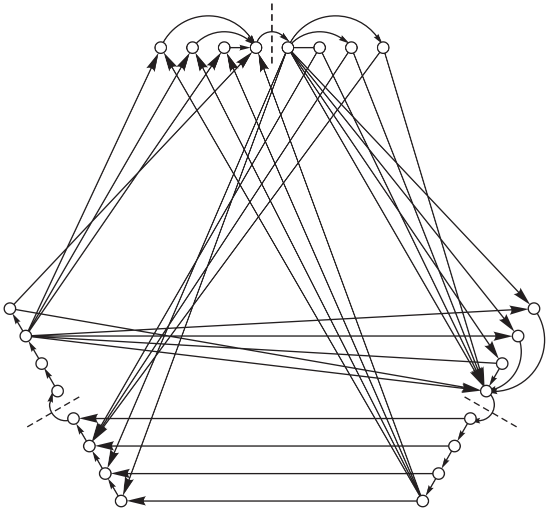

Figure 6. A typical trivault.
The dotted lines separate the two parts of each display member.

B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$ if and only if D is not a subdivision of a trivault without a niche.

Proof. Without loss of generality we may assume that D has no vertices of in- and out-degree 1.

Suppose first that D is a trivault as in the definition, without niches, and let us assume, to the contrary, there exist B, C as in the statement. Set $P_i := L_i \cup R_i$ for $i \in \{0,1,2\}$. As every cycle in D or $UG(D)$ which enters P_i via an in-arc, traverses along a subpath of P_i or $UG(P_i)$ and then exits P_i via an out-arc must traverse $P_i[c_i, b_i]$ it follows that B traverses at least two of the $P_i[c_i, b_i]$ and, for some i among these, some subpath of C enters *and* exits P_i in via out-arcs or via in-arcs (avoiding $P_i[c_i, b_i]$). Without loss of generality we may assume that it enters and exits P_i via out-arcs e, f. By definition of a trivault, R_i can only be a path. Also B must exit P_i via an out-arc pq. If one of e, f connect r in P_i to some vertex s on the L_j which contains q then p occurs before r on R_i because B traverses $P_i[c_i, b_i]$. Observe that B traverses $P_j[c_j, b_j]$, too. Hence, if L_j is a path, then q occurs after s on L_j, which yields a niche as in (a) of the definition, and if, otherwise, L_j is a star then $q = c_j$ and s is a leaf of L_j, so that, by definition, r is the unique in-neighbor of all the leaves of L_j, and all in-neighbors of c_j in P_i are on $R_i[r, x_i]$ – but p is not, contradiction. Hence we may assume that q is in some P_k, $k \in \{0,1,2\} - \{i\}$, whereas e, f connect $R_i(p, x_i]$ to P_j, where j is the one in $\{0,1,2\}-\{i,k\}$. But then we have a niche as described in (b) of the definition.

Conversely, suppose that B,C as in the statement do not exist. Let P_0,P_1,P_2 be the display of the reduction, and let L_i,R_i again be in- and out-trees as in Lemma 4.1. We deduce several claims on R_i, and once such a fact is proved the analogous statements for L_i obtained by exchanging the rules of L_j and R_j and the "directions of the objects" (in / out) is also true. For example, the analogous statement of the obvious fact that for $\{i,j,k\}=\{0,1,2\}$, every leaf of R_i is the tail of at least two arcs connecting to a vertex from $L_j \cup L_k$ is that every leaf of L_i is the head of at least two arcs connecting to a vertex from $R_j \cup R_k$.

Claim 1. Let $\{i,j,k\}=\{0,1,2\}$ and let X be a subdigraph of R_i such that $UG(X)$ is connected. If there are distinct e,f arcs from X to not necessarily distinct vertices in P_j then there is no path in $R_i-V(X)$ from the root of R_i to a vertex in the in-neighborhood of P_k. The analogous statement is true for L_i.

For if there was such a path from the root of R_i to an in-neighbor p of some vertex q in P_k then we take any arc from some r in P_k to some s in P_i and observe that $rP_i[s,p]P_k[q,r]$ would form a cycle B in D disjoint from the cycle in $UG(D)$ formed by e,f, and the subpaths in $UG(X),UG(P_j)$, respectively, connecting the end-vertices of e,f there. This proves Claim 1. ∎

Claim 2. Let $i\neq k$ in $\{0,1,2\}$ and let x be a leaf of R_i. Then

(i) x has at least one out-arc connecting it to P_k,
(ii) if there is another leaf y of R_i, then x (and y, respectively) has exactly one out-arc connecting it to P_k and the respective unique out-neighbors of x,y in P_k are equal, and
(iii) R_i has at most one vertex of out-degree larger than 1 in R_i.

The analogous statement is true for L_i.

Let $j\in\{0,1,2\}-\{i,k\}$. If (i) was not true then there were two distinct arcs e,f connecting x to P_j and an in-neighbor of P_k in R_i-x, which contradicts Claim 1. Hence (i) is true. If (ii) was not true then there are distinct e,f connecting x to P_k; by (i) – applied to y,j for x,k – y is an in-neighbor of P_j in R_i-x, which again contradicts Claim 1 (with k,j for j,k). Hence x has a unique out-neighbor u in P_k, and y has a unique out-neighbor v in P_k. By the definition of the reduction procedure, u and v are located in L_k; let w be the root of L_k. If u,v were distinct then either $L_k[u,w]$ does not contain v or $L_k[v,w]$ does not contain u, and, without loss of generality we may assume that $L_k[u,w]$ does not contain v. There is a path in $L_k - L_k[u,w]$ from some leaf z of L_k to v. There is an arc from some p in P_k to some q in P_i, so that $pP_i[q,x]P_k[u,p]$ forms a cycle B of D. By (i), applied to y,j for x,k, there is an arc from y to some a in P_j, and, by (i) applied to k,j,z,L_k for i,k,x,R_i, there is an arc from some b in P_j to z. Hence $yP_j[a,b]P_k[z,v]$ and the arc from y to v form a cycle in $UG(D)$ disjoint from B, which is not possible. This contradiction proves (ii). If (iii) was not true then there would exist three distinct leaves x,y,z in R_i such that the path in R_i connecting the root of R_i to z would be disjoint from the path X in $UG(R_i)$ connecting x,y; by (i), applied to j,x and j,y for k,x there were distinct arcs from $\{x,y\} \subseteq V(X)$ to P_j, and, by (i) applied to z for x, z is an in-neighbor of P_k. This contradicts Claim 1, which accomplishes the proof of Claim 2. ▮

Claim 3. Let $i \in \{0,1,2\}$, suppose that R_i is not a path and let r be the root of R_i. Then either R_i is a star with root r, or r has exactly one out-neighbor b in D and $R_i - r$ is a star with root b. The analogous statement holds for L_i.

Since R_i is not a path, there exists a vertex b of outdegree larger than 1 in R_i. By (iii) of Claim 2, all other vertices of R_i have out-degree 0 or 1. Suppose that x,y were distinct leaves and assume, to the contrary, that $R[b,x]$ had an inner vertex c. Then c must have an out-neighbor in P_j for some $j \neq i$. Let k be the index in $\{0,1,2\} - \{i,j\}$. As x has an out-neighbor in P_j by (i) of Claim 2, there are two arcs connecting $X := R_i[c,x]$ to P_j; there is an (r,y)-path in $R_i - V(X)$, so that y cannot be an in-neighbor of P_k by Claim 1, contradicting (i) of Claim 2. This shows that R_i is the union of $R_i[r,b]$ and a star with root b. If the statement of Claim 3 would fail then there existed a vertex $d \in R_i(r,b)$ which had an out-neighbor in P_k for some $k \in \{0,1,2\} - \{i\}$. Let j be the index in $\{0,1,2\} - \{i,k\}$. Then $X := R_i[\{b,x,y\}]$ forms a connected subgraph in $UG(D)$, and $R_i[r,d]$ would be disjoint from $R_i - V(X)$, again violating Claim 1. This proves Claim 3. ▮

Claim 4. Let $i \neq j$ be in $\{0,1,2\}$, a be the root of R_i, and b be the root of L_j.[7] Then there cannot be arcs pq, rs from P_i to P_j such that $R_i[a,p]$ does not contain r and $L_j[q,b]$ does not contain s.

For otherwise, we find a leaf x of R_i such that the path $R_i[r,x]$ is disjoint from $R_i[a,p]$ and a leaf y of L_j such that the path $L_j[y,s]$ is disjoint from $L_j[q,b]$. Let k be the one in $\{0,1,2\} - \{i,j\}$. x has an out-neighbor t in P_k, y has an in-neighbor u in P_k, and there is an arc from some v in P_j to some w in

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[7] Possibly R_i, L_j are paths, in which case a,b are the initial or terminal vertex, respectively.

P_i. Now $vP_i[w,p]P_j[q,v]$ forms a cycle in D disjoint from the cycle of $UG(D)$ formed by $R_i[r,x]P_k[t,u]L_j[y,s]$ and the arc rs. This proves Claim 4. ∎

Now we can finish the proof of the theorem. By the previous claims we see that each R_i, L_i is either as in Claim 3 or a path. If R_i, L_j are as in Claim 3 then, by (ii) of Claim 2 and the analogous statement for L_j, the unique out-neighbor of the leaves of R_i in L_j is the root of the star in L_j and the unique in-neighbor of the leaves in L_j is the root of the star in R_i. If R_i is as in Claim 3 and L_j is a path and x is the out-neighbor of R_i in L_j of minimal distance to the root of L_j, then, by Claim 4, x is the unique out-neighbor of the leaves of R_i in L_j. Similarly if R_i is a path and L_j is as in Claim 3. Hence D is a trivault, and, by Claim 1 and Claim 4, it has no niches as described in (b),(c) and in (a), respectively, of the definition. ∎

Corollary 4.5. *A strongly connected digraph with cycle transversal number* 2 *admits a cycle* B *in* D *and a cycle* C *in* $UG(D)$ *with* $V(B) \cap V(C) = \emptyset$ *if and only if* D *is not a subdivision of a vault without a niche, of a multiwheel, of a split multiwheel, or of a trivault without a niche.*

5. Cycle transversal number 1

Finally, we study the case where the cycle transversal number is 1 (for cycle transversal number 0, the answer in Problem 1.1 is obviously “no”). We prefix two simple Lemmas.

Lemma 5.1. *Let* H *be a simple bipartite graph with classes* A, B*, where* $d_H(x) \geq 2$ *for all* $x \in A$*. Then every cycle in* H *covers* B *if and only if* H *is a forest, or* $|B| = 2$ *and* H *is complete bipartite, or* H *is a cycle.*

Proof. The “if”-part is obvious. For the only-if part, suppose that every cycle of H covers B. We may assume that there exists a cycle $C = a_0b_0a_1b_1\ldots a_{\ell-1}b_{\ell-1}$ and that $\ell = |B| \geq 3$, for otherwise the statement is obvious. If there was a vertex in $x \in A - V(C)$ then we may assume, by symmetry, that x is adjacent to b_0, b_j for some $j \leq \ell/2$; but then the cycle $xb_0a_1b_1\ldots a_jb_jx$ misses b_{j+1}, as $j+1 \leq \ell/2+1 \leq \ell-1$, contradiction. So $A = \{a_0, \ldots, a_{\ell-1}\}$. If a_0 is adjacent to some b_j, where $j > 0$ and $j < \ell-1$, then the cycle $a_0b_0a_1b_1\ldots a_jb_ja_0$ misses b_{j+1}, contradiction, so that a_0, and, symmetrically, a_j for all $j \in \{0, \ldots, \ell-1\}$, has degree 2. ∎

Lemma 5.2. *Let* D *be an acyclic digraph and* $x \neq y$ *in* $V(D)$*. Then either the digraph obtained from* D *by adding a new arc from* x *to* y *or the digraph obtained from* D *by adding a new arc from* y *to* x *is acyclic.*

Proof. For otherwise, D contains a (y,x)-path and an (x,y)-path, and, hence, a cycle – contradiction. ∎

Let H be a subdigraph of some digraph D. An *H-bridge* of D is a subdigraph X either formed by a single arc of $A(D)-A(H)$ with both end-vertices in $V(H)$ (that may be a loop), or formed by the arcs incident with the vertices of some fixed component of $UG(D-V(H))$. The vertices in $V(X)\cap V(H)$ are the *vertices of attachment*, and the arcs in $A(X)$ incident with a vertex of $V(H)$ are the *arcs of attachment* of X.

Suppose that D is an acyclic digraph with a single source a and a single sink b. We now describe a polytime procedure by means of dynamic programming to answer the question "is there an (a,b)-path in D disjoint from some cycle in $UG(D)$?". Moreover, if the answer is "yes", then it gives back two objects certifying this. (The details of the latter part are left to the reader.)

As long as it is possible we repeat the following preprocessing step: If there is a vertex of both in- and out-degree 1 we contract an arc incident with it, and if there is an arc incident with a or b having a parallel then we delete it. The resulting graph D_3 has an (a,b)-path disjoint from some cycle in $UG(D_3)$ if and only if the original graph D has an (a,b)-path disjoint from some cycle in $UG(D)$. Hence we may assume that D has neither vertices of both in- and out-degree 1 nor parallel arcs incident with a or b.

If $V(D)=\{a,b\}$ then the answer to our question is obviously "no". Observe that for every $x\in V(D)-\{a,b\}$, there exists an (a,x)- and an (x,b)-path. Therefore, if there is an (a,b)-separator consisting of a single vertex c, then D is the union of the digraph D_1 formed by all vertices x for which there exists an (a,x)- and an (x,c)-path and the digraph D_2 formed by all vertices x for which there exist a (c,x)- and an (x,b)-path; D_1,D_2 have only c in common (as D is acyclic), and there exists an (a,b)-path in D disjoint from some cycle in $UG(D)$ if and only if there exists either an (a,c)-path P_1 in D_1 disjoint from some cycle in $UG(D_1)$ or a (c,b)-path P_2 in D_2 disjoint from some cycle in $UG(D_2)$ (or both). Hence we can reduce the problem to two smaller problems whose sizes add up to the size of the original problem.

So let us assume that the local connectivity $\kappa:=\kappa(a,b)$ from a to b in D is at least 2. Let $\mathcal{P}$ be a set of κ distinct internally disjoint (a,b)-paths. We modify $\mathcal{P}$ in such a way that, for each path from $\mathcal{P}$, the only arcs in D connecting the sink or source to one of its interior vertices are its initial- and terminal arc, respectively.

Let us consider a $\bigcup\mathcal{P}$-bridge X. If X consists of a loop then the answer to our question is obviously "yes". If X consists of an arc xy where x,y are on the same path P from $\mathcal{P}$ then $(x,y)\neq(a,b)$ by maximality of $|\mathcal{P}|=\kappa$, so that, by our inital modification of $\mathcal{P}$, both x,y would be interior vertices of

P, implying that the answer is "yes" as well. If X has two arcs of attachment whose end-vertices are interior vertices of the same path P then the answer is also "yes", and the same holds if $UG(X-\{a,b\})$ contains a cycle. Hence $UG(X-\{a,b\})$ is a tree, and, for every path P in $\mathcal{P}$, at most one arc of attachment of X connects to a vertex of $V(P)-\{a,b\}$.

Suppose, to the contrary, that, at this stage of the procedure, there still was a path P in $\mathcal{P}$ such that all vertices of attachment of X are in $V(P)$. Then X has at least 2 arcs (for otherwise we would have answered "yes" before). Let $e \in A(X)$. Since $X-\{a,b\}$ is a tree and since every vertex in $V(D)-\{a,b\}$ has a positive in- and out-degree, there exists a path P_e in X whose initial arc is e and whose terminal vertex y_e is a vertex of attachment of X, and there exists a path Q_e in X whose initial vertex x_e is a vertex of attachment of X and whose terminal arc is e. Since D is acyclic, Q_e, P_e have no vertices from $V(X)-V(e)$ and no arcs from $A(X)-\{e\}$ in common, and $R_e := Q_e \cup P_e$ is an (x_e, y_e)-path in X. R_e has length at least 2, as $|A(X)| \geq 2$. Since not both of the two end-arcs of R_e connect to vertices from $V(P)-\{a,b\}$, we either have $x_e = a$ or $y_e = b$, and, by maximality of $|\mathcal{P}|$, $(x_e, y_e) \neq (a,b)$. By symmetry, we therefore may assume, for some fixed $e \in A(X)$, that $x_e = a$ and $z := y_e \in V(P)-\{a,b\}$. Let g be the terminal arc of R_e and consider any $f \in A(X)-\{e\}$. If $y_f = b$ then the initial arc of R_f would connect to a vertex from $V(P)-\{a,b\}$ – so it would be equal to g, which is not possible. Hence $x_f = a$, implying that the terminal arc of R_f is g. Hence, for every arc $f \in A(X)$, R_f is an (a,z)-path whose terminal arc is g. In particular, a, z are the only vertices of attachment of X. If there were distinct f, f' with the same tail $c \in V(X)-\{a,z\}$ then $UG(P_f \cup P_{f'})$ would contain a cycle, which is impossible. Hence every vertex in $V(X)-\{a,z\}$ has out-degree 1 in D, implying that $X-\{a\}$ is an in-tree rooted at z, with at least one leaf d. Since the in-degree of d is at least 2, there must be distinct parallel arcs from a to d – but that has been ruled out before.

Let $\mathcal{X}$ be the set of all $\bigcup\mathcal{P}$-bridges, and let H be the bipartite graph with classes $\mathcal{X}, \mathcal{P}$, where there is an arc connecting $X \in \mathcal{X}$ and $P \in \mathcal{P}$ if $V(P)-\{a,b\}$ contains a vertex of attachment of X. As we have just seen, every $X \in \mathcal{X}$ has degree at least 2 in H. If there is a cycle in H not containing some $P \in \mathcal{P}$ then we can easily find a cycle C in $UG(D)$ disjoint from P, so that the answer is "yes" in this case.

Conversely, if there is a cycle C in $UG(D-\{a,b\})$ we may write it as $C = P_0' \cup X_0' \cup P_1' \cup X_1' \dots P_{\ell-1}' \cup X_{\ell-1}'$, where P_i' is a subpath of some $P_i \in \mathcal{P}$, X_i' is a subpath of the underlying graph of some $X_i \in \mathcal{X}$, and $X_i \neq X_{i+1}$ and $P_{i-1} \neq P_i$ since X_i has only one arc of attachment connecting to $V(P_i)-\{a,b\}$.

Therefore, the subgraph formed by the P_i, Q_i in H contains a cycle. These considerations imply that if H is a forest then the correct answer is “no”.

Hence we are down to the case that H is not a forest and every cycle of H covers $\mathcal{P}$. By Lemma 5.1, either $|\mathcal{P}| = 2$ and H is complete bipartite, or $|\mathcal{P}| \geq 3$ and H is a cycle. In either case, every $X \in \mathcal{X}$ has degree 2 in H; in particular, every $X \in \mathcal{X}$ has exactly two vertices of attachment in $V(D) - \{a,b\}$, and they are on distinct paths.

First look at the case that $\kappa = |\mathcal{P}| \geq 3$; we claim that the correct answer is always “no”. Let C be any cycle in $UG(D - \{a,b\})$. Since every X_i has only two vertices of attachment, C traverses every X_i only once, and it traverses every path (exactly) once. It follows that C contains all vertices of attachment distinct from a,b of all bridges in $\mathcal{X}$. Since C contains vertices of every path in $\mathcal{P}$ we see that an (a,b)-path P in $D - V(C)$ necessarily contains a vertex $x \in V(X) - \bigcup \mathcal{P}$ for some $X \in \mathcal{X}$; as P cannot contain a vertex of attachment of X distinct from a,b, P is internally disjoint from all members of $\mathcal{P}$, contradicting the maximality of $|\mathcal{P}|$. Hence such a path cannot exist, proving that “no” is the correct answer.

So we are left with the case that $\kappa = 2$. We show that in this case we can either answer correctly or reduce to a smaller instance unless all $\bigcup \mathcal{P}$-bridges consist of exactly one arc. Let P_1, P_2 be the two paths in $\mathcal{P}$. Let X be a bridge with more than one arc. Then it has exactly two arcs of attachment $e_1 \neq e_2$, where e_i connects to an interior vertex of P_i for $i \in \{1,2\}$.

First suppose that e_1 terminates in P_1 and e_2 terminates in P_2. Then the unique path Q in $UG(X - \{a,b\})$ with end-arcs e_1, e_2 must contain a vertex x with two out-arcs on Q. Let Q_1, Q_2 be paths in X starting with these arcs, respectively, and terminating in a vertex of attachment. There is a path R in X terminating in x and starting at some vertex of attachment, which clearly must be a. Hence Q_1, Q_2 cannot terminate in b (by maximality of $\mathcal{P}$), so that we may assume, without loss of generality, that the terminal arc of Q_i is e_i for $i \in \{1,2\}$. It follows that Q is the union of Q_1 and Q_2. If there was any vertex z in $X - \{a,b\} - V(Q)$ at all then we may choose it in such a way that it is a leaf of the tree $UG(X - \{a,b\})$; again, as for x, there is an (a,z)-path in X, so that there is no arc from z to b (maximality of $\mathcal{P}$); consequently, the only out-arc at z connects it to its neighbor in $X - \{a,b\}$, so that there are at least two arcs from a to z in X (because we suppressed vertices of in- and out-degree 1). But this has been ruled out earlier. It follows that R is formed by the unique arc from a to x. Now the maximality of $\mathcal{P}$ implies that there is no arc from any inner vertex of Q to b, so that Q_1 and Q_2 are distinct paths and there must be exactly one arc from a to each inner vertex of Q. This determines X (cf. Figure 7). Let x_i be the terminal vertex

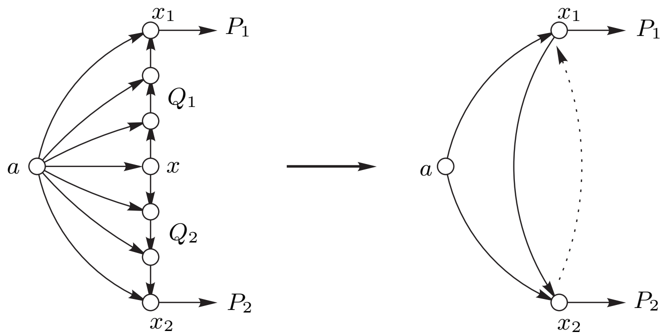

Figure 7. Reducing a nontrivial bridge I. The arrows from a to x_1,x_2 indicate paths of length 1 or 2, the dotted arc is an alternative, to ensure that the reduction is acyclic.

of e_i, for $i \in \{1,2\}$. If two distinct arcs of attachment of some bridges X_1,X_2, respectively, would connect to a vertex of $P_1(a,x_1)$ then X_1,X_2 were distinct and $P_1(a,x_1) \cup X_1 \cup X_2 \cup P_2(a,b)$ contained a cycle of $UG(D)$ disjoint from the (a,b)-path $aQ_1[x,x_1] \cup P_1[x_1,b]$. Hence we may assume that $P_1(a,x_1)$, and, symmetrically, $P_2(a,x_2)$ each connects to at most one arc of attachment in total (in particular, they each have at most one vertex, because we suppressed vertices of in- and out-degree 1). $D' := D - (V(X) - \{x_1,x_2,a\})$ is acyclic, and, by Lemma 5.2, we may add a new arc to D' connecting $x_1 \neq x_2$ the two vertices of attachment of X in D distinct from a such that the resulting digraph D_4 is acyclic (see again Figure 7). If D had an (a,b)-path P disjoint from some cycle C of $UG(D)$ then so does D_4 unless P contains some vertex from $V(X) - \{x_1,x_2,a\}$; but then $aQ_i[x,x_i]$ is an initial segment of P for some $i \in \{1,2\}$, so that C is disjoint from $X - \{x_1,x_2,a\}$ and from $P_i(a,x_i)$ as there is at most one arc of attachment in $P_i(a,x_i)$; hence $P_i[a,x_i]P[x_i,b]$ is an (a,b)-path in D_4 disjoint from C. Conversely, if D_4 contains an (a,b)-path P disjoint from some cycle C in $UG(D_4)$ then so does D, unless P contains the newly added arc; in that case, C is disjoint from $P_1(a,x_1)$ and from $P_2(a,x_2)$, and supposing that the new arc terminates in x_j, $j \in \{1,2\}$, we deduce that $P_j[a,x_j]P[x_j,b]$ is an (a,b)-path in D, disjoint from C. So we reduced to a smaller instance.

Hence we may assume that not both e_1,e_2 terminate in P_1,P_2, and, similarly, not both e_1,e_2 start in e_1. Without loss of generality, suppose that e_1 terminates in a vertex $d \in V(P_1)$ and e_2 starts in a vertex $c \in V(P_2)$. Let A be the set of vertices x for which there exists an (a,x)-path in X, and let B be those x for which there is an (x,b)-path in X. These sets are disjoint by maximality of $\mathcal{P}$. Let $x \in V(X - \{a,b,c,d\})$ be a vertex of out-degree at least

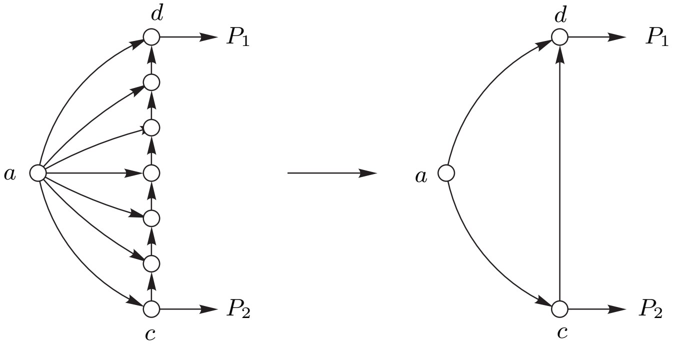

Figure 8. Reducing a nontrivial bridge II.
The arrows from a to c,d indicate paths of length 1 or 2.

2. Then there are two paths in X starting with distinct out-arcs at x and terminating in distinct vertices of attachment. It follows that at least one of them has to terminate in b, so that $x\in B$. Similarly, if x has in-degree at least 2 then it must be in A. Let Q be the unique path in $UG(X-\{a,b\})$ with end arcs e_1,e_2. If there was a vertex z in $A-\{a\}-V(Q)$ then we may choose it as a leaf of the tree $UG(X-\{a,b\})$; since z is incident with at least three arcs but not adjacent to b or to one of the vertices of attachment in e_1,e_2, there must be two arcs from a to z, which has been ruled out before, contradiction. Similarly, there cannot be a vertex in $B-\{b\}-V(Q)$, implying that $UG(X-\{a,b\})=Q$.

If B is empty then it follows from the above consideration of vertices in $V(X-\{a,b,c,d\})$ that Q is a path in D and there is a single arc from a to each inner vertex of Q; this determines X (cf. Figure 8). Since, for any inner vertex x of Q, $aQ[x,d]P_1[d,b]$ is an (a,b)-path in D, we are done unless $P_1(a,d)$ connects to at most one arc of attachment in total, just as in the prepreceeding paragraph. If the digraph D_5 obtained from $D-(V(Q)-\{c,d\})$ by adding a new arc from c to d (see Figure 8) is a “yes”-instance to our problem then so is D. Conversely, if D admits an (a,b)-path P disjoint from some cycle C in $UG(D)$ then so does D_5, unless P contains a vertex from $V(Q)-\{c,d\}$ and Q is not a subpath of P; but then the initial segment of P is $axQd$ for some $x\in V(Q)-\{c,d\}$, C is disjoint from $V(Q)-\{c,d\}$, and C is disjoint from $P_1(a,d)$ since $P_1(a,d)$ has at most one arc of attachment; hence $P_1[a,d]\cup P[d,b]$ is an (a,b)-path in D_5 disjoint from the cycle C in $UG(D_5)$. So we reduced to a smaller instance if $B=\emptyset$.

Analogously, we can reduce if $A=\emptyset$. Hence we may assume that both A and B are not empty, so that X contains a,b and at least one neighbor of each of A,B. Hence there must be an arc in Q distinct from e_1,e_2 which

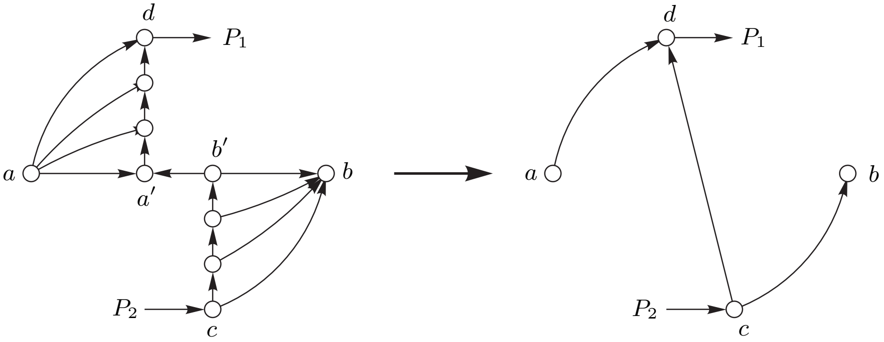

Figure 9. Reducing a nontrivial bridge III.
The arrows from a to c,d indicate paths of length 1 or 2.

connects some $a' \in A$ and some $b' \in B$. Clearly, this must be an arc from b' to a'. Now a' has in-degree at least 2 and out-degree 1, so that its neighbor on Q distinct from b' is an out-neighbor and, therefore, belongs to A, too. It follows by symmetry and inductive application of this argument that $Q_1 := Q[(A-\{a\}) \cup \{d\}]$ is an (a',d)-path in D and that $Q_2 := Q[(B-\{b\}) \cup \{c\}]$ is a (c,b')-path in D; there is a single arc from a to each vertex of $Q_1 - d$ and a single arc from each vertex of $Q_2 - c$ to b, so that X is determined (cf. Figure 9). Let D_6 be obtained from $D - (V(Q) - \{c,d\})$ by adding a new arc from c to d. As above, we are done unless $P_1(a,d)$ and $P_2(c,b)$ connect to at most one arc of attachment. If D_6 is a "yes"-instance of our problem then so is D. Conversely, if D has an (a,b)-path P disjoint from some cycle $C \in UG(D)$ then so has D_6, unless P contains a vertex from $V(Q) - \{c,d\}$ and P does not contain Q as a subpath. But then either (i) $aQ[x,d]$ is an initial segment of P for some $x \in A - \{a,d\}$, or (ii) $Q[c,y]b$ is a terminal segment of P for some $y \in B - \{b,c\}$. In either case, C is disjoint from $Q - \{c,d\}$. If (i) holds then P does not contain vertices from $cQx - x$ since D is acyclic, so that $P_1[a,d] \cup P[d,b]$ is an (a,b)-path in D_6, and, as above, is disjoint from C. If (ii) holds then, analogously, $P[a,c] \cup P_2[c,b]$ is an (a,b)-path in D_6, disjoint from C. Hence we reduced to a smaller instance.

Hence we may assume that every arc not on $P_1 \cup P_2$ connects an interior vertex of P_1 and one of P_2. Let c_i be the out-neighbor of a on P_i for $i \in \{1,2\}$. If c_1, c_2 are connected by an arc e then it is easy to see that D is a "yes"-instance if and only if $D - e$ is; so we may reduce in this case.

There must be an arc e_1 connecting c_1 and some d_2 on $P_2 - \{a,b\}$ and an arc e_2 connecting c_2 and some d_1 on $P_1 - \{a,b\}$, and we know that $d_i \neq c_i$ for $i \in \{1,2\}$ and $e_1 \neq e_2$ as c_1, c_2 are not adjacent. We take e_1, e_2, d_1, d_2 in such a way that the distance from a to d_i on P_i is minimum for $i \in \{1,2\}$. At least one

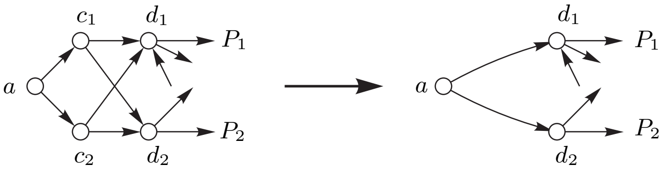

Figure 10. Reducing when e_i starts in c_i for $i \in \{1,2\}$.

of e_1, e_2 terminates in d_1 or d_2, respectively, since G is acyclic. Without loss of generality, we may assume that e_1 is from c_1 to d_2. If $P_2(a,d_2)$ connects to more than one arc of attachment then there exists a cycle in $UG(D)$ disjoint from the (a,b)-path $ac_1P_2[d_2,b]$, so that the correct answer is "yes". Hence we may assume that d_2 is the out-neighbor of c_2 on P_2 and that e_2 is the only arc of attachment at c_2.

If e_2 is from c_2 to d_1 then, analogously, "yes" is the correct answer unless d_1 is the out-neighbor of c_1 and e_1 is the only arc of attachment at c_1; in this case it is easy to see that D is a "yes"-instance if and only if the graph D^- obtained from $D - \{c_1, c_2\}$ by adding an arc from a to each of d_1, d_2 is a "yes"-instance (see Figure 10).

Hence we finally may assume that e_2 is an arc from d_1 to c_2. Obviously, if the contraction D/c_2d_2 is a "yes"-instance then so is D. However, the converse holds, too, for if P is an (a,b)-path in D disjoint from some cycle C in $UG(D)$ then not both $V(P)$ and $V(C)$ can intersect $\{c_2, d_2\}$, because if P contained c_2 then it must contain d_2 and if P does not contain c_2 but d_2 then c_2 has at most one neighbor in $D - V(P)$. Hence we reduced the problem to a smaller instance in this case, too.

This accomplishes the description of the procedure, and proves the following (details for actually finding the desired objects are left to the reader). Now if $\{a\}$ is a cycle transversal of a digraph D then the digraph D' obtained from D by splitting a into a^+, a^- is acyclic, and there exists an (a^+, a^-)-path P and a cycle in $UG(D')$ disjoint from P if and only if there exists a cycle B in D and a cycle in $UG(D)$ disjoint from B. Hence we proved the following.

Theorem 5.3. *There is a polynomial time algorithm for deciding if for a given strongly connected digraph D of cycle transversal number 1 there is a cycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and to find such B, C if the answer is "yes".*

Summarizing the results of the preceeding paragraphs and taking into account that McCuaigs algorithm [6, Section 9] finds, in polynomial time, either two disjoint cycles in a given digraph or determines its cycle transversal number, we deduce that there is a polynomial time algorithm for the decision Problem 1.1, which, in addition finds B, C if the answer is "yes".

Note added in proof

Together with Alessandro Maddaloni and Sven Simonsen we meanwhile proved, that the problem of deciding whether a (not necessarily strongly connected) digraph admits cycles B, C as in Problem 1.1 is $\mathcal{NP}$-complete. Moreover, we have constructed polynomial time algorithms for the case that the cycle transversal number is not equal to 1 and the case that the cycle transversal number is 1 but the number of cycle transversals is bounded by a constant [arXiv:1106.5885v1]. These algorithms heavily rely on the results of the present paper.

References

[1] J. Bang-Jensen and G. Gutin: *Digraphs. Theory, algorithms and applications,* Second edition, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London (2009).

[2] J. Bang-Jensen and M. Kriesell: Disjoint directed and undirected paths and cycles in digraphs, *Theoret. Comput. Sci.* **46–49** (2009), 5138–5144.

[3] G. A. Dirac: Some results concerning the structure of graphs, *Canad. Math. Bull.* **6** (1963), 183–210.

[4] S. Fortune, J. Hopcroft and J. Wyllie: The directed subgraph homeomorphism problem, *Theoret. Comput. Sci.* **10** (1980), 111–121.

[5] L. Lovász: On graphs not containing independent circuits (in Hungarian), *Mat. Lapok* **16** (1965), 289–299.

[6] W. McCuaig: Intercyclic digraphs, Graph structure theory (Seattle, WA, 1991), *Contemp. Math.* **147**, Amer. Math. Soc., Providence, RI (1993), 203–245.

[7] A. Metzlar: *Minimum transversal of cycles in intercyclic digraphs,* PhD thesis, University of Waterloo, Ontario, Canada (1989).

[8] A. Metzlar: Disjoint paths in acyclic digraphs, *J. Combin. Theory B* **57** (1993), 228–238.

[9] C. Thomassen: Disjoint cycles in digraphs, *Combinatorica* **3** (1983), 393–396.

[10] C. Thomassen: The 2-linkage problem for acyclic digraphs, *Discrete Math.* **55** (1985), 73–87.

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