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Theoretical Computer Science

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ARTICLE INFO

Communicated by Arnaud Casteigts

Keywords:

Flows
 (Arc-)disjoint paths with prescribed end vertices
 Acyclic digraph
 Polynomial time algorithm
 NP-complete problem
 Approximation algorithm
 Parameterised complexity

ABSTRACT

The **support** of a flow x in a network is the subdigraph induced by the arcs uv for which $x(uv) > 0$. We discuss a number of results on flows in networks where we put certain restrictions on structure of the support of the flow. Many of these problems are NP-hard because they generalize linkage problems for digraphs. For example deciding whether a network $\mathcal{N} = (D, s, t, c)$ has a maximum flow x such that the maximum out-degree of the support D_x of x is at most 2 is NP-complete as it contains the 2-linkage problem as a very special case.

Another problem which is NP-complete for the same reason is that of computing the maximum flow we can send from s to t along p paths (called a maximum p -**path-flow**) in \mathcal{N} . Baier et al. (2005) gave a polynomial time algorithm which finds a p -path-flow x whose value is at least $\frac{2}{3}$ of the value of a optimum p -path-flow when $p \in \{2, 3\}$, and at least $\frac{1}{2}$ when $p \geq 4$. When $p = 2$, they show that this is best possible unless $P=NP$. We show for each $p \geq 2$ that the value of a maximum p -path-flow cannot be approximated by any ratio larger than $\frac{9}{11}$, unless $P=NP$. We also consider a variant of the problem where the p paths must be disjoint. For this problem, we give an algorithm which gets within a factor $\frac{1}{H(p)}$ of the optimum solution, where $H(p)$ is the p 'th harmonic number ($H(p) \sim \ln(p)$). We show that in the case where the network is acyclic, we can find such a maximum p -path-flow in polynomial time for every p .

We determine the complexity of a number of related problems concerning the structure of flows. For the special case of acyclic digraphs, some of the results we obtain are in some sense best possible.

1. Introduction

Flows in networks form a very useful tool for modelling and solving many practical optimization problems. The maximum (s, t) -flow problem (where we seek a maximum flow from a source s to a sink t) is very well studied and many efficient algorithms are known (for an extensive collection of results on flows, see [1]). This combined with the Max Flow Min Cut theorem which characterizes the value of a maximum (s, t) -flow has led to efficient algorithms for numerous optimization problems. Maximum flow

[☆] This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.

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¹ Research supported by the Independent Research Fund Denmark under grant number DFF 7014-00037B.

² Financial support: DIGRAPHS ANR-19-CE48-0013.

³ Financial supports: DIGRAPHS ANR-19-CE48-0013 and EUR DS4H Investments ANR-17-EURE-0004.

<https://doi.org/10.1016/j.tcs.2024.114702>

Received 2 October 2023; Received in revised form 15 May 2024; Accepted 17 June 2024

Available online 21 June 2024

0304-3975/© 2024 Published by Elsevier B.V.

algorithms are also often used as subroutines in algorithms for more complicated - not necessarily - polynomial time algorithms for solving complex optimization problems. Another very nice and useful property of flows is that any flow, no matter how we choose the capacities (including choosing irrational numbers) can be decomposed into a linear number of much simpler flows, called **path-flows** and **cycle-flows** (see e.g. Chapter 4 in [4]).

There are, however, many problems concerning flows which cannot be solved using standard flow algorithms. The purpose of this paper is to study the complexity of a number of such problems all of which have a natural interpretation. All flows in this paper will be integer valued, that is, the flow on every arc is a non-negative integer. One problem that we study is the so-called maximum p -splittable flow problem [2] where the goal is to find the maximum value of an (s, t) -flow x which can be decomposed into p , not necessarily arc-disjoint, path-flows. When $p = 2$, Baier et al. [2] showed that problem is already NP-hard and cannot be approximated better than a factor of $\frac{2}{3}$, unless $P = NP$. They also gave a ρ -approximation algorithm for the p -splittable flow problem, with flows valued in \mathbb{R} , where $\rho = \frac{2}{3}$ when $p \leq 3$ and $\rho = \frac{1}{2}$ when $p \geq 4$. As we explain later, using a rounding strategy, this algorithm can be adapted into a $\frac{\rho}{2}$ -approximation for the problem with flows valued in \mathbb{N} . In this paper we give a $\frac{1}{H(p)}$ -approximation algorithm for this problem, where $H(p)$ is the p 'th harmonic number.⁴ This is an improvement on the $\frac{\rho}{2}$ -approximation above when $p \leq 30$. We also give non-approximability results for every fixed $p \geq 2$.

A further restriction on p -splittable flows could be that the p -paths must be arc-disjoint or even vertex-disjoint (apart from the end vertices s, t). We show that our approximation algorithm above applies to both of these cases also.

The paper is organized as follows. In Section 2 we recall some basic properties of flows as well as some classical complexity results. In Section 3 we study the complexity of flow problems where we want to restrict the maximum number of arcs that carry flow and are all leaving the same vertex. This problem turns out to be NP-complete, even for acyclic digraphs, as soon as the bound on the maximum out-degree of the support digraph is 2 or more. For non-acyclic digraphs, we show that even determining the existence of a flow x of value 9 where D_x has maximum out-degree at most 2 is NP-complete. In Section 4 we show that every 2-arc-strong (s, t) -flow network has a maximum flow whose support is also 2-arc-strong and show by a construction that this property does not generalize to higher arc-connectivities. The topic of Section 5 is finding a maximum flow which is as secure as possible towards arc-deletions, that is, we want a maximum flow D_x so that the value of a maximum flow in D_x only drops by a small amount when we delete a few arcs. In Section 6 we study how much flow we can send when we are only allowed to use p paths for some integer p . Again we also consider the cases where these paths need to be arc-disjoint, respectively internally vertex-disjoint. We show that in the case of acyclic networks, the arc-disjoint and vertex-disjoint cases can be solved in polynomial time by an XP-algorithm, parameterized by p , which is inspired by the solution in [13] of the weak- p -linkage problem. We also show that, unless $FPT = W[1]$,⁵ there can be no FPT-algorithm for this problem. In Section 7 we study the problem of finding a maximum flow which can be sent along paths so that no vertex is on more than p paths. For every fixed $p \geq 1$, this turns out to be NP-hard already for acyclic digraphs and capacities 1 and 2.

2. Preliminaries

Generally notation follows [4] so we only introduce a few things here for the convenience of the reader. The arc-set of a digraph D is denoted by $A(D)$ and its vertex-set is denoted by $V(D)$. The **maximum out-degree** and the **maximum in-degree** of D , denoted respectively by $\Delta^+(D)$ and $\Delta^-(D)$, are defined respectively as $\max_{v \in V(D)} d^+(v)$ and $\max_{v \in V(D)} d^-(v)$. Given two vertices s, t of a digraph D , we denote by $\lambda_D(s, t)$ the (s, t) -**arc-connectivity** of D , that is the maximum number of pairwise arc-disjoint paths from s to t in D .

Let $\mathcal{N} = (D, s, t, c)$ be a flow network and x an (s, t) -flow on \mathcal{N} . The **value** of the flow x , denoted by $|x|$, is the sum of the x -values of the arcs going out from s , that is $|x| = \sum_{u \in N^+(s)} x(us)$. The network \mathcal{N} is **acyclic** if D is an acyclic digraph.

Two very special flows are path-flows and cycle-flows. A **path-flow** (along P) is a flow x such that for some integer r we have $x(uv) = r$ for every arc $uv \in A(P)$ and $x(uv) = 0$ if $uv \notin A(P)$. A **cycle-flow** (along C) is defined similarly with P replaced by C above. The following very important property of flows is folklore. For a proof, see e.g. [4].

Lemma 1. *Every flow x in a network $\mathcal{N} = (D = (V, A), c)$ can be decomposed into (written as the arc-sum of) at most $|V| + |A|$ paths and cycle-flows such that at most $|A|$ of these are cycle-flows.*

Let $\mathcal{N} = (D, s, t, c)$ be a flow network and x an (s, t) -flow on \mathcal{N} . The **support digraph** of x , denoted by D_x , is the digraph obtained from D by removing all the arcs carrying no flow, that is $A(D_x) = \{uv \mid uv \in A(D) \text{ and } x(uv) > 0\}$.

Lemma 1 immediately implies the following classical lemma which allows us to turn D_x into an acyclic digraph.

Lemma 2. *Let x be a flow on a network $\mathcal{N} = (D, s, t, c)$. There exists a flow x' on \mathcal{N} with the same value as x such that $D_{x'}$ is acyclic. Moreover $D_{x'}$ is a subdigraph of D_x .*

⁴ The p 'th harmonic number is defined as $H(p) = \sum_{i=1}^p \frac{1}{i}$.

⁵ For definitions of the problem classes FPT and W[1] we refer the reader to [10].

Proof. If D_x is not acyclic, then let C be a (directed) cycle of D_x . Notice that C does not contain s , as we may assume that s is a source in D_x . Let u_0v_0 be an arc of C with minimum flow value along C . And now, let y be defined by $y(uv) = x(uv)$ if uv is not an arc of C and $y(uv) = x(uv) - x(u_0v_0)$ if uv is an arc of C . It is easy to check that y is a flow on \mathcal{N} , with the same value than x , as C does not contain s , and that D_y is a subgraph of D_x not containing u_0v_0 . Repeating this process of removing at least one arc from D_x while it contains a cycle stops with a flow x' with $D_{x'}$ being acyclic. \square

The p -LINKAGE problem is the following: Given a digraph D and $2p$ distinct vertices $s_1, \dots, s_p, t_1, \dots, t_p$; decide if D contains p vertex-disjoint paths P_1, \dots, P_p such that P_i is an (s_i, t_i) -path for $i \in [p]$. The WEAK- p -LINKAGE problem is the same as above but where we only require that the paths are arc-disjoint.

Theorem 3 (Fortune, Hopcroft and Wyllie [13]). *The p -LINKAGE problem and the WEAK- p -LINKAGE problem are both NP-complete for every fixed $p \geq 2$.*

Theorem 4 (Fortune, Hopcroft and Wyllie [13]). *The p -LINKAGE problem and the WEAK- p -LINKAGE problem are both solvable in polynomial time for every fixed $p \geq 1$ when the input is an acyclic digraph. If p is part of the input, then both problems become NP-complete.*

Theorem 5 (Slivkins [16]). *The p -LINKAGE problem and the WEAK- p -LINKAGE problem parameterised by p are both $W[1]$ -hard when the input is an acyclic digraph.*

The (3, B2)-SAT problem is the restriction of 3-SAT in which every literal appears exactly twice (i.e. every variable appears twice positively and twice negatively).

Theorem 6 ([8, Theorem 1]). *(3, B2)-SAT is NP-complete.*

3. Degree constrained flows

3.1. Bounded maximum out-degree

In this section, we are concerned with the following problem.

$(\Delta^+ \leq k)$ -MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$.

Output: The maximum value of a flow x on \mathcal{N} such that $\Delta^+(D_x) \leq k$.

When $k = 1$, the problem is known as the widest path problem or the maximum capacity path problem and is known to be polynomial time solvable by a modification of Dijkstra’s shortest path algorithm, see e.g. [1, Exercise 4.37]. We show that the problem turns out to be NP-hard for every fixed $k \geq 2$ even on acyclic networks. Note that, in the case where all the capacity values are 1, the problem is solvable in polynomial time. Indeed, replace every vertex $x \neq t$ by two vertices x^+ and x^- , add an arc from x^- to x^+ with capacity k and the arcs from $N^-(x)$ to x^- and the arcs from x^+ to $N^+(x)$ all with capacity 1. Then there is a one-to-one correspondence between the flows in the new network and the $(\Delta^+ \leq k)$ -flows in the initial one.

Theorem 7. *For every $k \geq 2$, $(\Delta^+ \leq k)$ -MAX-FLOW is NP-hard even when restricted to acyclic networks.*

Proof. We first prove the statement for $k = 2$ and then explain how to modify the proof to prove it for all $k \geq 3$. We will show how to reduce 3-SAT to the $(\Delta^+ \leq 2)$ -MAX-FLOW problem on acyclic networks. The first part of the reduction we use was used in several papers, see e.g. [3,6] it is also similar to the reduction used in Section 4 of [12].

Let $W[p, q]$ be the digraph with vertices $\{u, v, y_1, y_2, \dots, y_p, z_1, z_2, \dots, z_q\}$ and the arcs of the two (u, v) -paths $uy_1y_2 \dots y_pyv$ and $uz_1z_2 \dots z_qv$. Let H be the digraph on 4 vertices a_1, a_2, a_3, y and arcs a_1y, a_2y, a_3y . Let \mathcal{F} be an instance of 3-SAT with variables x_1, x_2, \dots, x_n and clauses C_1, C_2, \dots, C_m . By adding dummy clauses if necessary, we may assume that each variable x_i occurs at least once both in the negated and in non-negated forms in \mathcal{F} . The ordering of the clauses C_1, C_2, \dots, C_m induces an ordering of the occurrences of a variable x_j and its negation \bar{x}_j in these. With each variable $x = x_j$ we associate a copy of $W[p_i, q_i]$ where x_j occurs p_i times and \bar{x}_j occurs q_i times in the clauses of \mathcal{F} . Identify end vertices of these digraphs by setting $v_i = u_{i+1}$ for $i = 1, 2, \dots, n - 1$. Next, for each clause C_i we take a copy H_i of H and identify the vertices $a_{i,1}, a_{i,2}, a_{i,3}$ of H_i with vertices in the chain we build above as follows: assume C_i contains variables x_j, x_h, x_t (negated or not). If x_j is not negated in C_i and this is the r ’th copy of x_j (in the order of the clauses that use x_j), then we identify $a_{i,1}$ with $y_{j,r}$ and if C_i contains \bar{x}_j and this is the r ’th occurrence of \bar{x}_j , then we identify $a_{i,1}$ with $z_{j,r}$. We make similar identifications for $a_{i,2}, a_{i,3}$ with variables x_h, x_t .

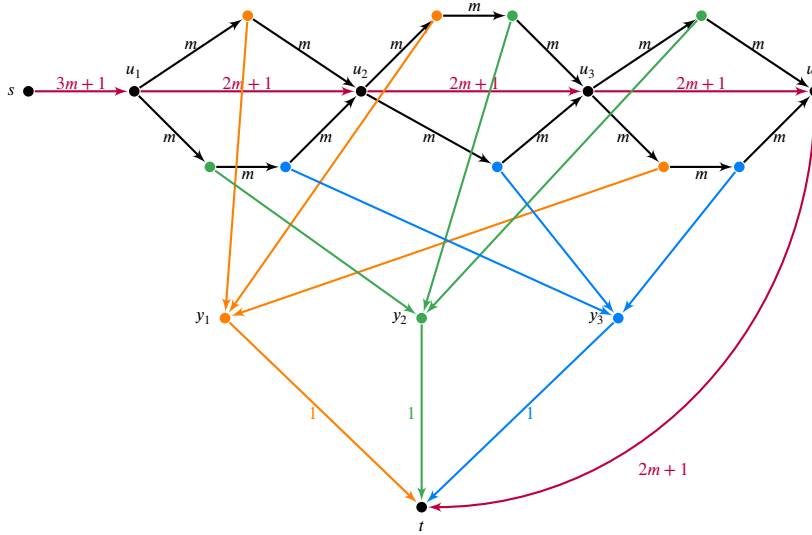


Fig. 1. The network $\mathcal{N}_{\mathcal{F}}$ when $\mathcal{F} = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$.

Remark 8. It was shown in [6] that \mathcal{F} is satisfiable if and only if the digraph $\tilde{D}_{\mathcal{F}}$ constructed so far has a (u_1, v_n) -path which visits at least one vertex from each of the sets $\{a_{i,1}, a_{i,2}, a_{i,3}\}$, $i \in [m]$ (actually in [6] the vertices y_1, \dots, y_m were not included but they play no role in (this part of) the argument).

To finish the construction of $D_{\mathcal{F}}$ we add a new vertex t , the arc $v_n t$ and all the arcs $y_i t$ for $i \in [m]$. Add a new vertex s and the arc su_1 . Finally we add the arcs $u_i v_i$ for $i \in [n]$. This concludes the description of the digraph $D_{\mathcal{F}}$ with special vertices s, t .

Now we construct a network $\mathcal{N}_{\mathcal{F}} = (D_{\mathcal{F}}, s, t, c)$ by adding the following capacities to the arcs of $D_{\mathcal{F}}$:

- the arc su_1 gets capacity $c(su_1) = 3m + 1$;
- the arc $v_n t$ gets capacity $c(v_n t) = 2m + 1$;
- the arcs of the form $u_i v_i$ get capacity $c(u_i v_i) = 2m + 1$;
- all arcs incident with y_i get capacity 1 for $i \in [m]$;
- all remaining arcs get capacity m .

Fig. 1 illustrates the construction of $\mathcal{N}_{\mathcal{F}}$. We claim that $\mathcal{N}_{\mathcal{F}}$ has an (s, t) -flow x of value $3m + 1$ such that $\Delta^+(D_x) \leq 2$ if and only if \mathcal{F} is satisfiable. This follows from the remark above: suppose first that \mathcal{F} is satisfiable and let ϕ be a truth assignment satisfying \mathcal{F} . Then we can select for each clause C_j the first literal ℓ_{j,r_j} , $r_j \in [3]$ which is satisfied by ϕ . By the remark, $D_{\mathcal{F}}$ has a (u_1, v_n) -path P which contains all the vertices $a_{1,r_1}, a_{2,r_2}, \dots, a_{m,r_m}$. Now we obtain an (s, t) -flow x of value $3m + 1$ and maximum out-degree 2 in its support as follows:

- send $2m + 1$ units along the path $su_1 u_2 \dots u_n t$
- use P and the paths $a_{j,r_j} y_j t$ $j \in [m]$ to send one unit of flow from s to each of the vertices y_j and then to t via the arc $y_j t$.

By construction D_x has maximum out-degree 2. To prove the other direction, suppose that x is an (s, t) -flow of value $3m + 1$ for which D_x has maximum out-degree 2. By the definition of capacities x must send $2m + 1$ units along the path $su_1 u_2 \dots v_n t$. So as $\Delta^+(D_x) = 2$ the remaining flow must be sent in a subdigraph D' of D_x so that each of u_1, u_2, \dots, u_n have out-degree at most 1 in D' . It is easy to see that this implies the existence of a (u_1, v_n) -path in $\tilde{D}_{\mathcal{F}}$ which visits each of the sets $\{a_{i,1}, a_{i,2}, a_{i,3}\}$, $i \in [m]$ at least once. Hence, by the remark, \mathcal{F} is satisfiable.

To modify the proof so that it works for any $k \geq 3$ we just need to replace the arcs of the form $u_i v_i$ with $k - 1$ paths $u_i r_{i,1} v_i, u_i r_{i,2} v_i, \dots, u_i r_{i,k-1} v_i$ and let all arcs of these paths have capacity $2m + 1$. Now it is easy to show that the new network has an (s, t) -flow x of value $m + (k - 1)(2m + 1)$ whose support has maximum out-degree k if and only if the 3-SAT formula \mathcal{F} is satisfiable. \square

Problem 9. Is there a way to approximate the $(\Delta^+ \leq k)$ -FLOW problem? In other words, if \mathcal{N} has a flow x^* satisfying $\Delta^+(D_{x^*}) \leq k$, is it possible to check in polynomial time that it has a flow x , $\Delta^+(D_x) \leq k$, such that $|x| \geq \rho(k) \cdot |x^*|$ for some function $\rho : \mathbb{N} \rightarrow]0, 1[$?

If such a function $\rho : \mathbb{N} \rightarrow]0, 1[$ exists, the result of the next section implies that $\rho(2) \leq \frac{8}{9}$ (see Corollary 12). For a similar type of questions on degrees of spanning trees in graphs, see the paper [14].

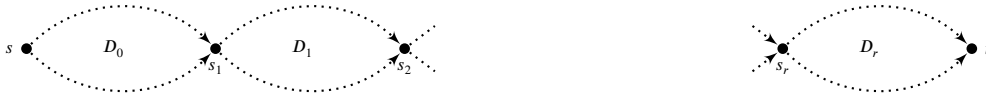


Fig. 2. The structure of the network \mathcal{N} .

3.2. Finding a flow with value close to the maximum out-degree

In this section, we take into account the value of the flow that we want to obtain and, for $\ell \geq 0$, we define the next problem.

$(\Delta^+ \leq k)$ -FLOW OF VALUE $k + \ell$

Input: A flow network $\mathcal{N} = (D, s, t, c)$.

Output: Does there exist a flow x of value at least $k + \ell$ on \mathcal{N} with $\Delta^+(D_x) \leq k$?

It is clear that $(\Delta^+ \leq k)$ -FLOW OF VALUE $k + \ell$ is polynomial for $\ell = 0$, as it is equivalent as the existence of a flow of value k . First we will see that the problem remains solvable in polynomial time for $\ell = 1$.

Theorem 10. *The problem $(\Delta^+ \leq k)$ -FLOW OF VALUE $k + 1$ is solvable in polynomial time.*

Proof. Let $\mathcal{N} = (D = (V, A), s, t, c)$ be an instance of $(\Delta^+ \leq k)$ -FLOW OF VALUE $k + 1$. We assume $2 \leq k \leq n - 1$ (where $n = |V(D)|$), the problem being clearly solvable in polynomial time when $k = 1$ or $k \geq n$. We first check whether \mathcal{N} admits an (s, t) -flow of value at least $k + 1$. If this is not the case, \mathcal{N} is clearly a negative instance. We can also assume that every vertex v (and every arc a) of D belongs to a path from s to t , as otherwise no acyclic flow from s to t sends flow through v (or a) and we may delete it. A vertex u of D is an (s, t) -vertex separator if there is no path from s to t in $D - u$. Assume first that D does not contain any (s, t) -vertex separator. Then we transform \mathcal{N} as follows. Denote by v_1, \dots, v_p the out-neighbours of s with $c(sv_i) \geq 2$. Then for $i = 1, \dots, p$ we build the network \mathcal{N}_i by adding first a new source s^* to D with an arc s^*s with capacity $k - 1$ and an arc s^*v_i with capacity $c(sv_i)$. We delete the arc sv_i , and then we do the usual vertex splitting operation (see e.g. [4, Section 4.2.4]): every vertex $v \notin \{s^*, s, t\}$ is replaced by two vertices v^-, v^+ and the arc v^-v^+ with capacity k , and every arc $uv \in A$ is replaced by the arc u^+v^- and it preserves its capacity. We call the arcs of the kind v^-v^+ , the **special arcs** of the new network \mathcal{N}_i . We claim that \mathcal{N} contains a flow x with value $k + 1$ and $\Delta^+(D_x) \leq k$ if, and only if, there exists $i \in [p]$ such that \mathcal{N}_i contains a flow of value $k + 1$.

Indeed, first assume that \mathcal{N}_i contains a flow x_i of value $k + 1$ for some i . Then for each arc uv of D different from sv_i , we assign to it the flow $x_i(u^+v^-)$ (or $x_i(sv_i^-)$ if $u = s$ or $x_i(u^+t)$ if $v = t$). We finally put on sv_i the flow $x_i(s^*v_i)$. It is easy to check that we obtain a flow x of value $k + 1$ in \mathcal{N} . Let us see that $\Delta^+(D_x) \leq k$. As every special arc of D_i carries at most k units of flow, it is clear that $d_{D_x}^+(v) \leq k$ for every $v \in D - s$. Moreover, as $k + 1$ units of flow leave s^* in \mathcal{N}_i and $c(s^*s) = k - 1$, we have $x(sv_i) = x_i(s^*v_i) \geq 2$. So, in \mathcal{N} , we have $x(sv_i) \geq 2$ and then $d_{D_x}^+(s) \leq k$. Thus, x is a flow on \mathcal{N} with value $k + 1$ and $\Delta^+(D_x) \leq k$.

Conversely, assume that \mathcal{N} contains a flow x with value $k + 1$ and $\Delta^+(D_x) \leq k$. As $d_{D_x}^+(s) \leq k$, there exists an out-neighbour v_i of s with $x(sv_i) \geq 2$. Then, we consider the network \mathcal{N}_i . Assume that it does not admit a flow of value $k + 1$. By the Max Flow Min Cut Theorem (see e.g. [4, Theorem 4.5.3]) this means that \mathcal{N}_i has a cut (X, \overline{X}) with $s^* \in X$, $t \in \overline{X}$ and capacity at most k . If some special arc u^-u^+ has $u^- \in X$ and $u^+ \in \overline{X}$, then this is the only arc from X to \overline{X} (as it has capacity exactly k). But this implies that u is an (s, t) -vertex separator in \mathcal{N} , contradicting our assumption. So no special arc goes across the cut. Suppose first that $s \in \overline{X}$. Then we must have $v_i \in X$ as $c(s^*v_i) \geq 2$ and $c(s^*s) = k - 1$. But then $(X - s^*, \overline{X})$ is a (v_i, t) -cut of capacity at most 1 in \mathcal{N} , contradicting that at least two units of the flow x go from v_i to t in \mathcal{N} . Thus we must have $s \in X$. Now define X' to be the subset of V consisting of s and all the vertices $u \in V$ with $u^-, u^+ \in X$. It is straightforward to see that (X', \overline{X}') is an (s, t) -cut of \mathcal{N} with the same capacity as (X, \overline{X}) , contradicting the existence of a flow with value $k + 1$ in \mathcal{N} .

There are at most n networks \mathcal{N}_i , and performing a search for a flow of value $k + 1$ in \mathcal{N}_i can be done in time $O(kn^2) = O(n^3)$ as we need at most $k + 1$ augmenting paths. So, in time $O(n^4)$ we can decide if there exists a $(\Delta^+ \leq k)$ -flow of value $k + 1$ in a network with no (s, t) -vertex separator.

Assume now that \mathcal{N} contains at least one (s, t) -vertex separator, and consider any path P from s to t in D . All the (s, t) -vertex separators of D belong to P , and we can enumerate them s_1, \dots, s_r according to the order in which they appear along P . We denote also s by s_0 and t by s_{r+1} . Finally, for $j = 0, \dots, r$ we denote by V_j the set of vertices of D lying on some path from s_j to s_{j+1} and by D_j the digraph induced by D on V_j . As every vertex and every arc of D are on a path from s to t , it is then not hard to see that the following holds. See Fig. 2 for an illustration.

- $V = \bigcup_{1 \leq j \leq r} V_j$
- for $1 \leq i < j \leq r + 1$ the sets V_i and V_j have a vertex in common only if $j = i + 1$ and this vertex is s_{i+1}
- for $1 \leq i < j \leq r + 1$ there is an arc from V_i and V_j only if $j = i + 1$ and this arc is incident to s_{i+1} .

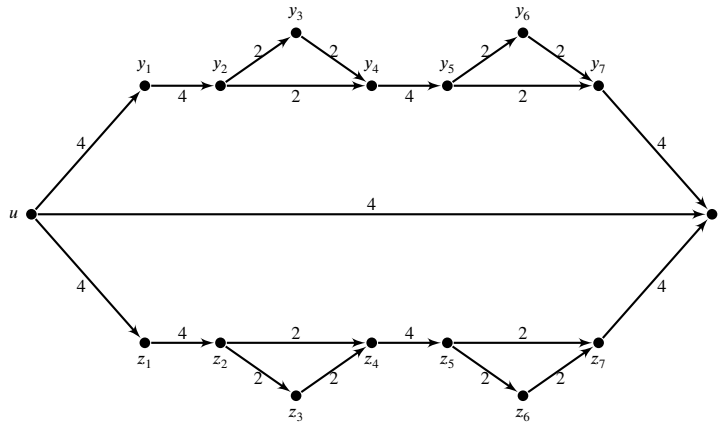


Fig. 3. The variable-gadget W .

To conclude the proof, notice that, by construction, each D_j has no (s_j, s_{j+1}) -vertex separator. For $j = 0, \dots, r$, we denote by \mathcal{N}_j the network (D_j, s_j, s_{j+1}, c_j) (where c_j is c restricted to the arcs of D_j). It is clear that \mathcal{N} contains a flow x of value $k + 1$ with $\Delta^+(D_x) \leq k$ if, and only if, each \mathcal{N}_j has such an (s_j, s_{j+1}) -flow for every $j = 0, \dots, r$. So, by the previous case, denoting by n_j the number of vertices of D_j , we can check in time $O(n_j^4)$ whether \mathcal{N}_j contains such a flow for $j = 0, \dots, r$. Using the convexity of the function $z \rightarrow z^4$ and the fact that $\sum n_j \leq 2n$, we conclude that it is possible to decide if \mathcal{N} contains a flow x with value $k + 1$ and $\Delta^+(D_x) \leq k$ in time $O(n^4)$. \square

We will now show that $(\Delta^+ \leq k)$ -FLOW OF VALUE $k + \ell$ is NP-complete for some fixed k, ℓ , namely $k = 2$ and $\ell = 7$. Note that this is not a strengthening of Theorem 7 as we do not restrict to acyclic networks.

Theorem 11. *The problem $(\Delta^+ \leq 2)$ -FLOW OF VALUE 9 is NP-complete.*

Proof. The following proof inspired by the proof of Theorem 9 in [15]. The problem clearly being in NP, we prove the hardness by a reduction from $(3, B2)$ -SAT, which is NP-complete by Theorem 6. Let $\mathcal{F} = (X, C)$ be an instance of $(3, B2)$ -SAT. We fix an arbitrary ordering x_1, \dots, x_n on the variables X and an ordering C_1, \dots, C_m on the clauses C .

We now describe the construction of a network $\mathcal{N}_{\mathcal{F}} = (D, s, t, c)$ such that $\mathcal{N}_{\mathcal{F}}$ admits a flow x of value at least 9 with $\Delta^+(D_x) \leq 2$ if and only if \mathcal{F} is satisfiable. Let W , our variable-gadget, be the digraph with vertex-set

$$V(W) = \{u, v\} \cup \{y_i, z_i \mid 1 \leq i \leq 7\}$$

and arc-set

$$A(W) = \{uy_1, uz_1, uv, y_2y_4, z_2z_4, y_5y_7, z_5z_7, y_7v, z_7v\} \cup \{y_iy_{i+1}, z_iz_{i+1} \mid 1 \leq i \leq 6\},$$

see Fig. 3 for an illustration and for the capacity $c(a)$ of every arc $a \in A(W)$.

For every variable $x_i \in X$ we add a distinct copy W^i of W (preserving the capacities of W). For every vertex $w \in W$, we let w^i designate the copy of w in W^i . For every $i \in [n - 1]$, we add the arc $v^i u^{i+1}$ with capacity 8.

For every clause $C_j \in C$, we add two vertices q_j and r_j , and for every $j \in \{1, \dots, m - 1\}$ we add the arc $r_j q_{j+1}$ with capacity 1. For every variable x_h appearing positively in $C_i, C_j, i < j$, and negatively in $C_k, C_\ell, k < \ell$, we add the following set of arcs with capacity 1:

$$\{q_i y_4^h, y_5^h r_i, q_j y_1^h, y_2^h r_j, q_k z_4^h, z_5^h r_k, q_\ell z_1^h, z_2^h r_\ell\}$$

We finally add the source s , the sink t , the arcs $su^1, v^m t$ with capacity 8, and the arcs $sq_1, r_m t$ with capacity 1. We let $\mathcal{N}_{\mathcal{F}} = (D, s, t, c)$ be the obtained network, see Fig. 4 for an illustration.

We now prove the equivalence between the instances \mathcal{F} and $\mathcal{N}_{\mathcal{F}}$. Assume first that \mathcal{F} is satisfied by a truth assignment ϕ , we will show that $\mathcal{N}_{\mathcal{F}}$ admits a flow x of value 9 such that $\Delta^+(D_x) \leq 2$. For every $j \in [m]$, we fix an arbitrary variable $x_{i(j)}$ such that C_j is satisfied by $x_{i(j)}$ or its negation in ϕ . If $x_{i(j)}$ satisfies C_j , we let w_j be the out-neighbour of q_j in $\{y_1^{i(j)}, y_4^{i(j)}\}$. Else if $\bar{x}_{i(j)}$ satisfies C_j , we let w_j be the out-neighbour of q_j in $\{z_1^{i(j)}, z_4^{i(j)}\}$. In both cases, we let w_j^i be the unique out-neighbour of w_j (observe that w_j^i is an in-neighbour of r_j). We define x as follows:

- $x(su^1) = x(v^m t) = 8$;
- $x(sq_1) = x(r_m t) = 1$;

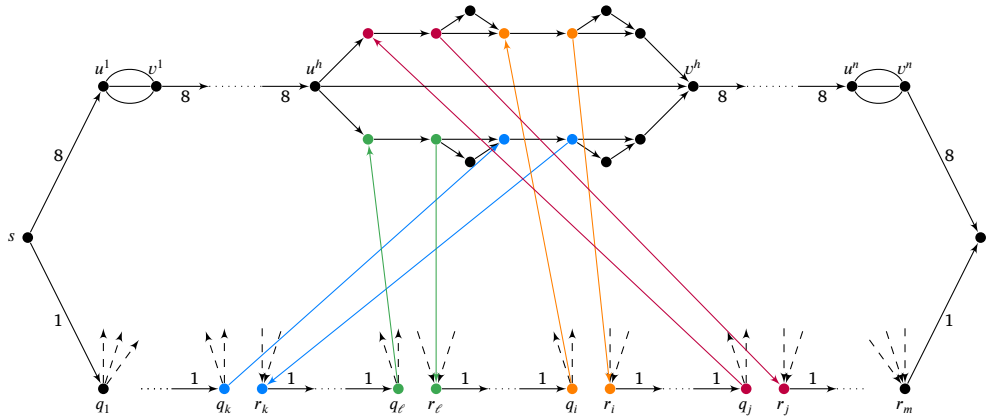


Fig. 4. An illustration of the network \mathcal{N}_F . Three undirected edges between the same pair of vertices represent a copy of W . Arcs in a copy of W have the same capacities as in Fig. 3. Arcs incident to vertices in $\{q_i, r_i \mid 1 \leq i \leq m\}$ have capacity 1.

- $\forall i \in [n], x(u^i v^i) = 4;$
- $\forall i \in [n - 1], x(v^i u^{i+1}) = 8;$
- $\forall j \in [m - 1], x(r^j q^{j+1}) = 1;$
- $\forall i \in [n]$ such that $\phi(x_i)$ is true, $x(u^i z_1^i) = x(z_1^i z_2^i) = x(z_4^i z_5^i) = x(z_7^i v^i) = 4$ and $x(z_2^i z_3^i) = x(z_3^i z_4^i) = x(z_5^i z_6^i) = x(z_6^i z_7^i) = x(z_5^i z_7^i) = 2;$
- $\forall i \in [n]$ such that $\phi(x_i)$ is false, $x(u^i y_1^i) = x(y_1^i y_2^i) = x(y_4^i y_5^i) = x(y_7^i v^i) = 4$ and $x(y_2^i y_3^i) = x(y_3^i y_4^i) = x(y_5^i y_6^i) = x(y_6^i y_7^i) = x(y_5^i y_7^i) = 2;$ and
- $\forall j \in [m], x(q_j w_j) = x(w_j r_j) = x(w_j r_j) = 1.$

By definition, x is a flow of value 9. We now prove that $\Delta^+(D_x) \leq 2$. By construction, the only vertices that may have out-degree more than 2 in D_x are vertices in $\{y_2^i, y_3^i, z_2^i, z_5^i \mid i \in [n]\}$ identified as w'_j for some $j \in [m]$. Assume that, for some indices i, j , w'_j belongs to $\{y_2^i, y_3^i, z_2^i, z_5^i\}$. By choice of w_j and w'_j , either C_j is satisfied by x_i and $w'_j \in \{z_2^i, z_5^i\}$ or C_j is satisfied by \bar{x}_i and $w'_j \in \{y_2^i, y_3^i\}$. In both cases, by definition of x , we conclude that u'_j has out-degree 1 in D_x . Hence $\Delta^+(D_x) \leq 2$ as desired.

Conversely, assume now that \mathcal{N}_F admits a flow y of value 9 such that $\Delta^+(D_y) \leq 2$. Among all such flows y , we choose x for which $|A(D_x)|$ is minimum. The two following claims show that x satisfies some structural properties.

Claim 11.1. For every $i \in [n]$, $x(u^i v^i) = 4$ and $x(v^{i-1} u^i) = 8$, where v^0 is identified as s . Moreover, $x(u^i y_1^i) = x(y_7^i v^i)$ and $x(u^i z_1^i) = x(z_7^i v^i)$.

Proof. We proceed by induction on $i \in [n]$. Since $|x| = 9$, we necessarily have $x(su^1) = 8$.

Assume now that, for some $i \in [n]$, $x(v^{i-1} u^i) = 8$. Since $\Delta^+(D_x) \leq 2$, and because 8 units of flow enter u^i , exactly two arcs of $\{u^i v^i, u^i y_1^i, u^i z_1^i\}$ carry 4 units of flow, and the third arc does not carry any flow.

Suppose that $x(u^i y_1^i) = 4$. Hence, we have $x(y_1^i y_2^i) = 4$. Since $\Delta^+(D_x) \leq 2$, and because 4 units of flow enter y_2^i , we must have $x(y_2^i y_3^i) = x(y_2^i y_4^i) = x(y_3^i y_4^i) = 2$ and $x(y_2^i r_j) = 0$ (where j is the index of the second clause C_j containing x_i). Hence 4 units of flow enter y_4^i , and repeating the same arguments we obtain $x(y_7^i v^i) = 4$, which shows $x(u^i y_1^i) = x(y_7^i v^i)$.

Similarly, if $x(u^i z_1^i) = 4$, we have $x(z_1^i z_2^i) = x(y_7^i v^i) = 4$, $x(z_2^i z_3^i) = x(z_2^i z_4^i) = x(z_3^i z_4^i) = 2$ and $x(z_2^i r_\ell) = 0$ (where ℓ is the index of the second clause C_j containing \bar{x}_i).

Hence at least 8 units of flow enter v^i . Since v^i has exactly one leaving arc (namely $v^i u^{i+1}$ if $i \leq n - 1$ and $v^i s$ otherwise) with capacity 8, there must be exactly 8 units of entering v^i . Thus, when $i \leq n - 1$, we have $x(v^i u^{i+1}) = 8$, and one of $u^i v^i, y_7^i v^i, z_7^i v^i$ does not carry any flow.

It remains to show that $x(u^i v^i) > 0$. Assume for a contradiction that $x(u^i v^i) = 0$, that is $x(u^i y_1^i) = x(u^i z_1^i) = 4$. Let x' be the flow defined as follows:

$$x'(a) = \begin{cases} 4 & \text{if } a = u^i v^i \\ 0 & \text{if } a \in \{u^i z_1^i, z_2^i z_3^i, z_2^i z_4^i, z_3^i z_4^i, z_4^i z_5^i, z_5^i z_6^i, z_5^i z_7^i, z_6^i z_7^i, z_7^i v^i\} \\ x(a) & \text{otherwise.} \end{cases}$$

By definition, we have $|x'| = |x|$, $\Delta^+(D_{x'}) \leq 2$ and $|A(D_{x'})| = |A(D)| - 8$. Hence x' contradicts the choice of x , and the claim follows. \diamond

Claim 11.2. There exists a path-flow P of x that contain all vertices $q_1, r_1, q_2, r_2, \dots, q_m, r_m$ in this order.

Proof. We proceed by induction on $j \in [m]$. Since $|x| = 9$, we necessarily have $x(sq_1) = 1$. Let P be the unique path-flow containing the arc sq_1 .

Assume that, for some $j \in [n]$, P contains all vertices $q_1, r_1, \dots, r_j - 1, q_j$ in this order. As one unit of flow enters q_j , the successor of q_j in P is a vertex $w_j \in V(W^i)$ for some $i \in [n]$. Let w'_j be the successor of w_j in P , and observe that $w'_j \in \{z_2^i, z_5^i, y_2^i, y_3^i\}$. Assume that $w'_j \in \{y_2^i, y_3^i\}$ (so $w_j \in \{y_1^i, y_4^i\}$), the other case being symmetric. By Claim 11.1, we have $x(u^i y_1^i) = x(y_7^i v^i) \in \{0, 4\}$. Since both y_1^i and y_4^i have only one leaving arc with capacity 4, we must have $x(u^i y_1^i) = x(y_7^i v^i) = 0$.

If $w'_j = y_2^i$, then r_j must be the successor of w'_j , for otherwise $x(y_7^i v^i) > 0$, a contradiction. Else if $w'_j = y_3^i$, we claim that r_j is the successor of w'_j in P . Assume for a contradiction that its successor belongs to $\{y_3^i, y_4^i\}$. In both cases, P contains the arc $y_4^i y_5^i$. Since $x(y_7^i v^i) = 0$, the successor of y_5^i must be $r_{j'}$ for some $j \in [m]$. By construction, we must have $j' < j$. Hence, by induction hypothesis on P , P contains a cycle, a contradiction.

This shows that P contains $q_1, r_1, q_2, r_2, \dots, q_j, r_j$ in this order. If $j \leq m - 1$, P contains $q_1, r_1, q_2, r_2, \dots, q_j, r_j, q_{j+1}$ in this order as q_{j+1} is the only out-neighbour of r_j , so the claim follows. \diamond

We are now ready to prove that \mathcal{F} is satisfiable. Let ϕ be the truth assignment which is true on variable x_i if and only if $x(u^i z_1^i) = 4$. Let us show that \mathcal{F} is satisfied by ϕ . Let $C_j \in \mathcal{C}$ be any clause. By Claim 11.2, the path-flow P contains q_j . Let $w_j \in \{z_1^i, z_4^i, y_1^i, y_4^i\}$ (for some $i \in [n]$) be its successor in P . If $w_j \in \{y_1^i, y_4^i\}$, by construction x_i appears positively in C_j . By Claim 11.1, and because both y_1^i and y_4^i have only one leaving arc with capacity 4, we must have $x(u^i y_1^i) = 0$. Hence, $x(u^i z_1^i) = 4$ and ϕ is true on x_i , so C_j is satisfied by ϕ . Else if $w_j \in \{z_1^i, z_4^i\}$, by construction x_i appears negatively in C_j . By Claim 11.1, and because both z_1^i and z_4^i have only one leaving arc with capacity 4, we must have $x(u^i z_1^i) = 0$. Hence, ϕ is false on x_i , so C_j is satisfied by ϕ . This concludes the proof that $(\Delta^+ \leq 2)$ -FLOW OF VALUE 9 is NP-hard. \square

We easily derive the following inapproximability result.

Corollary 12. *Unless $P=NP$, $(\Delta^+ \leq 2)$ -MAX-FLOW cannot be approximated by any ratio larger than $\frac{8}{9}$.*

We note that the proof of Theorem 11 can be adapted to show that the problem $(\Delta^+ \leq k)$ -FLOW OF VALUE $2k^2 + 1$ is NP-complete for every fixed $k \geq 2$. To do so, modify the variable-gadget W as follows: (i) replace the two paths from y_2 to y_4 (respectively y_5 to y_7 , z_2 to z_4 , and z_5 to z_7) by k paths with capacity 2, (ii) replace the arc uv by $k - 1$ paths each with capacity $2k$, and (iii) set the capacity of the eight remaining arcs to $2k$. Finally, in the network $\mathcal{N}_{\mathcal{F}}$, the arcs with capacity 8 receive capacity $2k^2$. The same arguments show that \mathcal{F} is a positive instance of $(3, B2)$ -SAT if and only if the obtained network admits a $(\Delta^+ \leq k)$ -flow of value $2k^2 + 1$.

Hence, for fixed k , if ℓ is large enough, deciding the existence of a $(\Delta^+ \leq k)$ -flow of value $k + \ell$ is an NP-complete problem. We indeed believe that ℓ being large enough is a necessary condition for this problem to be NP-complete, and pose the following conjecture.

Conjecture 1. *For every fixed integer ℓ , there exists $k_\ell \in \mathbb{N}$ such that, for every $k \geq k_\ell$, $(\Delta^+ \leq k)$ -FLOW OF VALUE $k + \ell$ is solvable in polynomial time.*

3.3. Bounded maximum in- and out-degree

In this section we consider the problems of the two previous sections, but now we ask D_x to have bounded maximum in- and out-degrees (and not only bounded maximum out-degree). We will justify that the complexity results are exactly the same as in the previous sections. We start with the following one.

$(\Delta^+ \leq k_1, \Delta^- \leq k_2)$ -MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$

Output: The maximum value of a flow x on \mathcal{N} such that $\Delta^+(D_x) \leq k_1$ and $\Delta^-(D_x) \leq k_2$.

If k_1 or k_2 is equal to one, then $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -MAX-FLOW consists of finding the maximum c such that there exists a path from s to t in which every arc has capacity at least c . This can be done in polynomial time. Now if $k_1, k_2 \geq 2$, $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -MAX-FLOW is NP-hard even when restricted to acyclic networks. We use the same reduction as in the proof of Theorem 7. In the built network, for every flow x with $\Delta^+(D_x) \leq 2$, every vertex but t has in-degree at most two. Let us replace the single vertex t by the in-branching depicted in Fig. 5. In the obtained network, for every flow x with $\Delta^+(D_x) \leq 2$, we have $\Delta^-(D_x) \leq 2$. Since $k_2 \geq 2$, this shows the hardness of the problem.

Analogously, one can check that the reduction made in the proof of Theorem 11 shows that deciding whether a network \mathcal{N} admits a $(\Delta^+ \leq 2, \Delta^- \leq 2)$ -flow of value 9 is an NP-complete problem.

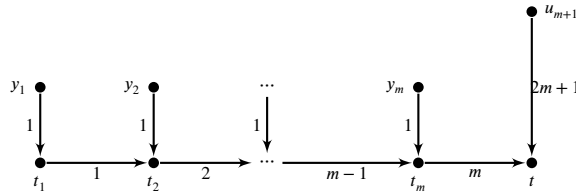


Fig. 5. A gadget that ensures $d^-(t) \leq 2$ in the support of every flow.

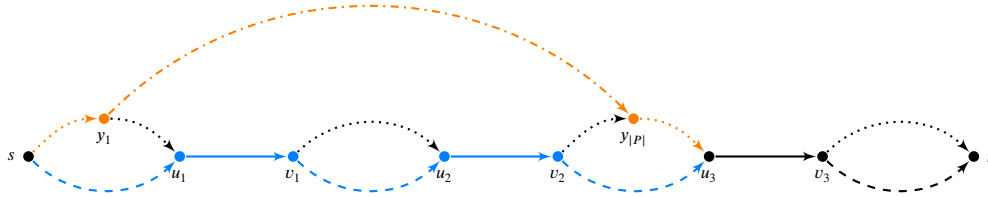


Fig. 6. The support D_x and the construction of x' . On this example, we have $\ell = 3$ and $j = 2$. The solid arcs represent the (s, t) -cut-arcs. The dotted and dashed arcs represent the paths Q_1^j and Q_2^j respectively. The path P is represented by dash-dotted lines. The paths P_1' and P_2' are represented respectively in blue and orange.

4. Flows with high connectivity

In this section we turn our focus on maximum flows $|x|$ for which $\lambda_{D_x}(s, t)$ is large enough. We first prove that a trivial necessary condition on a network $\mathcal{N} = (D, s, t, c)$ to admit a maximum flow x with $\lambda_{D_x}(s, t) \geq 2$ is indeed sufficient. In a network $\mathcal{N} = (D, s, t, c)$, an (s, t) -cut-vertex (respectively an (s, t) -cut-arc) is a vertex in $V(D) \setminus \{s, t\}$ (respectively an arc in $A(D)$) which is contained in all the paths of D from s to t .

Theorem 13. *If $\lambda_D(s, t) \geq 2$, then there exists a maximum flow x on \mathcal{N} such that $\lambda_{D_x}(s, t) \geq 2$. In particular, it is polynomial time solvable to decide if a network \mathcal{N} admits a maximum flow x with $\lambda_{D_x}(s, t) \geq 2$.*

Proof. As having $\lambda_D(s, t) \geq 2$ is a necessary condition to get a maximum flow x on \mathcal{N} such that $\lambda_{D_x}(s, t) \geq 2$, and as it is possible to decide in polynomial time if $\lambda_D(s, t) \geq 2$ or not, the first part of the statement implies the announced polynomial time algorithm.

Let us assume then that $\lambda_D(s, t) \geq 2$ and let us consider a maximum flow x on \mathcal{N} such that D_x has a minimum number of (s, t) -cut-arcs. Let us show then that D_x has no (s, t) -cut-arc.

Towards a contradiction, assume that D_x contains some (s, t) -cut-arcs. Consider a path Q from s to t in D_x . By definition of an (s, t) -cut-arc, every (s, t) -cut-arc of D_x belongs to Q . We label these (s, t) -cut-arcs $u_1v_1, \dots, u_\ell v_\ell$ according to their ordering along Q . In what follows, we identify t with $u_{\ell+1}$ and s with v_0 .

Every vertex of D_x is on a path from s to t . As every path from s to t contains all the (s, t) -cut-arcs $u_i v_i$, we can partition the vertices of D_x into (X_0, \dots, X_ℓ) where $X_i = \{y \in V(D_x) : y \text{ belongs to a path from } v_i \text{ to } u_{i+1} \text{ in } D_x\}$ for every $i \in \{0, \dots, \ell\}$. For every $i \in \{0, \dots, \ell\}$, note that there exist two arc-disjoint paths Q_1^i, Q_2^i from v_i to u_{i+1} , for otherwise there exists a (v_i, u_{i+1}) -cut-arc in D_x , which is also an (s, t) -cut-arc.

Since u_1v_1 is not a (s, t) -cut-arc in D , D contains a path $P = y_1, \dots, y_{|P|}$ such that $V(P) \cap X_0 = \{y_1\}$ and $V(P) \cap (X_1 \cup \dots \cup X_\ell) = \{y_{|P|}\}$.

Let $j \geq 1$ be the index such that $y_{|P|} \in X_j$. We assume without loss of generality that Q_0^1 contains y_1 and Q_j^1 contains $y_{|P|}$. Let Q' be the path made of the concatenation of all the Q_i^1 s, and Q be its subpath from y_1 to $y_{|P|}$. We define the (s, t) -flow x' of \mathcal{N} as follows:

$$\forall e \in A(D), x'(e) = \begin{cases} 1 & \text{if } e \in A(P) \\ x(e) - 1 & \text{if } e \in A(Q) \\ x(e) & \text{otherwise.} \end{cases}$$

Clearly x' is an (s, t) -flow and $|x'| = |x|$. Observe that $D_{x'}$ contains two arc-disjoint paths from s to u_{j+1} : one is the concatenation $P_1' = Q_0^2 u_1 v_1 Q_1^2 u_2 v_2, \dots, u_j v_j Q_j^2$ and the other is the concatenation P_2' of $Q_0^1[s, y_1]$, P and $Q_j^1[y_{|P|}, u_{j+1}]$. Note that P_2' is disjoint from Q so it is actually a path in $D_{x'}$. Also, P_1' intersects Q exactly on the arcs $u_i v_i$ for $i \leq j - 1$, and all these arcs carry at least two units of flow since they are (s, t) -cut-arcs (and $\lambda_D(s, t) \geq 2$ implies $|x| \geq 2$). See Fig. 6 for an illustration.

Hence every (s, t) -cut-arc of $D_{x'}$ must be on every path from u_{j+1} to t , implying that it is also a (s, t) -cut-arc in D_x . Since u_1v_1 is not a (s, t) -cut-arc in $D_{x'}$, x' contradicts the choice of x . \square

In the following we show that it is not possible to generalize the first part of Theorem 13 for higher values of $\lambda_D(s, t)$.

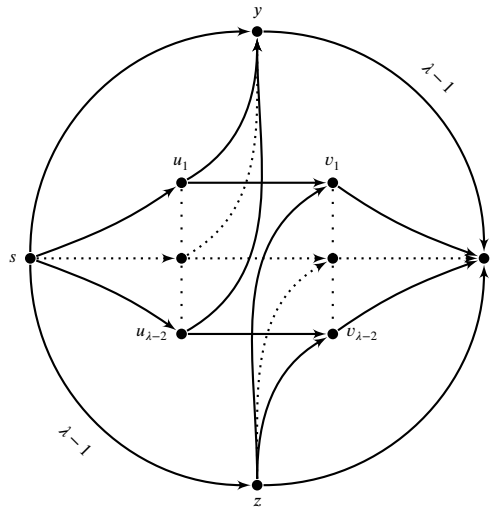


Fig. 7. Example of a network (all arcs have capacity 1 except the two with indicated capacities $\lambda - 1$). The network has λ arc-disjoint paths from s to t , but for every maximum flow x (there is only one) there exist only two arc-disjoint paths from s to t in D_x .

Theorem 14. For every fixed integer $\lambda \geq 3$, there exists a network $\mathcal{N}_k = (D, s, t, c)$ such that $\lambda_D(s, t) = \lambda$ and for every maximum (s, t) -flow x , $\lambda_{D_x}(s, t) \leq 2$.

Proof. Let D be the digraph made of $\lambda - 2$ (s, t) -paths of length 3 $su_i v_i t$, $i \in [\lambda - 2]$ and two (s, t) -paths of length 2 $sy t$ and $sz t$. Then add all the arcs of $\{u_i y \mid i \in [\lambda - 2]\} \cup \{z v_i \mid i \in [\lambda - 2]\}$. All the arcs have capacity 1 except sz and yt which have capacity $\lambda - 1$. See Fig. 7 for an illustration.

We clearly have $\lambda_D(s, t) = \lambda$, let us show that for every maximum (s, t) -flow x , $\lambda_{D_x}(s, t) = 2$. Observe first that there exists a flow x^* of value $2\lambda - 2$, which is defined as follows:

$$x^*(uv) = \begin{cases} \lambda - 1 & \text{if } uv \in \{sz, yt\} \\ 0 & \text{if } uv \in \{u_i v_i \mid i \in [\lambda - 2]\} \\ 1 & \text{otherwise.} \end{cases}$$

Let x be a maximum (s, t) -flow, then we claim that D_x does not contain any arc of the form $u_i v_i$, for $i \in [\lambda - 2]$. Assume for a contradiction that it does. Let P be the path-flow of x containing the arc $u_i v_i$. Since u_i has in-degree 1 and su_i has capacity 1, we have $x(u_i y) = 0$. Since y has in-degree $\lambda - 1$, we have $x(yt) \leq \lambda - 2$. Hence the flow entering t is at most $|x| \leq (\lambda - 1) + (\lambda - 2) = 2\lambda - 3$, a contradiction to the maximality of x since $|x^*| = 2\lambda - 2 > |x|$.

This shows that D_x does not contain any arc of the form $u_i v_i$. Hence $D_x \setminus \{sz, yt\}$ does not contain any (s, t) -path, implying that $\lambda_{D_x}(s, t) \leq 2$ as desired. \square

5. Safe maximum flows

We now discuss problems concerning stability of (s, t) -flows in a given network towards arc-deletions. Related problems for other structures were considered in [3,5,7,9,11].

If we delete a set A' of one or more arcs from the support D_x of a maximum (s, t) -flow x , then the restriction of x to $A(D_x) \setminus A'$ may not be an (s, t) -flow anymore, but it can be modified to a new maximum flow \tilde{x} in $D_x[A(D_x) \setminus A']$ of value at least $|x| - \sum_{uv \in A'} x(uv)$ in polynomial time.

Consider finding in a fixed network \mathcal{N} a maximum flow x^* which is as persistent as possible against (a few) arcs deletions. That is, the value of x^* restricted to the arc set $A \setminus A'$, where $|A'| \leq k$ is as high as possible. Clearly, if we delete one or more arcs that are in the same minimum (s, t) -cut of \mathcal{N} , then every maximum flow will be hurt by the same amount (the sum of the capacities of the deleted arcs) but this is by no means the case for arcs which are not contained in some min cut: one max flow may send all its flow through some arc of very high capacity (not in any min cut) and hence be hurt very much if that arc is deleted, while another max flow may send its flow along many pairwise arc-disjoint paths.

For a given flow network $\mathcal{N} = (D = (V, A), s, t, c)$ we can determine in polynomial time the subset $A_{mincut} \subseteq A$ of arcs that belong to some minimum (s, t) -cut in \mathcal{N} .

k -ARC-PERSISTENT-MAX-FLOW

Input: A flow network $\mathcal{N} = (D = (V, A), s, t, c)$ and an integer k .

Output: A maximum flow x such that the minimum value of a maximum flow \tilde{x} in $D_x[A(D_x) \setminus A']$ over all subsets $A' \subseteq A \setminus A_{\text{mincut}}$ of k arcs is as large as possible.

This problem turns out to be NP-hard even when $k = 1$ and all capacities are 1. Indeed, let $\mathcal{N} = (D, s, t, c)$ be a network and assume that s has exactly one leaving arc ss' and t has exactly one entering arc $t't$, so the maximum flow is 1. Assume also that there exist two arc-disjoint paths from s' to t' , so A_{mincut} is exactly $\{ss', t't\}$. Then the problem consists of deciding whether \mathcal{N} admits an (s, t) -flow x such that, for every arc $uv \notin A_{\text{mincut}}$, D_x contains an (s, t) -path disjoint from uv . We claim that this problem consists exactly of finding, in $D - \{s, t\}$, three arc-disjoint paths P_1, P_2 and P_3 such that P_1, P_2 go from s' to t' and P_3 goes from t' to s' , which is known to be NP-hard. This follows from results in [13] and the fact that we can reduce a problem on vertex-disjoint paths to a problem on arc-disjoint paths via the vertex splitting operation.

To see this, assume first that we have such three paths P_1, P_2 and P_3 . Let x be the flow obtained by sending exactly one unit of flow along each path and one unit of flow along ss' and $t't$. Then, for every arc $uv \notin A_{\text{mincut}} = \{ss', t't\}$, $D_x \setminus \{uv\}$ contains at least one of the paths sP_1t and sP_2t .

Conversely, if \mathcal{N} admits such a flow x , then it can be decomposed into a path-flow sP_1t (where P_1 goes from s' to t') and some cycle-flows C_1, \dots, C_r . Note that P_1, C_1, \dots, C_r are pairwise arc-disjoint because all the capacities are 1. The digraph D' on vertices $V(D) \setminus \{s, t\}$ with arc-set $A(C_1) \cup \dots \cup A(C_r)$ is eulerian, and s, t must belong to the same connected component of D' , for otherwise the last arc of P_1 leaving the connected component of s is an (s, t) -cut-arc in D_x . Hence, since D' is eulerian, it admits two arc-disjoint paths P_2, P_3 such that P_2 goes from s' to t' and P_3 goes from t' to s' .

For larger values of k , one can just add $k - 1$ arcs (of paths of length two if we forbid multiple arcs) from s to t to obtain the hardness result.

The perspicacious reader may see that, in the proof above, the hardness of the k -ARC-PERSISTENT-MAX-FLOW problem mainly comes from the fact that an optimal solution may contain cycle-flows. The following problem then naturally arises.

Problem 15. What is the complexity of the 1-ARC-PERSISTENT-MAX-FLOW problem for acyclic digraphs?

There is an analogous version involving vertex deletions. Again, if a vertex v is incident to one or more arcs across some minimum (s, t) -cut, then deleting v and its incident arcs will affect the value of all such minimum (s, t) -cuts by the same amount. Let V_{mincut} be the set of vertices of \mathcal{N} which are incident to at least one arc in a minimum (s, t) -cut.

k -VERTEX-PERSISTENT-MAX-FLOW

Input: A flow network $\mathcal{N} = (D = (V, A), s, t, c)$ and an integer k .

Output: A maximum flow x such that the minimum value of a maximum flow \tilde{x} in $D_x[V(D_x) - V']$ over all subsets $V' \subseteq V - V_{\text{mincut}}$ of k vertices is as large as possible.

Using exactly the same arguments as above, one can show that already 1-VERTEX PERSISTENT-MAX-FLOW PROBLEM is NP-hard with all capacities are 1, by reducing from the vertex-disjoint version of the problem of finding three paths P_1, P_2 , and P_3 such that P_1, P_2 go from s' to t' and P_3 goes from t' to s' , which is known to be NP-hard [13].

The following might be an interesting generalisation of arc-connectivity.

Question 16. What is the complexity of finding, for a given network \mathcal{N} and an integer K , the minimum number of arcs that one has to delete before the max-flow value in the resulting network is below K ?

Observe that, when $K = 1$, this problem is exactly the min-cut problem. This is why it is actually a generalisation of the arc-connectivity. Note also that, if all the capacities are 1, all the minimum (s, t) -cut have the same number of arcs, so the solution is exactly $c(S^*, \bar{S}) - K + 1$ (where (S^*, \bar{S}^*) is any min-cut).

6. Flows that are decomposable into a given number of paths

6.1. General case

In this section we consider the following problems. The first one has been studied previously under the name of the p -splittable maximum flow problem in [2].

p -DECOMPOSABLE-MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$ and an integer p .

Output: The maximum value of a flow x such that x can be decomposed into at most p path-flows.

p -ARC-DECOMPOSABLE-MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$ and an integer p .

Output: The maximum value of a flow x such that x can be decomposed into at most p pairwise arc-disjoint path-flows.

p -VERTEX-DECOMPOSABLE-MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$ and an integer p .

Output: The maximum value of a flow x such that x can be decomposed into at most p pairwise vertex-disjoint path-flows.

Note that every solution for the second problem is actually a solution for the first one, but that the converse is not true. Analogously, every solution to the third one is a solution to the second one.

It was proved in [2] that 2-DECOMPOSABLE-MAX-FLOW is NP-Hard and that it cannot be approximated by any ratio larger than $\frac{2}{3}$ unless P=NP. We extend their result by showing NP-hardness and non-approximability results of p -DECOMPOSABLE-MAX-FLOW for every fixed $p \geq 2$.

Theorem 17. *For any fixed $p \geq 2$, the p -DECOMPOSABLE-MAX-FLOW problem is NP-hard. Moreover, unless P=NP, it cannot be approximated by any ratio larger than $\rho(p) = \min(\rho_1(p), \rho_2(p))$, where $\rho_1(p), \rho_2(p)$ are defined as follows:*

$$\rho_1(p) = \begin{cases} \frac{5}{6} & \text{if } p \equiv 0 \pmod{4} \\ \frac{5p-1}{6p-2} & \text{if } p \equiv 1 \pmod{4} \\ \frac{5p-2}{6p-2} & \text{if } p \equiv 2 \pmod{4} \\ \frac{6p}{5p-3} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

$$\rho_2(p) = \begin{cases} \frac{4}{5} & \text{if } p \text{ is even} \\ \frac{4p-2}{5p-3} & \text{otherwise.} \end{cases}$$

In particular, $\rho(2) = \frac{2}{3}$, $\rho(3) = \frac{3}{4}$, $\rho(p) \xrightarrow{p \rightarrow +\infty} \frac{4}{5}$, and $\rho(p) \leq \frac{9}{11}$ in general.

Proof. Let us fix p , we will first prove that the p -DECOMPOSABLE-MAX-FLOW problem cannot be approximated by any ratio larger than $\rho_1(p)$, and then that it cannot by any ratio larger than $\rho_2(p)$.

In both cases, we will reduce from the WEAK-2-LINKAGE problem, which is NP-hard by Theorem 3. Let $\mathcal{I} = (D, s_1, s_2, t_1, t_2)$ be an instance of the WEAK-2-LINKAGE problem. We assume that there exists an (s_2, t_2) -path in D , for otherwise \mathcal{I} is clearly a negative instance (and it can be checked in polynomial time). We also assume that there exists two arc-disjoint paths from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, for otherwise \mathcal{I} is clearly a negative instance.

In both case, we will build a network $\mathcal{N} = (D', s, t, c)$ from \mathcal{I} and show that the solution of p -DECOMPOSABLE-MAX-FLOW on \mathcal{N} is at least o^+ if \mathcal{I} is a positive instance, and at most o^- otherwise, where $o^- < o^+$ are two specific values depending only on p . This will show that p -DECOMPOSABLE-MAX-FLOW is NP-Hard and that it cannot be approximated by any ratio larger than $\frac{o^-}{o^+}$ (unless P=NP).

We now make a construction for which $\frac{o^-}{o^+} = \rho_1(p)$. We will distinguish four cases depending on the value of p modulo 4. They are all similar but we detail each of them for completeness.

- Assume first that $p = 4q$ for some $q \in \mathbb{N}$.

We build \mathcal{N}_0 as follows. We start from $2q$ disjoint copies D^1, \dots, D^{2q} of D . For every vertex v of D , we denote its copy in D^i by v^i . Then we add a source s , a sink t , and the arcs $ss_1^i, ss_2^i, t_1^i t, t_2^i t$ for every $i \in [2q]$. All the arcs of \mathcal{N}_0 have capacity 2, except the arcs ss_1^i and $t_1^i t$ (for $i \in [2q]$) which have capacity 1.

Assume first that \mathcal{I} is a positive instance of WEAK-2-LINKAGE, which means that D contains two arc-disjoint paths P_1, P_2 going respectively from s_1, s_2 to t_1, t_2 . For each $i \in [2q]$, let P_1^i and P_2^i be the copies of P_1 and P_2 respectively in D^i . Let Q_1^i and Q_2^i be the concatenations $ss_1^i, P_1^i, t_1^i t$ and $ss_2^i, P_2^i, t_2^i t$ respectively. The flow consisting of exactly two units of flow on every path Q_2^i and exactly one unit of flow on every path Q_1^i shows that the solution of p -DECOMPOSABLE-MAX-FLOW on \mathcal{N}_0 is at least $o^+ = 6q$.

Assume now that \mathcal{I} is a negative instance of WEAK-2-LINKAGE, and let x be a flow of maximum value that can be decomposed into at most p path-flows. For every $i \in [2q]$, let $x_i = x(ss_1^i) + x(ss_2^i)$. Note that $x_i \leq 3$, we will first show that the maximality of x implies $x_i \geq 2$.

If $x_i = 1$, then x contains exactly one path-flow going through D^i , and it has value 1. Since D contains a path from s_2 to t_2 , we can replace this path-flow by a path-flow of value 2 to contradict the maximality of x . If $x_i = 0$, x decomposes into exactly p path-flows, for otherwise we can just add one path-flow to x going through D^i to contradict the maximality of x . Then, there exists $j \neq i$ such that x contains three distinct path-flows of value exactly 1 going through D^j . Replacing one of these path-flows by a path-flow of value 2 going through D^j contradicts the maximality of x .

Let n be the number of indices i such that $x_i = 3$. The number of path-flows x is decomposed into is at least $3n + (2q - n)$. Since this number is at most $p = 4q$, it implies that $n \leq q$. Note also that $|x| = 3n + 2(2q - n) = 4q + n$. Hence the value of x is at most $o^- = 5q$.

We have $\frac{o^-}{o^+} = \frac{5q}{6q} = \frac{5}{6}$ as desired.

- Assume now that $p = 4q + 1$ for some $q \in \mathbb{N}$.

We let \mathcal{N}_1 be the network obtained from \mathcal{N}_0 by adding the arc st with capacity 1. If \mathcal{I} is a positive instance, using the same flow as in the previous case plus one path-flow of value 1 through the arc st , we get that the solution of p -DECOMPOSABLE-MAX-FLOW on \mathcal{N}_1 is at least $o^+ = 6q + 1$.

Assume now that \mathcal{I} is a negative instance, and let x be a flow of maximum value that can be decomposed into at most p path-flows $P_1, \dots, P_{p'}$. At least one the P_i s has value 1, otherwise for every $i \in [2q]$, there exists exactly one path-flow (which has value 2) going through D^i . Hence $p' = 2q < p$ and we can add to x the path-flow of value 1 going through the arc st , which contradicts the maximality of x .

Hence we may assume without loss of generality that $P_{p'}$ is a path-flow of value 1, and that $P_{p'}$ is st if one of the P_i s is exactly the path on two vertices st . Therefore the flow x' made of the $p' - 1$ path-flows $P_1, \dots, P_{p'-1}$ is an (s, t) flow of \mathcal{N}_0 that can be decomposed into at most $p - 1$ path-flows. The previous case implies that $|x'| \leq 5q$. Hence $|x| = |x'| + 1 \leq 5q + 1 = o^-$.

We have $\frac{o^-}{o^+} = \frac{5q+1}{6q+1} = \frac{5p-1}{6p-2}$ as desired.

- Assume now that $p = 4q + 2$ for some $q \in \mathbb{N}$.

We build \mathcal{N}_2 exactly as we build \mathcal{N}_0 , except that we take $2q + 1$ copies of D instead of $2q$.

If \mathcal{I} is a positive instance, using exactly the same argument as we did for \mathcal{N}_0 , we can build a flow on \mathcal{N}_2 that can be decomposed into p path-flows and that has value $o^+ = 3(2q + 1) = 6q + 3$.

Assume now that \mathcal{I} is a negative instance, and let x be a flow of maximum value that can be decomposed into at most p path-flows. For every $i \in [2q + 1]$, let $x_i = x(ss_1^i) + x(ss_2^i)$. For the same reasons as before, we have $x_i \in \{2, 3\}$. Let n be the number of indices i such that $x_i = 3$. The number of path-flows x decomposed into is at least $3n + (2q + 1 - n)$, implying that $n \leq q$. Note also that $|x| = 3n + 2(2q + 1 - n) = 4q + n + 2$. Hence the value of x is at most $o^- = 5q + 2$.

We have $\rho(p) = \frac{o^-}{o^+} = \frac{5q+2}{6q+3} = \frac{5p-2}{6p}$ as desired.

- Assume finally that $p = 4q + 3$ for some $q \in \mathbb{N}$.

We let \mathcal{N}_3 be the network obtained from \mathcal{N}_2 by adding the arc st with capacity 1. Using the same ideas we used in the second case, with \mathcal{N}_2 playing the role of \mathcal{N}_0 , we obtain $o^+ = 6q + 4$ and $o^- = 5q + 3$.

Hence we obtain $\frac{o^-}{o^+} = \frac{5q+3}{6q+4} = \frac{5p-3}{6p-2}$ as desired.

This concludes the first part of the proof. We now make a construction for which $\frac{o^-}{o^+} = \rho_2(p)$. Assume first that $p = 2q$ is even. We build $\mathcal{N}' = (D, s, t, c')$ exactly as \mathcal{N}_0 from q copies D^1, \dots, D^q of D . We just change the capacities, so every arc has capacity 3 except ss_1^i and $t_1^i t$ (for $i \in [q]$) which have capacity 2.

If \mathcal{I} is a positive instance, then we can send 5 units of flow through each copy of D , using exactly p path-flows, and we get a flow of value $5q$. Assume below that \mathcal{I} is a negative instance and let x be a maximum flow in \mathcal{N}' .

For every $i \in [q]$, let $x_i = x(ss_1^i) + x(ss_2^i)$. The maximality of x implies that $x_i \in \{3, 4, 5\}$ for every $i \in [q]$. Indeed, if all the path-flows of x have value 3, then x decomposed into at most $q - 1$ path-flows, and we can just add one additional path-flow of value 3 through D^i . Else if x sends no flow through a copy D^i of D , then we can replace a path-flow of x with value 1 or 2 by a path-flow of value 3 through D^i . In both cases, we contradict the maximality of x .

Now, if x has exactly one path-flow through D^i , then it has value 3 (since there exists a path from s_2 to t_2). If x has two path-flows through D^i , then $x_i = 4$ because \mathcal{I} is a negative instance. Moreover, if $x_i = 5$ then x must have 3 path-flows through D^i (since \mathcal{I} is a negative instance).

For each $j \in \{3, 4, 5\}$, let n_j be the number of indices i such that $x_i = j$. We have $|x| = 3 \cdot n_3 + 4 \cdot n_4 + 5 \cdot n_5$, and the minimum number of path-flows x decomposes into is $n_3 + 2 \cdot n_4 + 3 \cdot n_5$. If $n_5 > n_3$ then x decomposes into at least $2q + 1$ path-flows, a contradiction since $p = 2q$. Then we have $n_3 \geq n_5$ and $|x| \leq 4 \cdot (n_3 + n_4 + n_5) = 4q$. Hence when p is even, unless $P = NP$, there is no approximation with a ratio larger than $\rho_2(p) = \frac{4q}{5q} = \frac{4}{5}$.

When $p = 2q + 1$ is odd, we just add to \mathcal{N}' an arc from s to t with capacity 1. If \mathcal{I} is a positive instance, we have a flow of value $5q + 1$. When \mathcal{I} is a negative instance, one can easily show that there exists a maximum flow x that contains the path-flow st with value 1. Hence, forgetting this path-flow, we obtain a flow on \mathcal{N}' , and the result above implies $|x| \leq 4q + 1$. Hence, unless $P = NP$, there is no approximation with a ratio larger than $\rho_2(p) = \frac{4q+1}{5q+1} = \frac{4p-2}{5p-3}$. \square

Theorem 18. For any fixed p , both problems p -ARC-DECOMPOSABLE-MAX-FLOW and p -VERTEX-DECOMPOSABLE-MAX-FLOW are NP-hard and cannot be approximated by any ratio $\rho > \frac{2}{3}$ unless $P = NP$.

Proof. We reduce from WEAK-2-LINKAGE as in the proof of Theorem 17 for $p = 2$. Since the out-degree of s is 2, any solution to the p -ARC-DECOMPOSABLE-MAX-FLOW problem is actually a solution to the 2-ARC-DECOMPOSABLE-MAX-FLOW. This shows the result for the arc-disjoint version.

The arguments for the vertex-disjoint version are the same, except that we reduce from 2-LINKAGE instead of 2-WEAK-LINKAGE. \square

It was shown in [2] that one can obtain a ρ -approximation for the value of maximum p -decomposable flow in polynomial time, where $\rho = \frac{2}{3}$ if $p \in \{2, 3\}$ and $\rho = \frac{1}{2}$ when $p \geq 4$. It is important to note that, in their work, the flows are valued in \mathbb{R} , while ours are valued in \mathbb{N} . Still their result can be adapted to flows valued in \mathbb{N} , using the following rounding strategy, to obtain a $\frac{\rho}{2}$ -approximation.

First compute in polynomial time a flow y (valued in \mathbb{R}) decomposable into $r \leq p$ path-flows P_1, \dots, P_r , such that $|y| \geq \rho \cdot |y^*|$, where y^* is a maximum p -decomposable flow valued in \mathbb{R} . A fortiori, we have $|y| \geq \rho \cdot |x^*|$, where x^* is a maximum p -decomposable flow valued in \mathbb{N} . Now let x be the p -decomposable flow valued in \mathbb{N} with path-flows P'_1, \dots, P'_r where, for each $i \in \{1, \dots, r\}$, P_i and P'_i use the same path and $|P'_i| = \lfloor |P_i| \rfloor$. We thus obtain $|x| \geq \rho \cdot |x^*| - p$. If $|x| < p$, we replace x by any flow of value p which decomposes into p path-flows (we assume that such a flow exists, for otherwise the problem is clearly solvable in polynomial time). We thus have $|x| \geq \max(p, \rho \cdot |x^*| - p)$. If $|x^*| \geq \frac{2p}{\rho}$, then $|x| \geq \rho \cdot |x^*| - p \geq \rho \geq \frac{\rho}{2} |x^*|$. Else if $|x^*| \leq \frac{2p}{\rho}$, then $|x| \geq p \geq \frac{\rho}{2} |x^*|$.

The following result improves on the $\frac{\rho}{2}$ -bound for small values of p . We will also show that it can be derived into an approximation result for the disjoint versions of the problem.

Theorem 19. For any fixed p , the p -DECOMPOSABLE-MAX-FLOW problem can be approximated by a ratio $\rho = \frac{1}{H(p)}$ where $H(p) = \sum_{i=1}^p \frac{1}{i}$ is the p^{th} term of the harmonic series.

Proof. Let us fix p , we will show that Algorithm 1 below is a $\frac{1}{H(p)}$ -approximation for the p -DECOMPOSABLE-MAX-FLOW problem. Note first that Algorithm 1 runs in polynomial time. In particular, observe that the for loop of line 3 may be replaced by a dichotomy strategy, so if c_{\max} is larger than the size of the network this is not a problem.

Algorithm 1 $\frac{1}{H(p)}$ -approximation for p -DECOMPOSABLE-MAX-FLOW.

Input: A flow network $\mathcal{N} = (D = (V, A), s, t, c)$.

Output: A flow x that is p -decomposable with value at least $\frac{1}{H(p)} \cdot |x^*|$, where $|x^*|$ is the value of an optimal solution.

```

1: for every  $i \in \{1, \dots, p\}$  do
2:    $x_i \leftarrow$  the empty flow.
3:   for  $v = 1$  to  $c_{\max}$  where  $c_{\max} = \max\{c(uv) \mid uv \in A\}$  do
4:     Build  $D_v$  from  $D$  as follows : for each arc  $uv$ , if  $c(uv) < v$ , then remove  $uv$ , otherwise replace  $uv$  by  $\lfloor \frac{c(uv)}{v} \rfloor$  multiple arcs.
5:     if there exists  $i$  arc-disjoint paths from  $s$  to  $t$  in  $D_v$  then
6:        $x_i \leftarrow$  the flow made by sending exactly  $v$  units of flow on every such path.
7:     end if
8:   end for
9: end for
10: return A flow  $x_i$  among  $x_1, \dots, x_p$  with maximum value.
```

Let $\mathcal{N} = (D, s, t, c)$ be an instance of the p -DECOMPOSABLE-MAX-FLOW problem. Let x be the flow computed by Algorithm 1 on \mathcal{N} and x^* be an optimal solution. Note that x is p -decomposable because any flow composed of i arc-disjoint paths in D_v is actually a i -decomposable flow in D .

Assume that $\rho \cdot |x^*| > |x|$ for some $\rho \in [0, 1]$, then we will show that $\rho > \frac{1}{H(p)}$ must hold. Observe that, by definition of a p -decomposable flow, x^* can be decomposed into p path-flows x_1^*, \dots, x_p^* of values $c_1^* \geq \dots \geq c_p^*$ respectively (some of them may be empty). For each $i \in [p]$, we define c_i as the largest integer $v \in \{1, \dots, c_{\max}\}$ such that D_v contains at least i arc-disjoint paths from s to t .

For each $i \in \{1, \dots, p\}$, let us show that $c_i \geq c_i^*$. If $c_i^* = 0$, this is clear. Else if $c_i^* \geq 1$ then x_1^*, \dots, x_i^* converts into a collection of i non-empty arc-disjoint paths from s to t in $D_{c_i^*}$. Note that they are actually arc-disjoint because each path-flow x_1^*, \dots, x_i^* has value at least c_i^* , so if k of them share an arc uv , then $c(uv) \geq k \cdot c_i^*$ and $D_{c_i^*}$ contains at least k multiple arcs uv . This shows $c_i \geq c_i^*$.

Observe also that x_i , the flow computed by Algorithm 1 at the i^{th} step of the first for-loop, has value exactly $i \cdot c_i$. Since x is exactly a flow x_i with maximum value, we deduce the following inequalities for every $i \in [p]$:

$$\rho \cdot |x^*| > |x| \geq i \cdot c_i \geq i \cdot c_i^*$$

Multiplying each inequality by $\frac{1}{i}$ and summing all the resulting inequalities, we obtain:

$$\sum_{i=1}^p \left(\frac{1}{i} \cdot \rho \cdot |x^*| \right) > \sum_{i=1}^p c_i^*.$$

Since $|x^*|$ is exactly $\sum_i c_i^*$, we deduce from the inequality above that $\rho > \frac{1}{H(p)}$ as desired. \square

Theorem 20. For any fixed p , both problems p -ARC-DECOMPOSABLE-MAX-FLOW and p -VERTEX-DECOMPOSABLE-MAX-FLOW can be approximated by a ratio $\rho = \frac{1}{H(p)}$.

Proof. The proofs are really similar the proof of Theorem 19 so we only briefly describe them.

For the arc-disjoint version, we will consider Algorithm 1 with the following modification. On line 4, we build D_v by removing every arc uv with capacity $c(uv) < v$, and we do not modify the other arcs. For the vertex-disjoint version, we do the same modification and we look for vertex-disjoint paths instead of arc-disjoint paths in line 5.

In both cases, we consider an optimal solution x^* that is the (vertex or arc)-disjoint union of p path-flows x_1^*, \dots, x_p^* (some of them may be empty). The flow sent on each path x_i^* is exactly the minimum capacity of its arc-set. For every $i \in [p]$, if c_i is the largest capacity v such that D_v contains i (vertex or arc)-disjoint paths, we obtain that $i \cdot c_i \geq i \cdot c_i^*$. We then conclude as in the proof of Theorem 19. \square

Remark 21. Algorithm 1 is a $\frac{2}{3}$ -approximation for the 2-DECOMPOSABLE-MAX-FLOW problem, and this is best possible by Theorem 17. For $p = 3$ Algorithm 1 is a $\frac{6}{11}$ -approximation algorithm and the bound from Theorem 17 is $\frac{3}{4}$ so there may exist a better approximation algorithm.

Problem 22. What is the best approximation guarantee one can obtain for the p -DECOMPOSABLE-MAX-FLOW problem when $p > 2$?

6.2. Restriction to acyclic networks

In this section, we consider the NP-hard problems of the previous section restricted to acyclic networks. Some remain NP-hard whereas others turn out to be polynomial time solvable.

Given a source s in a digraph D and an ordered set of vertices $W = (v_1, \dots, v_{|W|})$, a W -tricot is an ordered set of $|W|$ paths $(Q_1, \dots, Q_{|W|})$ that pairwise intersect exactly on $\{s\}$, and such that Q_i goes from s to v_i . The **value** of a W -tricot $T = (Q_1, \dots, Q_{|W|})$ is $(c_1, \dots, c_{|W|})$ where c_i is the minimum capacity along Q_i . The **total value** of T is exactly $\sum_{i=1}^{|W|} c_i$. Given two W -tricot T and T' with values $(c_1, \dots, c_{|W|})$ and $(c'_1, \dots, c'_{|W|})$ respectively, we consider that the value of T is at least as large as the value of T' if $c_i \geq c'_i$ holds for every i .

Given a network $\mathcal{N} = (D, s, t, c)$, if $W \subseteq N^-(t)$, the (s, t) -flow associated with a W -tricot $T = (Q_1, \dots, Q_{|W|})$ is the flow made by sending exactly c_i units of flow on every path Q_i (extended to t) of T . Note that this needs to assume that the capacity $c_i(t, t)$ is at least c_i for each end-vertex t_i of Q_i . In Algorithm 2, we ensure that this is true by subdividing every arc entering t .

Theorem 23. When restricted to acyclic networks, the problem p -VERTEX-DECOMPOSABLE-MAX-FLOW can be solved in time $O(n^{f(p)})$ for some computable function f .

Proof. Our proof is inspired from the one due to Fortune et al. [13] when they showed that the p -LINKAGE problem restricted to acyclic networks is in XP when parameterised by p . Let us show that Algorithm 2 is correct. Note that this algorithm computes a maximum flow x such that D_x is the vertex-disjoint union of exactly p paths (if such a flow exists). For the general problem, where D_x is the vertex-disjoint union of at most p paths, we just execute Algorithm 2 for every value $p' \leq p$, and choose the maximum computed flow.

From now on, we consider D as the digraph where each arc su and vt has been subdivided (after line 1). When we subdivide an arc, we set the capacities of the new arcs to the capacity of the original one. In particular, note that this operation does not change the value of a p -vertex-disjoint maximum flow. To prove that Algorithm 2 is correct, we will show by induction on $i \in [r]$ that after iteration i of the for-loop of line 9, every element of $L[W_i]$ is indeed a W_i -tricot. Moreover, for every W_i -tricot T , we will prove that there exists $T' \in L[W_i]$ such that $\text{value}(T') \geq \text{value}(T)$. This will imply the result since, for every p -vertex-disjoint flow x , its support D_x is actually a W -tricot, where $W = V(D_x) \cap N^-(t)$. Let us fix $i \in [r]$ and assume that both statements hold for any $\ell < i$.

First let T' be any element of $L[W_i]$ that has been added at step $\alpha < i$ of the for-loop of line 9. At iteration α , by induction, $T = (Q_1, \dots, Q_p)$ is a W_α -tricot. Let v be the only vertex in $W_i \setminus W_\alpha$, then v does not belong to any path Q_h otherwise there would be a path from v to the end of Q_h , which belongs to W_α . This shows that T' must be a W_i -tricot.

Now let \tilde{T} be any W_i -tricot with value $(\tilde{c}_1, \dots, \tilde{c}_p)$. Let z be the last vertex of W_i according to the computed ordering v_1, \dots, v_n . If $z \in N^+(s)$, then $W_i \subseteq N^+(s)$ and the W_i -tricot computed in the for-loop of line 4 is exactly \tilde{T} . Henceforth we assume that z has a predecessor $y \neq s$ in \tilde{T} . Let W^y be the p -tuple obtained from W_i by replacing z by y . In the computed ordering on the p -tuples, W^y must be smaller than W_i (because z is larger than y). Let ℓ be the index such that $W_\ell = W^y$. By induction, $L[W_\ell]$ contains a W_ℓ -tricot T with value (c_1, \dots, c_p) at least as large as the value of $\tilde{T} - \{z\}$. Hence, for each $h \in [p]$, $c_h \geq \tilde{c}_h$. At iteration ℓ , at some step we consider the tricot T , where the vertex y plays the role of u and the vertex z plays the role of v . Note that there is no path from z to W_ℓ , because z is the largest vertex of W_i (according to the computed acyclic ordering) and because y is an in-neighbour of z . Hence, at this moment, we consider W' which is actually W_i and T' , built from T , with value (c'_1, \dots, c'_p) . By construction of T' , we have $c'_h = c_h \geq \tilde{c}_h$ when $h \neq j$ and $c'_j = \min\{c_j, c(yz)\} \geq \tilde{c}_j$ (where j is the index of the path containing y in T , as in the algorithm). Thus, at the end of iteration ℓ , either T' is an element of $L[W_i]$ larger than \tilde{T} or $L[W_i]$ already contains an element even larger than T' .

Now we justify that Algorithm 2 runs in time $O(n^{f(p)})$ for some computable function f . Let us bound the number of iterations of each for-loop of the algorithm. Note that the number of p -tuples is bounded by $\binom{n+m}{p}$ (recall that we subdivided some arcs in the beginning of the algorithm). Also note that, for every p -tuple W and every W -tricot T with value (c_1, \dots, c_p) , each coordinate c_i must correspond to the capacity of an arc. Thus, the number of possible values for T is bounded by m^p . Hence the number of tricots in $L[W]$ is at most m^p since $L[W]$ never contains two W -tricots with the same value. Altogether, we get that Algorithm 2 runs in time $O(n^{f(p)})$ where f is some computable function. \square

Algorithm 2 p -VERTEX-DECOMPOSABLE-MAX-FLOW restricted to acyclic networks.

Input: A flow network $\mathcal{N} = (D = (V, A), s, t, c)$ such that D is acyclic.

Output: A maximum flow x such that D_x is the vertex-disjoint union of exactly p paths.

```

1: Subdivide every arc  $su$  and  $vt$ .
2: Tricot[]  $L$  : a list indexed by the  $p$ -tuples of  $V(D) \setminus \{s\}$ . Each cell  $L[W]$  is a set of  $W$ -tricots.
3: Initially, for every  $p$ -tuple  $W$ ,  $L[W] \leftarrow \emptyset$ .
4: for every  $p$ -tuple  $W = (s_1, \dots, s_p) \subseteq N^+(s)$  do
5:    $L[W] \leftarrow \{ \text{the only } W\text{-tricot } (ss_1, \dots, ss_p) \}$ .
6: end for
7:  $v_1, \dots, v_n \leftarrow$  an acyclic ordering of  $V(D)$  for which  $N^+[s]$  are the first vertices.
8:  $W_1, \dots, W_r \leftarrow$  the lexicographic ordering of the  $p$ -tuples of  $V(D) \setminus \{s\}$  (w.r.t.  $v_1, \dots, v_n$ ).
9: for  $i = 1$  to  $r$  do
10:  for every tricot  $T \in L[W_i]$  do
11:    Denote  $T = (Q_1, \dots, Q_p)$ .
12:    for every vertex  $u \in W_i$  do
13:       $Q_j \leftarrow$  the path of  $T$  ending on  $u$ .
14:      for every vertex  $v \in N^+(u)$  do
15:        if there is no path from  $v$  to  $W_i$  in  $D$  then
16:           $W' \leftarrow W_i \setminus \{u\} \cup \{v\}$ .
17:          Let  $T' = (Q'_1, \dots, Q'_p)$  be the  $W'$ -tricot where  $Q'_j = Q_j \cup \{v\}$  and  $Q'_h = Q_h$  for  $h \neq j$ .
18:          if for every  $\tilde{T} \in L[W']$  value( $\tilde{T}$ )  $\not\geq$  value( $T'$ ) then
19:             $L[W'] \leftarrow L[W'] \cup T'$ .
20:          end if
21:        end if
22:      end for
23:    end for
24:  end for
25: end for
26: Find the tricot  $T \in \bigcup_{W \subseteq N^+(s)} L[W]$  with maximum total value.
27: return the flow associated with  $T$ .
```

Corollary 24. *When restricted to acyclic networks, the p -ARC-DECOMPOSABLE-MAX-FLOW problem can be solved in time $O(n^{f(p)})$ for some computable function f .*

Proof. Let $\mathcal{N} = (D, s, t, c)$ be an instance of the p -ARC-DECOMPOSABLE-MAX-FLOW problem, with D being acyclic. We may assume that there is no arc from s to t in D , for otherwise we just subdivide it. We let D' be the line digraph of D , that is $V(D') = A(D)$ and

$$A(D') = \{uv \mid u, v \in A(D), \text{ the head of } u \text{ coincides with the tail of } v\}.$$

We then add to D' a source s' and a sink t' , and all arcs of $\{s'u \mid u \text{ is a leaving arc of } s \text{ in } D\} \cup \{vt' \mid v \in A(D) \text{ is an entering arc of } t\}$. We finally define the capacities on $A(D')$ as follows:

$$\forall uv \in A(D'), c'(uv) = \begin{cases} c(v) & \text{if } u = s', \\ c(u) & \text{if } v = t', \\ \min(c(u), c(v)) & \text{otherwise.} \end{cases}$$

Let $\mathcal{N}' = (D', s', t', c')$. It is easy to see that we can solve the p -ARC-DECOMPOSABLE-MAX-FLOW problem on \mathcal{N} by solving the p -VERTEX-DECOMPOSABLE-MAX-FLOW problem on \mathcal{N}' . As we saw in Theorem 23 this can be done in time $O(|V(D')|^{f(p)})$ so we can solve the p -ARC-DECOMPOSABLE-MAX-FLOW in time $O(m^{f(p)}) = O(n^{f(p)})$. \square

Question 25. Is there an analogue of Theorem 23 for p -DECOMPOSABLE-MAX-FLOW?

Theorem 26. *When p is part of the input, the problems p -DECOMPOSABLE-MAX-FLOW, p -ARC-DECOMPOSABLE-MAX-FLOW and p -VERTEX-DECOMPOSABLE-MAX-FLOW are NP-hard for acyclic networks, even when all capacities are 1 or 2.*

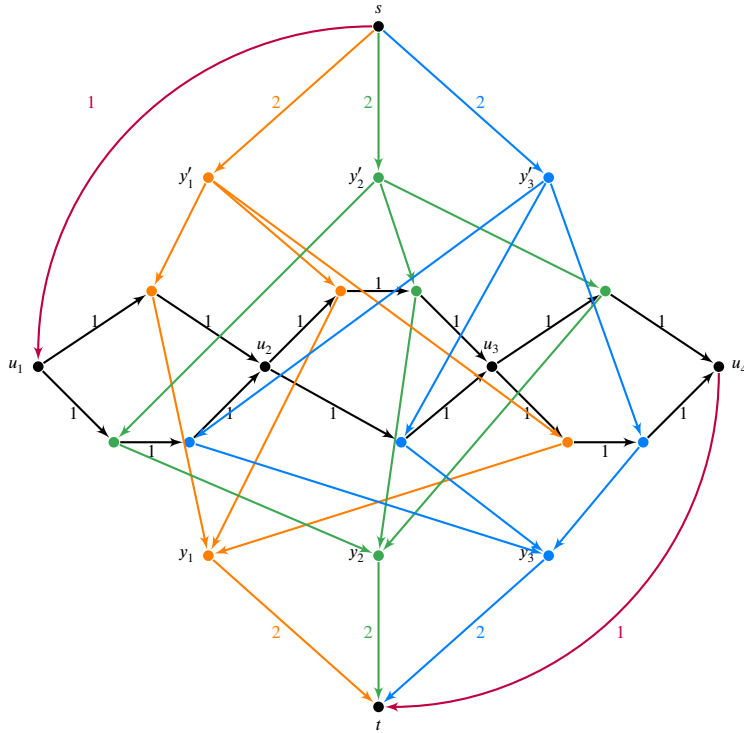


Fig. 8. The network $\hat{\mathcal{N}}_{\mathcal{F}}$ when $\mathcal{F} = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$.

Proof. We prove the result for the vertex-disjoint version. The hardness of the arc-disjoint version then follows from the usual splitting operation. This also shows the hardness of p -DECOMPOSABLE-MAX-FLOW since, in the reduction, every optimal solution contains exactly one path-flow of value 1 (and all other path-flows have value 2). Therefore, since all capacities are 1 or 2, the path-flows must be arc-disjoint.

We show that 3-SAT reduces to our problem. The reduction uses a modification of the gadget used in the proof of Theorem 3 in [12]. Let C be an instance of 3-SAT. We first construct a digraph $\hat{D}_{\mathcal{F}}$ which is similar to the digraph $\tilde{D}_{\mathcal{F}}$ used in the proof of Theorem 7. The only difference is that the digraph H has 5 vertices y', y, a_1, a_2, a_3 and the arcs form the three (y', y) -paths $y'a_1y, y'a_2y, y'a_3y$.

The analogous version of Remark 8 also holds when we want the (u_1, v_n) -path in $\tilde{D}_{\mathcal{F}}$ to avoid at least one vertex of each set $\{a_{i,1}, a_{i,2}, a_{i,3}\}$.

Now we construct $\hat{D}_{\mathcal{F}}$ from $\tilde{D}_{\mathcal{F}}$ by adding two new vertices s, t and the following arcs $su_1, sy'_1, \dots, sy'_m, v_n t, y_1 t, \dots, y_m t$. Finally form the network $\hat{\mathcal{N}}_{\mathcal{F}} = (\hat{D}_{\mathcal{F}}, s, t, c)$ by giving the arcs $\{sy'_j, y'_j a_{i,k}, a_{i,k} y_j, y_j t \mid j \in [m], k \in [3]\}$ capacity 2 and all other arcs capacity 1. Fig. 8 illustrates the construction of $\hat{\mathcal{N}}$.

We claim that $\hat{\mathcal{N}}_{\mathcal{F}}$ has an (s, t) -flow of value $2m + 1$ which can be decomposed into $m + 1$ vertex-disjoint path-flows if and only if \mathcal{F} is satisfiable.

Suppose first that ϕ is a truth assignment which satisfies \mathcal{F} . Fix one true literal for each clause and let $\ell_i \in [3]$ be the index of the true literal we chose for C_i . Let P be the (u_1, v_n) -path which for each $j \in [n]$ follows the subpath $u_j z_1 \dots z_{q_j} v_j$ if $\phi(x_j)$ is true and otherwise follows the subpath $u_j y_1 \dots y_{p_j} v_j$. Now we can send 2 units of flow along each of the paths $sy'_i a_{i,\ell_i} y_i t$ and one unit along the path sPt . By construction, all the $m + 1$ paths we used are vertex-disjoint.

Suppose now that $\hat{\mathcal{N}}_{\mathcal{F}}$ has an (s, t) -flow of value $2m + 1$ which can be decomposed into $m + 1$ vertex-disjoint path-flows. Let sQt be the path on which we send one unit of flow and note that Q must be a (u_1, v_n) -path. For each $j \in [n]$ this path will use either the subpath $u_j y_1 \dots y_{p_j} v_j$ or the subpath $u_j z_1 \dots z_{q_j} v_j$. If Q uses the first subpath we set x_j to false and otherwise we set x_j to true. Since we have m internally disjoint (s, t) -paths which avoid all vertices of Q , for each $i \in [m]$, at least one of the $a_{i,h}$ s where $h \in [3]$ is not on Q . By the analogous version of Remark 8, this shows that our truth assignment satisfies \mathcal{F} . \square

The following result shows that Theorem 23 is somehow best possible, in the sense that p -VERTEX-DECOMPOSABLE-MAX-FLOW, restricted to acyclic digraphs, cannot be solved in time $O(f(p) \cdot n^{O(1)})$ for any computable function f , unless $\text{FPT} = \text{W}[1]$.

Theorem 27. *The three problems*

- p -VERTEX-DECOMPOSABLE-MAX-FLOW,
- p -ARC-DECOMPOSABLE-MAX-FLOW, and
- p -DECOMPOSABLE-MAX-FLOW,

are all W[1]-hard when parameterised by p .

Proof. We first show that the p -VERTEX-DECOMPOSABLE-MAX-FLOW problem on acyclic digraphs is W[1]-hard when parameterised by p by reducing from p -LINKAGE parameterised by p on acyclic digraphs. This problem is W[1]-hard by Theorem 5. We will then show how to adapt it for p -ARC-DECOMPOSABLE-MAX-FLOW and p -DECOMPOSABLE-MAX-FLOW.

Let $\mathcal{I} = (D, (s_i)_{i \in [p]}, (t_i)_{i \in [p]})$ be an instance of the p -LINKAGE problem for acyclic digraphs. We first add a source s and a sink t to D , and all the arcs $ss_i, t_i t$ for $i \in [p]$. Clearly, the obtained digraph remains acyclic. For every $i \in [p]$, the capacity of ss_i and $t_i t$ is i . We set the capacity of every other arc to p .

Let \mathcal{N} be the obtained network. It is clear that \mathcal{I} is a positive instance if and only if the solution of p -VERTEX-DECOMPOSABLE-MAX-FLOW on \mathcal{N} is $\frac{1}{2}p(p+1)$. This shows the result.

For p -ARC-DECOMPOSABLE-MAX-FLOW, we do exactly the same reduction but we reduce from WEAK- p -LINKAGE. For p -DECOMPOSABLE-MAX-FLOW, we reduce from WEAK- p -LINKAGE but we change the capacities to ensure that the path-flows are pairwise disjoint. So we set the capacities of ss_i and $t_i t$ to $p+i$ for every $i \in [p]$, and we set the capacity of every other arc to $2p$. \square

7. Separable flows

Let x be a flow in a network $\mathcal{N} = (D, s, t, c)$ that is the union of p path-flows Q_1, \dots, Q_p . Then x is a q -vertex-separable flow if each vertex of $V(D) \setminus \{s, t\}$ belongs to at most q different paths Q_i . Analogously, x is a q -arc-separable flow if each arc of D belongs to at most q different paths Q_i . We consider the following two problems.

q -VERTEX-SEPARABLE-MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$.

Output: The maximum value of a flow x such that x can be decomposed into path-flows with the property that each vertex of $D - \{s, t\}$ belongs to at most q such path-flows.

q -ARC-SEPARABLE-MAX-FLOW

Input: A flow network $\mathcal{N} = (D, s, t, c)$.

Output: The maximum value of a flow x such that x can be decomposed into path-flows with the property that each arc of D belongs to at most q such path-flows.

Note that a flow x is 1-vertex separable if it is made of p path-flows Q_1, \dots, Q_p that are pairwise intersecting exactly on $\{s, t\}$. Analogously, x is 1-arc separable if Q_1, \dots, Q_p are arc-disjoint. Note that the number p of paths is not constrained. In the following we show that the two problems above are NP-hard even when the network is acyclic and capacities are in $\{1, 2\}$. This is in contrast with Theorem 23: unless $P=NP$, neither q -ARC-SEPARABLE-MAX-FLOW nor q -VERTEX-SEPARABLE-MAX-FLOW, restricted to acyclic networks, can be solved in time $O(n^{f(q)})$ for any computable function f .

Theorem 28. For every fixed $q \geq 1$, both q -ARC-SEPARABLE-MAX-FLOW and q -VERTEX-SEPARABLE-MAX-FLOW are NP-hard even when restricted to acyclic networks with capacities $\{1, 2\}$.

Proof. We will show the result for q -VERTEX-SEPARABLE-MAX-FLOW. The hardness of q -ARC-SEPARABLE-MAX-FLOW then follows easily by the usual splitting operation.

Let us fix q . We will reduce from p -VERTEX-DECOMPOSABLE-MAX-FLOW, with p being part of the input, which is NP-hard on acyclic networks with capacities in $\{1, 2\}$ by Theorem 26. The construction in the proof of Theorem 26 shows that the problem remains hard if p is equal to the out-degree of s .

So let $\mathcal{N} = (D, s, t, c)$ be a network instance of p -VERTEX-DECOMPOSABLE-MAX-FLOW with p being equal to the out-degree of s in D , D is acyclic and the capacities are in $\{1, 2\}$. We build D' from D as follows. For every vertex $v \in V(D) \setminus \{s, t\}$, we add to D $2(q-1)$ new vertices $v_1^-, \dots, v_{q-1}^-, v_1^+, \dots, v_{q-1}^+$. Then we add every arc of the path $sv_i^- v_i^+ t$ (for $i \in [q-1]$). Note that D' remains acyclic. Then we form $\mathcal{N}' = (D', s, t, c)$ by giving all the new arcs capacity 2.

We claim that \mathcal{N} admits an (s, t) -flow x made of at most p internally vertex-disjoint paths of value $|x|$ if and only if \mathcal{N}' admits a q -vertex-separable (s, t) -flow x' of value $|x'| = |x| + 2n(q-1)$, where $n = |V(D) \setminus \{s, t\}|$. This equivalence shows the result.

Assume first that \mathcal{N} admits an (s, t) -flow x made of at most p internally vertex-disjoint path-flows. We call these path-flows as the original ones. Note that this flow is also a flow of \mathcal{N}' . We complete this flow by sending 2 units of flow along every path of the form $sv_i^- v_i^+ t$ (for $v \in V(D) \setminus \{s, t\}$ and $i \in [q-1]$). We call these $n(q-1)$ path-flows the new ones. The obtained flow x' has value

$|x| + 2n(q - 1)$ as desired. Since the original path-flows are disjoint, and because every vertex (different from s, t) belongs to at most $q - 1$ new path-flows, we get that every vertex belongs to at most q path-flows. Hence x' is q -vertex-separable.

Conversely, assume now that \mathcal{N}' admits a q -separable flow of value $v + 2n(q - 1)$. Since it is q -separable, it is decomposable into path-flows such that each vertex but s, t belongs to at most q of them. Let x' be such a flow and $Q'_1, \dots, Q'_{p'}$ be such a decomposition into path-flows for which the number of path-flows of value 2 of the form $sv_i^-vv_i^+t$ is maximized. It is straightforward that, in this case, the number of such path-flows is exactly $n(q - 1)$. Let $Q_1, \dots, Q_{p' - n(q - 1)}$ be the other path-flows, then $p' - n(q - 1) \leq p$ because p is the out-degree of s in D . We claim that there are pairwise disjoint (except on $\{s, t\}$). Assume not, meaning that Q_i and Q_j are intersecting on $v \in V(D) \setminus \{s, t\}$. Note that v belongs to $(q - 1)$ other path-flows of $Q'_1, \dots, Q'_{p'}$ (the ones of the form $sv_i^-vv_i^+t$). Thus v belongs to $q + 1$ path-flows of $Q'_1, \dots, Q'_{p'}$, a contradiction. Finally note that the flow made of the path-flows $Q_1, \dots, Q_{p' - n(q - 1)}$ has value v , which concludes the proof. \square

CRedit authorship contribution statement

J. Bang-Jensen: Writing – original draft, Writing – review & editing. **S. Bessy:** Writing – original draft, Writing – review & editing. **L. Picasarri-Arrieta:** Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors are thankful to the anonymous referees who went deeply into the proofs of the paper, and for their valuable comments, which improved the quality of the manuscript.

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