Complexity of 2-SAT

2-SAT is the version of SAT where every clause has 2 literals

$(x_1 \vee \bar{x}_2) \wedge (x_1 \vee \bar{x}_3) \wedge (x_3 \vee \bar{x}_4) \wedge (x_2 \vee x_4)$

We will show:

<u>Theorem</u>

2-SAT can be solved in polynomial (in fact linear) time

Idea: reduce to a digraph problem

Given $f = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

where each C_i is over the literals

$x_1, x_2, \ldots x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$

Construct D_f by letting

$V(D_f) = \{v_1, v_2, \ldots, v_n, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$

when the vertex v_i ($\bar{v}_i$) corresponds to x_i ($\bar{x}_i$)

For every clause $(p \vee q)$ with

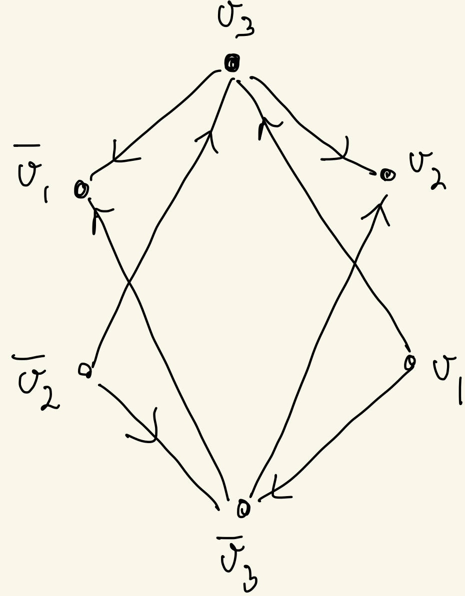
$p, q \in \{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$

$A(D_f)$ contains the arcs $\bar{p} \to q$ and $\bar{q} \to p$

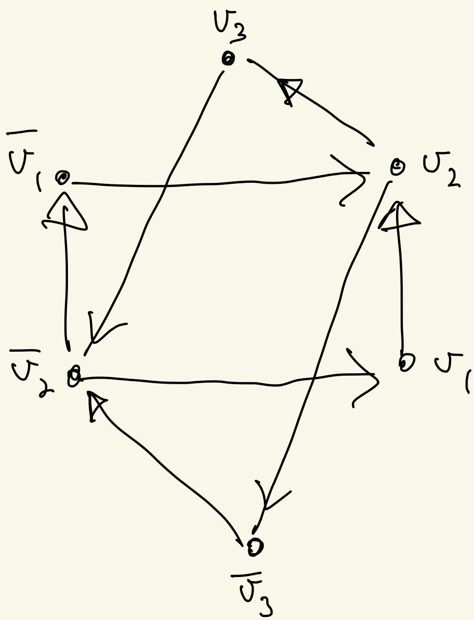
For example if $C_j = (x_5 \vee \bar{x}_7)$ then

$A(D_f)$ contains the arcs $\bar{v}_5 \to \bar{v}_7$ and $v_7 \to v_5$

(recall $\bar{\bar{x}}_i = x_i$



$$f = (\bar{x}_1 \vee \bar{x}_3) \vee (x_2 \vee \bar{x}_3) \vee (\bar{x}_1 \vee x_3) \vee (x_2 \vee x_3)$$



$$f = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

Properties of D_f:

(1) if $p \rightsquigarrow q$ then $\bar{q} \rightsquigarrow \bar{p}$

Proof: note that

$$a \to b \in A(D_f) \iff (\bar{a} \vee b) \in f \iff \bar{b} \to \bar{a}$$

now the claim follows by induction of the length of P.

$$p \to a \to b \to c \to q$$

$$\bar{q} \to \bar{c} \to \bar{b} \to \bar{a} \to \bar{p}$$

(2) If D_f has a (p,q)-path P, then for every satisfying truth assignment t

$$p(t) = 1 \Rightarrow q(t) = 1$$

$a \to b \iff (\bar{a} \vee b) \in f$ so

$$a(t) = 1 \iff \bar{a}(t) = 0 \Rightarrow b(t) = 1$$

+ induction on length of P $\square$.

<u>Corollary</u> if t satisfies f then $\forall$ strong component D' of D_f and $\forall$ choice of distinct $p, q \in V(D')$ we have $p(t) = q(t)$ and $\bar{p}(t) = \bar{q}(t)$.

Now we are ready to prove

<u>Lemma</u>

f is satisfiable

$\Updownarrow$

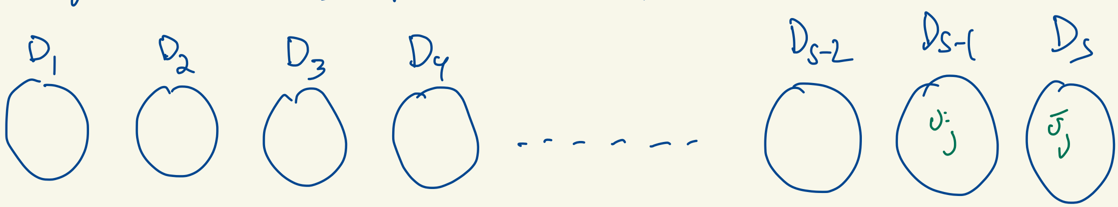
no strong component of D_f contains both v_i and $\bar{v}_i$ (correspond to x_i and $\bar{x}_i$ in f)

- Suppose t satisfies f but v_i and $\bar{v}_i$ are in same component of D_f
- then $v_i \rightsquigarrow \bar{v}_i$ so

$$x_i(t)=1 \Rightarrow \bar{x}_i(t)=1$$
$$x_i(t)=0 \Rightarrow \bar{x}_i(t)=0$$

Suppose now that no strong component of D_f contains both v_i and $\bar{v}_i$ for some $i \in [n]$

acyclic ordering of strong components



Start in D_s and set all literals whose corresponding vertex is in D_s true

go backwards in the order $D_{s-1}, D_{s-2}, \dots, D_2, D_1$

When processing D_i:

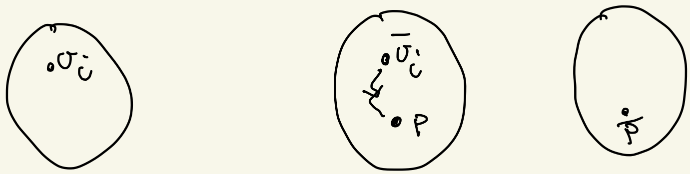
if v ($\bar{v}$) is in D_i and $\bar{v}$ (v) has already been processed (is in D_j for some $j > i$) then assign value 0 to all literals whose vertex is in D_i. Otherwise assign all literals in D_i value 1

This is equivalent to the following truth assignment to each variable x_i:

Let $v_i \in D_r$ and $\bar{v}_i \in D_t$ (note $r \neq t$)

If $t > r$ then $v_i(t) = 0$ and otherwise $v_i(t) = 1$

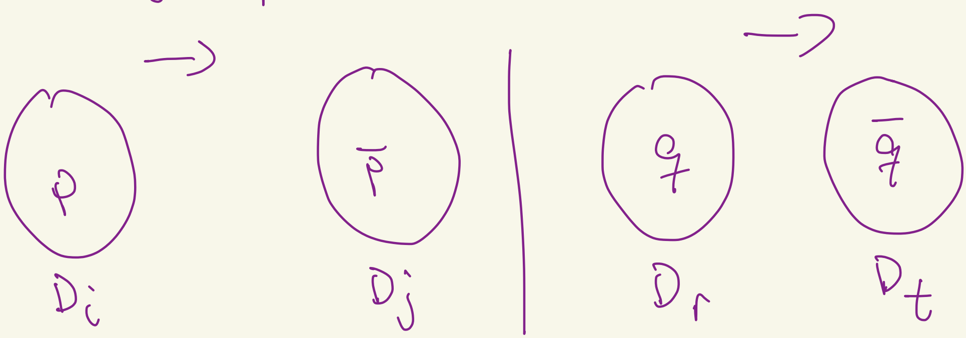
To see this suppose we have assigned $\bar{x}_i$ value 0, then



v_i can reach p so $\bar{p}$ can reach v_i contradiction

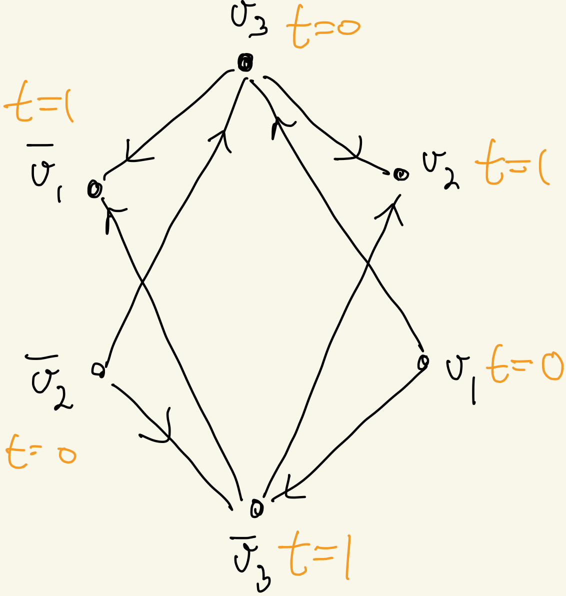
We obtain a satisfying truth assignment by the process above:

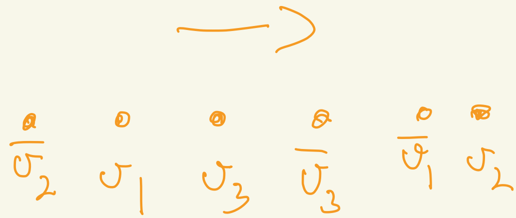
If $C_i = (p \vee q)$ has $p(t) = q(t) = 0$ then



but D_E contains the arcs $\bar{p} \to q$ and $\bar{q} \to p$

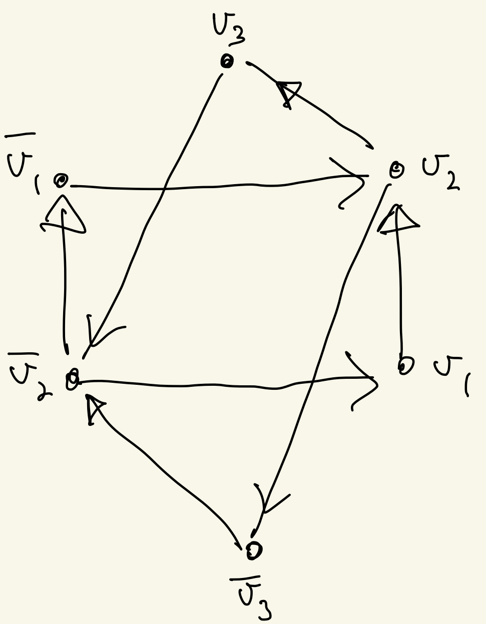
so $t \leq i < j \leq r < t$ ↯





$\Rightarrow$ f satisfiable

$$f = (\bar{x}_1 \vee \bar{x}_3) \vee (x_2 \vee \bar{x}_3) \vee (\bar{x}_1 \vee x_3) \vee (x_2 \vee x_3)$$



D_f has only one strong component

$\Downarrow$

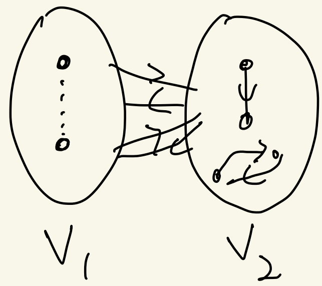
f not satisfiable

$$f = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

Application of 2-SAT to partitioning problems

1) recognizing split digraphs

V_1 independent
V_2 semicomplete



Given $G=(V,E)$ make 2-SAT instance with variables $x_1, x_2 \ldots x_n$ $\quad x_i \sim v_i \in V$

$\forall v_i \neq v_j \in V$ if v_i and v_j adjacent add $(\bar{x}_i \vee \bar{x}_j)$ to $\mathcal{F}$
otherwise add $(x_i \vee x_j)$ to $\mathcal{F}$

$\boxed{\text{claim } \mathcal{F} \text{ is satisfiable} \iff G \text{ is a split digraph}}$

$\Rightarrow$ put v_i in $V_1 \iff x_i(t)=1$ (so $v_i \in V_2 \iff x_i(t)=0$)
V_1 is independent as $v_i - v_j \Rightarrow (\bar{x}_i \vee \bar{x}_j) \in \mathcal{F}$ so at least one of v_i, v_j is in V_2

V_2 is semicomplete as $v_i \cdots v_j \Rightarrow (x_i \vee x_j) \in \mathcal{F}$ so at least one of v_i, v_j is in V_1

$\Leftarrow$ suppose G is a split digraph with partition V_1, V_2

set $x_i = 1 \iff v_i \in V_1$

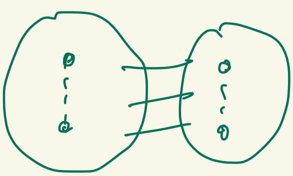
then each clause $(x_i \vee x_j)$ is satisfied since it came from $v_i \cdots v_j$

and each clause $(\bar{x}_r \vee \bar{x}_s)$ is satisfied as it came from $v_r - v_s$

so at least one of v_r, v_s is in V_2

implying $\bar{x}_r$ or $\bar{x}_s$ is true

Similarly with

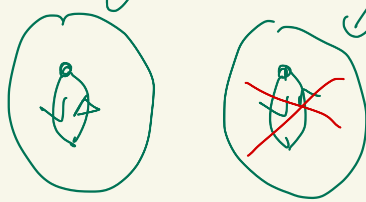


bipartite

and

complete

oriented



etc

lower bounds on the two sets

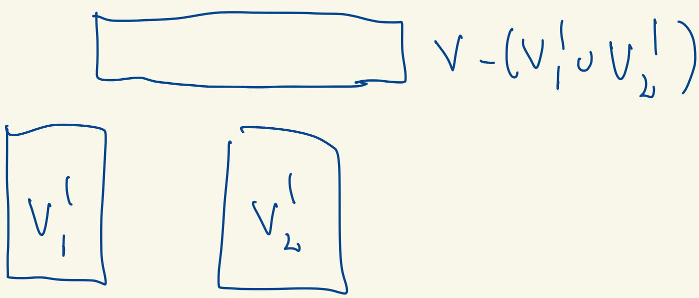
Given k_1, k_2 does G have a partition V_1, V_2 so that V_i has property $\mathbb{P}_i$?

<u>Theorem</u> the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partion problem is polynomial when each $\mathbb{P}_i$ belongs to

Complete, oriented, semicomplete, tournament, independent, symmetric

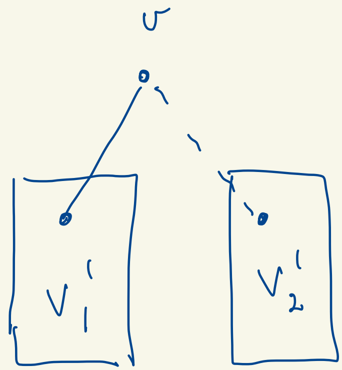
Example for split digraphs:

1. Guess V_1' independent and V_2' semicomplete

2. 

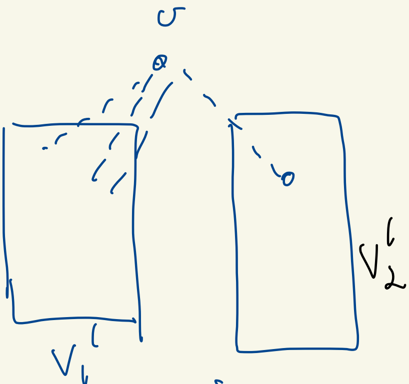
while $\exists\, v \in V - (V_1' \cup V_2')$:

if (a)



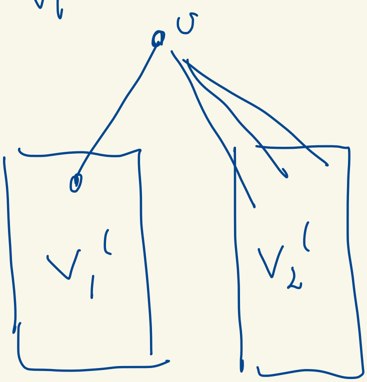
then no solution for current V_1', V_2'
so try new V_1', V_2' of size k_1, k_2 respectively

if (b)



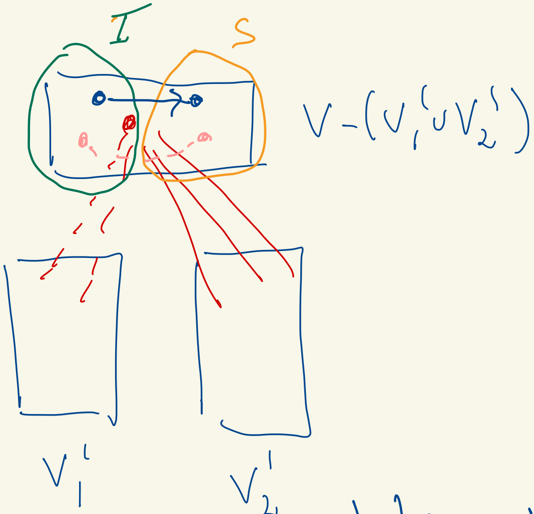
put v in V_1' : $V_1' \leftarrow V_1' + v$

if (c)



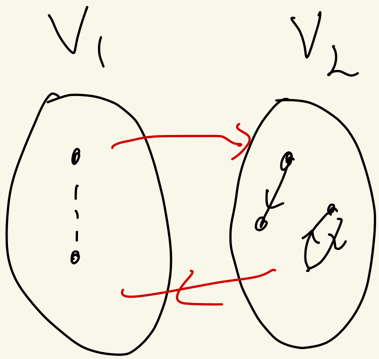
put v in V_2' $V_2' \leftarrow V_2' + v$

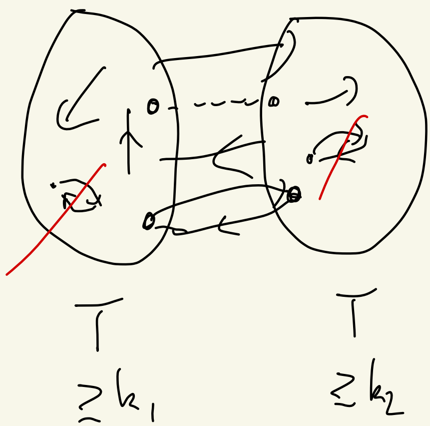
after this we either reached case (a) for some vertex showing that the initial V_1', V_2' does not work or



Now D has a good partition V_1, V_2 with $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ if and only if $D[V-(V_1'\cup V_2')]$ is a split digraph

use 2-SAT to decide this


V_1
V_2


T
T
$\geq k_1$
$\geq k_2$