

2-partitions of digraphs, Shandong university course on digraphs, 2026

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Let $\mathbb{P}_1, \mathbb{P}_2$ be two (di)graph properties

A $(\mathbb{P}_1, \mathbb{P}_2)$ -**partition** of a (di)graph D is a 2-partition (V_1, V_2) of $V(D)$ such that V_1 induces a (di)graph with property \mathbb{P}_1 and V_2 a (di)graph with property \mathbb{P}_2 .

For example a $(\delta^+ \geq 1, \delta^+ \geq 1)$ -**partition** is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least 1.

Similarly a **(strong, acyclic)-partition** is a 2-partition (V_1, V_2) such that $D\langle V_1 \rangle$ is strongly connected and $D\langle V_2 \rangle$ is an acyclic digraph.

Examples for undirected graphs:

- (independent,complete)-partition **split graphs**
- (independent,independent)-partition **bipartite graphs**
- (complete,complete)-partition **complements of bipartite graphs**

To avoid trivial partitions where one vertex on one side is enough, we consider $[k_1, k_2]$ -**partitions**, that is, partitions (V_1, V_2) of V such that $|V_1| \geq k_1$ and $|V_2| \geq k_2$.

For given positive integers k_1, k_2 the $(\mathbb{P}_1, \mathbb{P}_2)$ - $[k_1, k_2]$ -partition problem consists in deciding whether a given digraph D has a $(\mathbb{P}_1, \mathbb{P}_2)$ - $[k_1, k_2]$ -partition.

When $k_1 = k_2 = 1$ we usually just write $(\mathbb{P}_1, \mathbb{P}_2)$ -partition.

Let \mathcal{H} and \mathcal{E} denote the following two sets of natural properties of digraphs all of which can be checked in polynomial time:

$\mathcal{H} = \{\text{acyclic, complete, arcless, oriented (no 2-cycle), semicomplete, symmetric, tournament}\}$

These properties are all **hereditary**, that is, closed under induced subdigraphs

$\mathcal{E} = \{\text{strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}\}$.

These properties are all **enumerable**, that is, one can enumerate in polynomial time all its inclusion-wise maximal subdigraphs having the property.

Complexity for arbitrary input digraphs

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	\mathbb{B}^+	\mathbb{B}^-	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	A	C	X
strong	NPc	NPc ^L	NPc ^L	NPc ^L	NPc ^L	NPc ^L	NPc ^L	NPc	P	P	P
conn.	NPc ^R	P	P	P	P	NPc	NPc	NPc	P	P	P
\mathbb{B}^+	NPc ^R	P	P	NPc	P	NPc	P	NPc	P	P	P
\mathbb{B}^-	NPc ^R	P	NPc	P	P	P	NPc	NPc	P	P	P
$\delta \geq 1$	NPc ^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\delta^+ \geq 1$	NPc ^R	NPc	NPc	P	NPc	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc ^R	NPc	P	NPc	NPc	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	NPc	NPc	P	P	P
A	P	P	P	P	P	P	P	P	NPc	P	NPc
C	P	P	P	P	P	P	P	P	P	P	P
X	P	P	P	P	P	P	P	P	NPc	P	P

Properties: conn. : connected; \mathbb{B}^+ : out-branchable; \mathbb{B}^- : in-branchable; A: acyclic; C: complete; X: any property in 'being independent', 'being oriented', 'being semi-complete', 'being a tournament' and 'being symmetric'.

Complexities: P: polynomial-time solvable; NPc : NP-complete for all values of k_1, k_2 ;

NPc^L : NP-complete for $k_1 \geq 2$, and polynomial-time solvable for $k_1 = 1$.

NPc^R : NP-complete for $k_2 \geq 2$, and polynomial-time solvable for $k_2 = 1$.

Theorem

Let \mathbb{H} be a checkable hereditary property, \mathbb{E} be an enumerable property, and let k_1 and k_2 be two positive integers. One can decide in polynomial time whether a given digraph D has a (\mathbb{H}, \mathbb{E}) - $[k_1, k_2]$ -partition.

Proof: We shall describe a polynomial-time procedure that for any fixed set U_1 of k_1 vertices of D decides whether D has an (\mathbb{H}, \mathbb{E}) - $[k_1, k_2]$ -partition (V_1, V_2) with $U_1 \subseteq V_1$.

Then applying this algorithm to the $O(n^{k_1})$ k_1 -subsets of $V(D)$, we obtain the desired algorithm.

- First, we enumerate the maximal subdigraphs of $D - U_1$ with property \mathbb{E} . This can be done in polynomial time because \mathbb{E} is enumerable.
- Now for each such subdigraph F , (there is a polynomial number of them), we check whether $|F| \geq k_2$ and if $D - F$ has property \mathbb{H} . This can be done in polynomial time because \mathbb{H} is checkable.
- In the affirmative, we return 'Yes', and in the negative we proceed to the next subdigraph.
- If no more subdigraph remains, we return 'No'.

The above procedure clearly runs in polynomial time.

We need to show that D has an (\mathbb{H}, \mathbb{E}) - $[k_1, k_2]$ -partition (V_1, V_2) with $U_1 \subseteq V_1$ if and only if there is a maximal subdigraph F of $D - U_1$ with property \mathbb{E} of order at least k_2 such that $D - F$ has property \mathbb{H} .

- If there is a maximal subdigraph F of $D - U_1$ with property \mathbb{E} of order at least k_2 such that $D - F$ has property \mathbb{H} , then $(V(D - F), V(F))$ is clearly an (\mathbb{H}, \mathbb{E}) - $[k_1, k_2]$ -partition (V_1, V_2) with $U_1 \subseteq V_1$.
- Conversely, assume there is an (\mathbb{H}, \mathbb{E}) - $[k_1, k_2]$ -partition (V_1, V_2) with $U_1 \subseteq V_1$. Then $D \langle V_2 \rangle$ has property \mathbb{E} and thus is contained in a maximal subdigraph F of $D - U_1$ with property \mathbb{E} .

One can easily check that the algorithm described in the proof of Theorem 1 runs in time $O(n^{k_1+c})$ for some constant c .

A natural question is then to ask whether the problem could be FPT with respect to (k_1, k_2) , that is, in time $f(k_1, k_2)n^c$ for some constant c and computable function f .

If not, one may ask if it can be solved in FPT time with respect to k_1 or k_2 only, that is, in time $g(k_i)n^{h(k_3-i)}$ for some computable functions g and h .

A base digraph for NP-completeness proofs

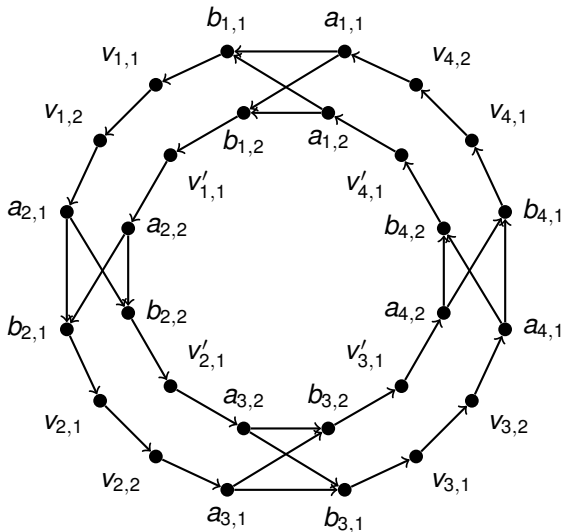


Figure: A ring digraph

For $1 \leq j \leq m$, we associate to the j th clause

$C_j = (\ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3})$ the set W_j consisting of three vertices of $R(\mathcal{F})$ representing the occurrences of the literals of C_j in \mathcal{F} .

Theorem

Let \mathcal{F} be a 3-SAT formula and let $R(\mathcal{F})$ be the corresponding ring digraph. Then the following holds:

- *$R(\mathcal{F})$ contains a directed cycle which avoids at least one vertex from each of the sets W_1, \dots, W_m if and only if \mathcal{F} is a 'Yes'-instance of 3-SAT.*
- *$R(\mathcal{F})$ contains two disjoint directed cycles R_1, R_2 , each of which intersects all the sets W_1, \dots, W_m if and only if \mathcal{F} is a 'Yes'-instance of NAE-3-SAT.* □

Complecity for strongly connected input digraphs

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	\mathbb{B}^+	\mathbb{B}^-	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	A	C	H
strong	NPc	P	NPc*	NPc*	P	NPc ^L	NPc ^L	NPc	P	P	P
conn.	P	P	P	P	P	P	P	P	P	P	P
\mathbb{B}^+	NPc*	P	P	NPc*	P	NPc ^L	P	NPc ^L	P	P	P
\mathbb{B}^-	NPc*	P	NPc*	P	P	P	NPc ^L	NPc ^L	P	P	P
$\delta \geq 1$	P	P	P	P	P	P	P	P	P	P	P
$\delta^+ \geq 1$	NPc ^H	P	NPc ^H	P	P	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc ^H	P	P	NPc ^H	P	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	P	NPc ^H	NPc ^H	P	NPc	NPc	NPc	P	P	P
A	P	P	P	P	P	P	P	P	NPc	P	NPc
C	P	P	P	P	P	P	P	P	P	P	P
H	P	P	P	P	P	P	P	P	NPc	P	P

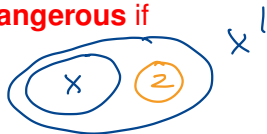
The legend is the same as in the first table, but we have one more complexity type: NPc* : NP-complete for $k_1, k_2 \geq 2$, and polynomial-time solvable for $k_1 = 1$ or $k_2 = 1$. We also emphasize with P, the problems that are polynomial-time solvable on strong digraphs and NP-complete in the general case.

2-partitions of Tournaments

A digraph D is called **k -out-critical** if $\delta^+(D) = k$ and no subset of its vertices can be removed without decreasing the minimum out-degree of the resulting digraph.

Let $X \subseteq V$ be a set of vertices in a digraph D with minimum out-degree at least k . A set $X' \subseteq V$ is called **X -out-critical** if $X \subseteq X'$, $\delta^+(D \setminus X') \geq k$ and $\delta^+(D \setminus (X' - Z)) < k$ for every $\emptyset \neq Z \subseteq X' - X$.

A vertex $v \in V(T)$ is said to be **k -out-dangerous** if $d^+(v) < 2k - 1$.



Lemma

Let k be a fixed integer and let D be a semicomplete digraph with minimum out-degree at least k . Then the number of k -out-dangerous vertices of D is at most $4k - 3$.

Lemma

Let D be a semicomplete digraph such that $\delta^+(D) \geq k$ and let $X \subseteq V(D)$. Then for every X -out-critical set X' in D we have $|X'| \leq \frac{k^2+3k+2}{2} + |X|$. In particular every k -out-critical set in D has size at most $\frac{k^2+3k+2}{2}$.

Theorem

For every fixed integer k there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k, \delta^+ \geq k)$ -partition of a given semicomplete digraph D or correctly outputs that none exists.

Proof:

It suffices to prove that we can test, for a given partition (O_1, O_2) of the out-dangerous vertices, whether there is a solution with $O_i \subseteq V_i$.



- Let X be an O_1 -out-critical set such that $X \subseteq V - O_2$. If no such X exists, there is no solution with $O_i \subseteq V_i$.
- Starting from the partition $(V_1, V_2) = (X, V - X)$, and moving one vertex at a time, move vertices of $V_2 - O_2$ which have $d_{T \setminus \langle V_2 \rangle}^+(v) < k$ to V_1 .
- If, at any time, this results in a vertex $v \in O_2$ having $d_{T \setminus \langle V_2 \rangle}^+(v) < k$, or $V_2 = \emptyset$, then there is no good partition with $O_i \subseteq V_i, i = 1, 2$ and the algorithm terminates.
- Otherwise the algorithm will terminate with $O_2 \subseteq V_2 \neq \emptyset$ and hence it has found an $(\delta^+ \geq k, \delta^+ \geq k)$ -partition (V_1, V_2) with $O_i \subseteq V_i, i = 1, 2$.

The correctness of \mathcal{B} follows from the fact that we only move vertices that are not dangerous and each such vertex has at least $2k - 1$ out-neighbours in D .

Hence, as the vertex that we move does not have k out-neighbours in V_2 , it must have at least k out-neighbours in V_1 , so $\delta^+(D \setminus V_1) \geq k$ will hold throughout the execution of \mathcal{B} .

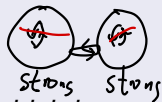
By Lemma 3, the number of out-dangerous vertices is at most $4k - 3$ and hence the number of (O_1, O_2) -partitions is at most 2^{4k-3} which is a constant when k is fixed.

Thus we obtain the desired polynomial time algorithm by running the subalgorithm \mathcal{B} for all possible partitions (O_1, O_2) of the out-dangerous vertices and all possible O_1 -critical sets. \square

Theorem

The following 2-partition problems are \mathcal{NP} -complete for the class of semicomplete digraphs and polynomial for tournaments.

- (a) Partitioning into two strong tournaments.
- (b) Partitioning into two tournaments both of which have minimum out-degree at least one.
- (c) Partitioning into two tournaments so that one has minimum in-degree at least one and the other has minimum out-degree at least one.



Spanning bipartite digraphs

Let $D = (V, A)$ be a digraph. For a given 2-partition (V_1, V_2) of V we denote by $B_D(V_1, V_2)$ the spanning bipartite subdigraph induced by the arcs with one end in V_1 and the other in V_2 .

Observation (Alon): For every k there exists a digraph D with minimum out-degree k such that for every 2-partition (V_1, V_2) of $V(D)$ some vertex of $B_D(V_1, V_2)$ has out-degree zero.

This follows from a construction of Thomassen of k -out-regular digraphs with no even cycle.

Spanning bipartite digraphs of minimum out-degree at least 1

Theorem

It is polynomial to decide whether a given digraph D has a 2-partition (V_1, V_2) so that $B_D(V_1, V_2)$ has minimum out-degree at least one.

Such a partition exists if and only if every terminal strong component contains an even directed cycle.

Theorem

For fixed every choice of natural numbers k_1, k_2 such that $k_1 + k_2 \geq 3$ it is NP-complete to decide whether a given digraph D has a 2-partition (V_1, V_2) so that in $B_D(V_1, V_2)$ every vertex of V_i has minimum out-degree at least k_i for $i = 1, 2$.

Spanning strong bipartite digraphs

Theorem

For every non-negative integer K there exists an eulerian K -strong digraph D such that for every 2-partition (V_1, V_2) the bipartite digraph $B_D(V_1, V_2)$ is not strong.

Theorem

For every non-negative integer K it is NP-complete to decide whether a given K -strong eulerian digraph D has a 2-partition (V_1, V_2) such that the bipartite digraph $B_D(V_1, V_2)$ is strong.

Out-colourings of bipartite tournaments

Let D be a digraph. A k -colouring of $V(D)$ is a **k -out-colouring** if no out-neighbourhood is monochromatic.

Proposition

For all positive integers k, r there exists a bipartite tournament $B_{k,r}$ with $\delta^+(B_{k,r}) = k$ which has no r -out-colouring.

Theorem

It is NP-complete to decide whether a bipartite tournament admits a 2-out-colouring

Out-colourings of tournaments

Theorem

Every tournament T with $\delta^+(T) \geq 3$ different from the Paley tournament P_7 admits a 2-out-colouring.

A 2-out-colouring with colour classes V_1 and V_2 is **balanced** if we have $||V_1| - |V_2|| \leq 1$.

Theorem

Every semicomplete digraph T which admits a 2-out-colouring also admits a balanced 2-out-colouring except if T is one of 10 semicomplete digraphs on 6 vertices.

Partitioning a tournament evenly

Theorem (Alon, Bang-Jensen, Bessy)

There exist two absolute positive constants c_1, c_2 so that the following holds.

- Let $T = (V, E)$ be a tournament with minimum out-degree at least $2k + c_1\sqrt{k}$. Then there is a balanced partition $V = V_1 \cup V_2$ of V so that $\delta^+(T\langle V_1 \rangle)$, $\delta^+(T\langle V_2 \rangle)$ and $\delta^+(T\langle V_1, V_2 \rangle)$ are all at least k .*
- For infinitely many values of k there is a tournament with minimum out-degree at least $2k + c_2\sqrt{k}$ so that for any partition $V = V_1 \cup V_2$ of V into two disjoint sets, at least one of the quantities $\delta^+(T\langle V_1 \rangle)$, $\delta^+(T\langle V_2 \rangle)$, $\delta^+(T\langle V_1, V_2 \rangle)$ is smaller than k .*

Theorem (Alon, Bang-Jensen, Bessy)

There exists an absolute constant c so that the following holds for every positive integer k . Let $T = (V, A)$ be a tournament with minimum out-degree at least $2k + c\sqrt{k}$ and minimum in-degree at least $2k + c\sqrt{k}$. Then there is a balanced partition $V = V_1 \cup V_2$ of V so that every vertex has at least k out-neighbors in V_1 and in V_2 , and every vertex has at least k in-neighbors in V_1 and in V_2 .

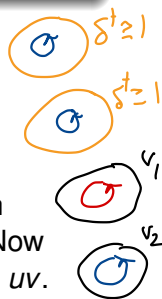
Disjoint cycles

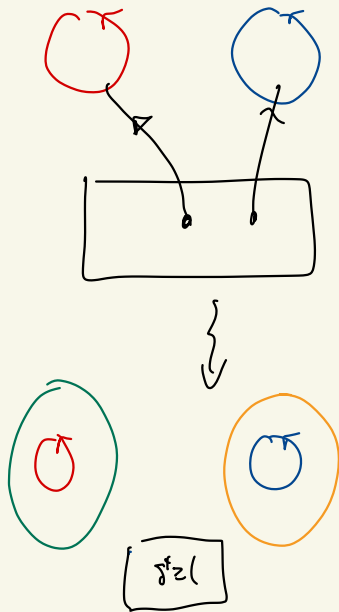
Theorem (Thomassen)

Every digraph of minimum out-degree at least 3 has a pair of disjoint cycles.

Proof:

- May assume every vertex of D has out-degree 3 (otherwise delete $d^+(v) - 3$ arcs out of v).
- May assume D has no 2-cycle
- If $N^-(v)$ is acyclic and u is a vertex of in-degree 0 in $D[N^-(v)]$, then no vertex dominates both u and v . Now delete arcs leaving u except uv and contract the arc uv . The new digraph is smaller and has minimum out-degree at least 3. So we are done by induction (every 3-regular tournament has disjoint 3-cycles).





- Now we know that $D[N^-(v)]$ contains a cycle for each $v \in V$.
- As D has no 2-cycle and $|A(D)| = 3|V(D)|$, every vertex has in-degree exactly 3 and D is 3-regular.
- By the argument above the digraph D' which we obtain by reversing all arcs has a cycle in each $N_{D'}^-(v)$
- Thus every $N_D^+(v)$ contains a cycle and this is disjoint from the cycle in $N_D^-(v)$.

