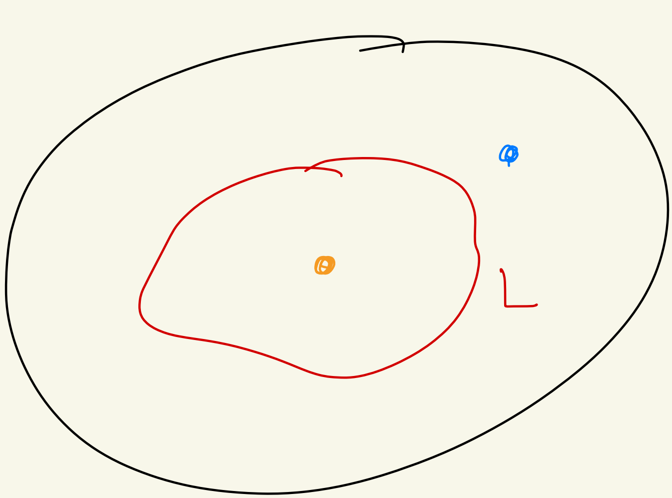
Complexity

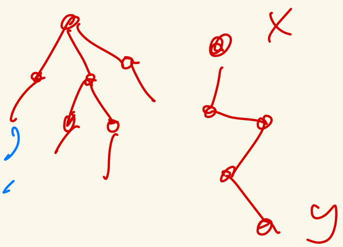
Decision problem



$\{0,1\}^*$ (all binary strings)

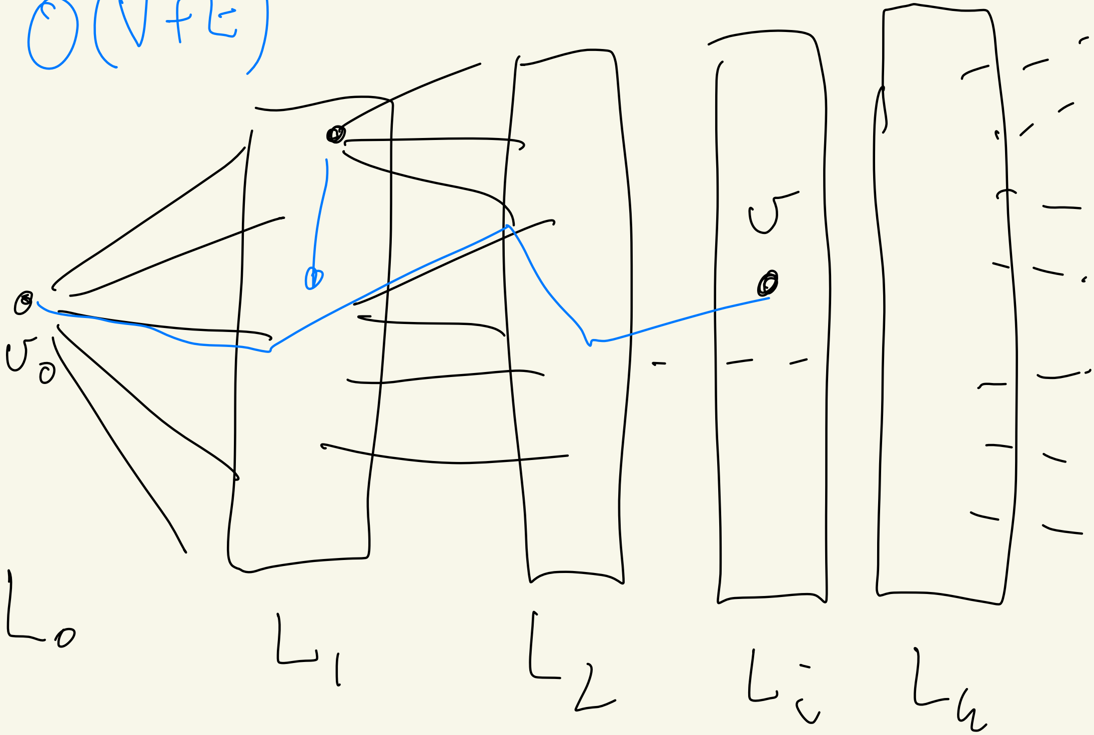
$\boxed{\text{Given } x \in \{0,1\}^* \text{ is } x \in L\text{?}}$

P1: Given $G=(V,E)$
is G connected?



Breath First Search BFS

$O(V+E)$

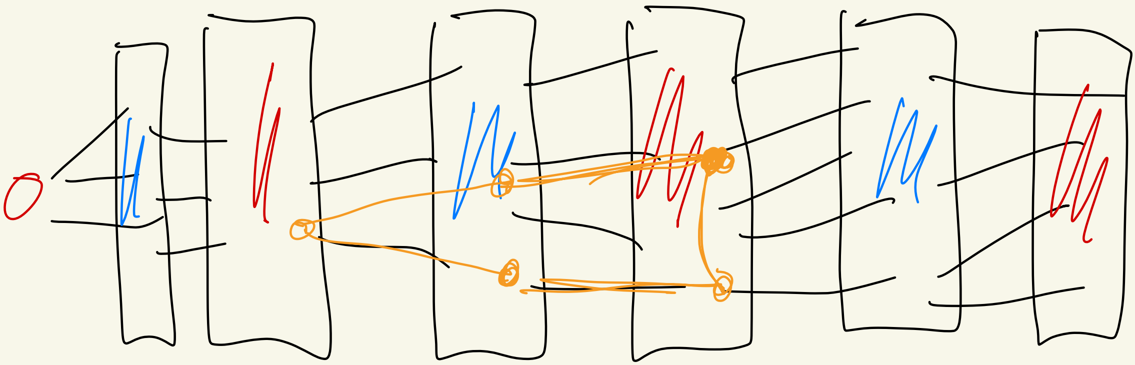


distance classes

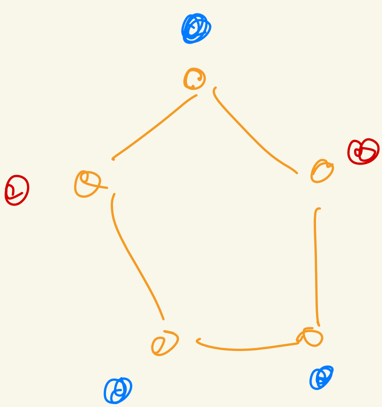
G is connected $\Longleftrightarrow$ every vertex is collected (labelled)

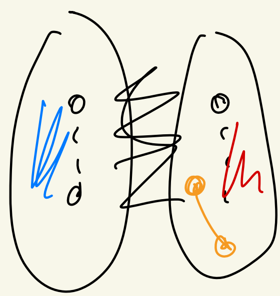
P2: Is G bipartite?

assume G connected



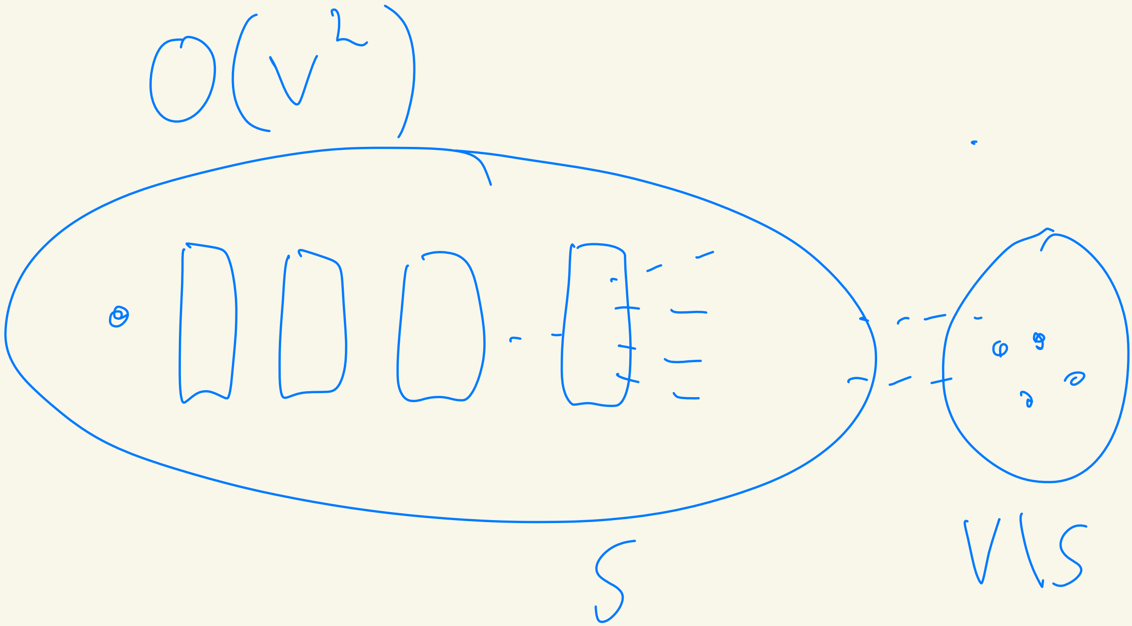
G is bipartite





P = decision problems that can be solved in polynomial time

($\exists$ polynomial p s.t. input x can be decided in time $O(|x|^c)$ for some constant c



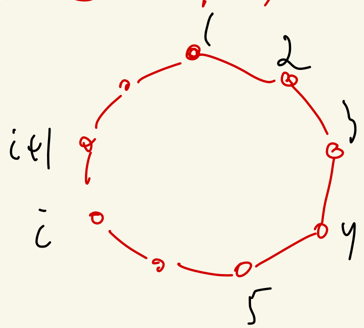
NP = decision problems that can be verified in polynomial time

(if $x \in L$ then we have a short 'proof' of this)

$$x \xrightarrow{A(x)} \{\text{yes}, \text{no}\}$$

$P \subseteq NP$

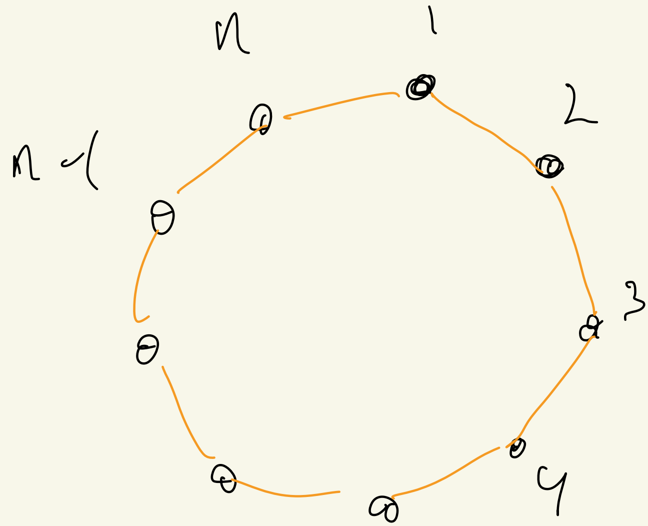
P3: Given $G=(V,E)$ is G hamiltonian?



contains all of V

P3 is in NP:

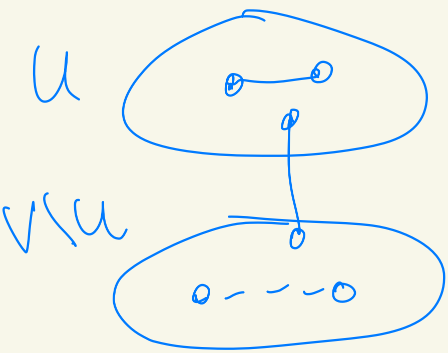
if G has a hamilton cycle then show a permutation of V



if G has <u>no</u> hamilton cycle

<u>thus</u> <u>No</u> permutation is good

Vertex cover



Given $G=(V,E)$, $k \in \mathbb{N}$ does G have a VC of size $\leq k$?

3-SAT

$\in$ NP

n variables ($0,1$) $x_1, x_2, \ldots, x_n$

m clauses $C_1, C_2, \ldots C_m$

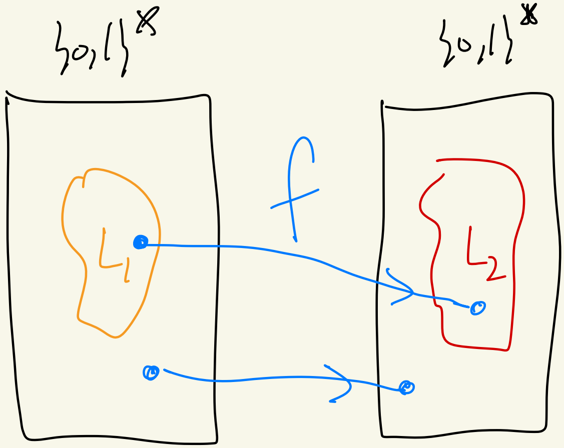
$|C_i| = 3 \quad C_i = (x_1 \vee \bar{x}_2 \vee x_3)$

Question Can we assign truth $(0,1)$ values to $x_1, x_2, \ldots x_n$ s.t. $\forall i \in [m]$ C_i is true

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3)$$

$x_1 = 1, \; x_2 = 0, \; x_3 = 1$

Polynomial reduction of L_1 to L_2



$L_1 \leq_p L_2$

$f(x) \in L_2 \iff x \in L_1$

L is <u>NP-complete</u>

1. $L \in NP$
2. $\forall L' \in NP : L' \leq_p L$

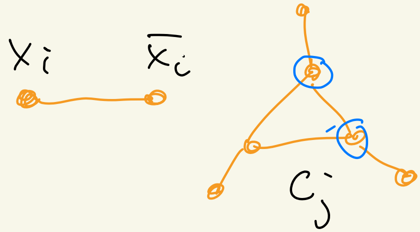
$$\underbrace{x \xrightarrow[n^c]{f} f(x) \xrightarrow[n^k]{g} g(f(x))}_{n \to n^{c \cdot k}}$$

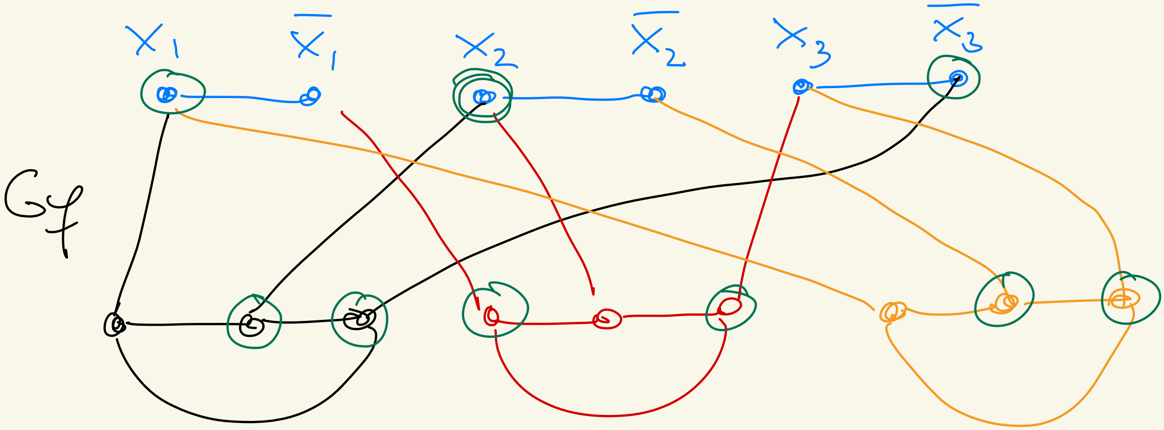
<u>Thm (Cook)</u>

3-SAT is NP-complete

$f = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3})$

3-SAT $\leq_p$ Vertex cover





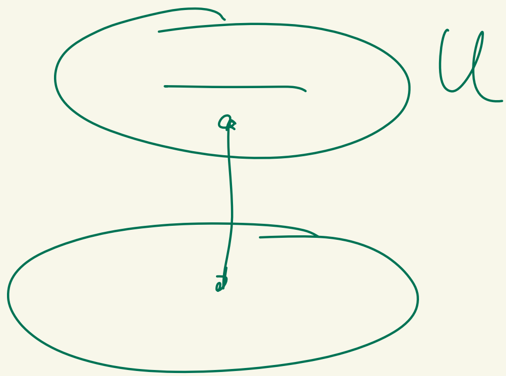
Claim G_f has a vertex cover of size $n+2m$

$\Updownarrow$

f is satisfiable

Why only decision problems?

Vertex cover



Optimization version:

Find a minimum size vertex cover

$(G, 1)$ ✗

$(G, 2)$ ✗

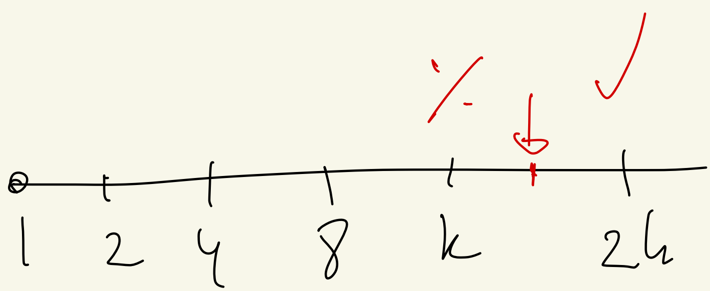
$(G, 4)$ ✗

$(G, 8)$ ✗

⋮

(G, k) ✗

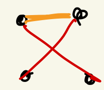
$(G, 2k)$ ✓

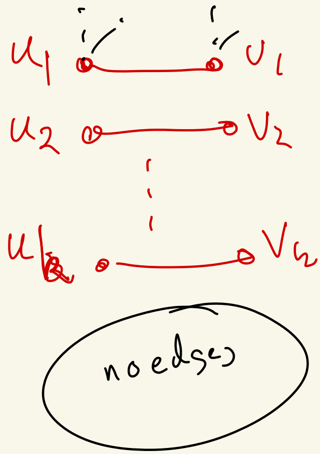


$(G, \frac{3k}{2})$?

2-approximation algorithm for VC

produce a maximal matching greedily





$U = \{u_1, u_2 \ldots u_k\} \cup \{v_1, v_2 \ldots v_k\}$

$\downarrow$

is a VC

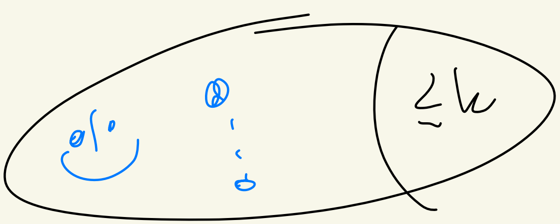
$|U| = 2k$

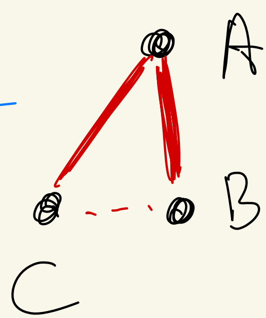
U^* min VC

$|U^*| \geq k = \frac{1}{2}|U|$

FPT algorithm for VC of size k

Bar Fight problem



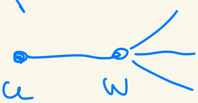


Rules

0. if $d(v)=0$ — let v in the bar

1. if $d(v) > k$ — remove v, put in cover and $k \to k-1$

2. if $d(u)=1$



remove w (take w in the cover) $k \to k-1$

$$G_0 \xrightarrow{\text{rule 1}} G_1 \xrightarrow{\text{rule 1}} G_2 \xrightarrow{\text{rule 0}} \cdots \to G_r$$

no rule applies

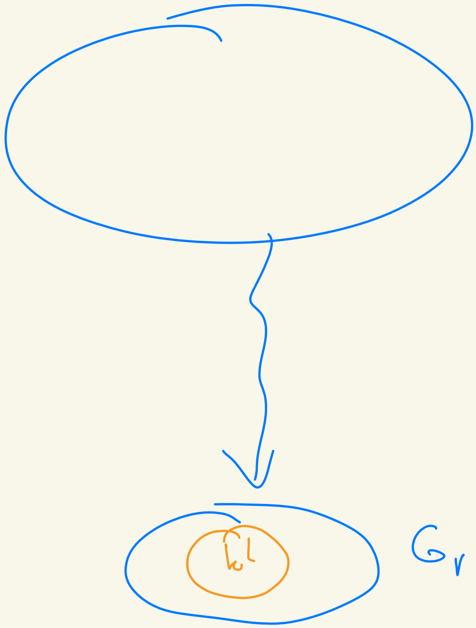
Now in remaining G_r

min degree ≥ 2

$|E| \leq k^2$ <u>or no solution</u>

$\leq k$ vertices, each $\leq k$

$$|V| = \frac{1}{2}\sum_{v \in V} 2 \leq \frac{1}{2}\sum d(v) = |E| \leq k^2$$



G_0

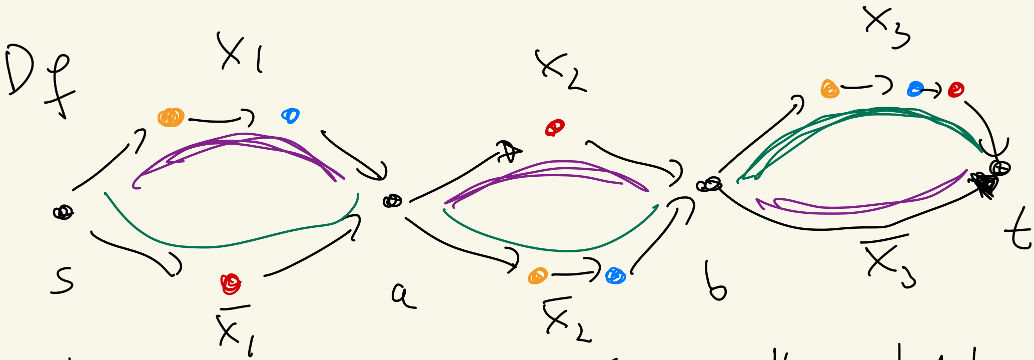
G_r $|V| \le k'^2$ $|E| \le k'^2$ $k' \le k$

Brute force: try all k' subsets

$$\le \binom{k'^2}{k'} \text{ subsets to try}$$

$$O(k^{2k}) \qquad 10^{20}$$

$f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$



<u>Claim</u> D_f has an (s,t)-paths which contains all colours

$\Updownarrow$

f is satisfiable

V_1, V_2, V_3 all size 3

From ~~~~~~

set $x_i = 1$ if upper part
0 otherwise