Finding good 2-partitions of digraphs I. Hereditary properties

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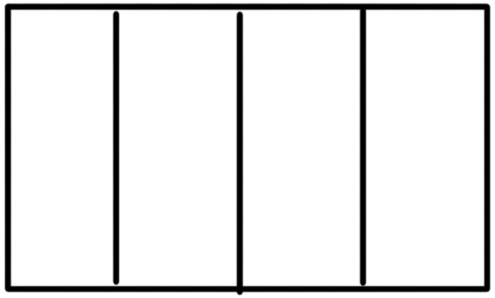
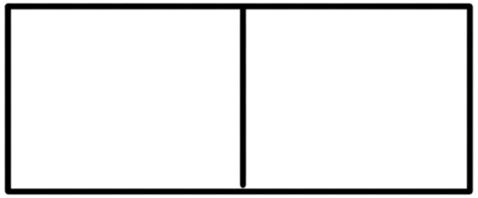
$\in P$?

NP-C ?

patition :

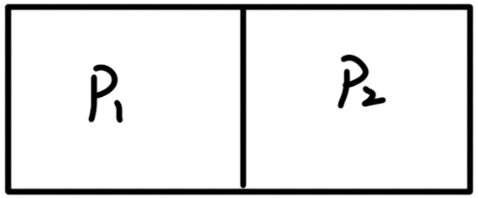


2- patition :



Let P_1, P_2 be two digraph properties,

(P_1, P_2)- patition $\Rightarrow$



$D\langle V \rangle$: subdigraph induced by V.

A property $\mathbb{P}$ is **hereditary** if the set of digraphs having the property is closed under taking induced subdigraphs, i.e. if a digraph has the property $\mathbb{P}$, then all its induced subdigraphs also have the property $\mathbb{P}$. It is easy to see that all properties in $\mathcal{H}$ are hereditary, while e.g., being connected is not a hereditary property.

A property $\mathbb{P}$ is **enumerable** if given a digraph one can enumerate in polynomial time all its inclusion-wise maximal subdigraphs having property $\mathbb{P}$. In particular, this requires that the number of maximal subdigraphs of a digraph with property $\mathbb{P}$ is polynomial.

$\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

acyclic : no directed cycles.

complete : any pair vertices $u, v \in V$, $u \rightleftarrows v \in A(D\langle V\rangle)$

independent: does not contain any edges.

oriented : digraph without directed 2-cycles.

semicom. : digraph with no non-adjacent vertices.

tournament : semicom. digraph which is also an oriented graph.
(complete graph's oriented)

symmetric : if $uv \in A(D) \Rightarrow vu \in A(D)$

acyclic $\mathcal{A}$: no directed cycles

(P_1, P_2)-$[k_1, k_2]$-patition : V_1, V_2 patition V

$D\langle V_1\rangle \to P_1$

$D\langle V_2\rangle \to P_2$

$|V_1| \geq k_1$

$|V_2| \geq k_2$

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

Theorem 4.4. *The* $(\mathbb{A},\mathbb{P})$-$[k_1,k_2]$*-partition problem is NP-complete for* $\mathbb{P}\in\{\mathbb{A},\mathbb{I},\mathbb{O},\mathbb{S},\mathbb{T},\mathbb{Z}\}$ *and every choice of positive integers* k_1,k_2. *This holds even when the input is restricted to strongly connected digraphs.* $\rightarrow$ NP ✓

$$3\text{-SAT} \xrightarrow{\text{reduce}} (\mathbb{P}_1,\mathbb{P}_2)\text{- patition}$$

3-SAT: $\mathcal{F} = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ $\rightarrow$ NAE-3-SAT : each C_i has 0 and 1

$\Leftrightarrow$ each C_i, $1 \le 1\# \le 2$

$x_1, \dots, x_n$

NP-C

$C_i = (l_{i,1}, l_{i,2}, l_{i,3})$

2-IN-3-SAT : each C_i, $1\# = 2$

$l_{i,j} \in \{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\}$

instance $\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3} \Rightarrow$

variables x_1, x_2, x_3, x_4 $\rightarrow D$

clauses C_1, C_2, C_3

NAE : $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

$\mathcal{F} = \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{0}{x_2} \vee \overset{1}{x_3})}_{C_1} \wedge \underbrace{(\overset{1}{x_1} \vee \overset{0}{x_2} \vee \overset{1}{x_4})}_{C_2} \wedge \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{1}{\bar{x}_2} \vee \overset{0}{\bar{x}_3})}_{C_3}$

2-IN $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

$\mathcal{F} = \underbrace{(\overset{1}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3})}_{C_1} \wedge \underbrace{(\overset{0}{x_1} \vee \overset{1}{x_2} \vee \overset{1}{x_4})}_{C_2} \wedge \underbrace{(\overset{1}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})}_{C_3}$

truth assign. of $F \Leftrightarrow$ good patition of D.

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

Theorem 4.4. *The $(\mathbb{A}, \mathbb{P})$-$[k_1, k_2]$-partition problem is NP-complete for $\mathbb{P} \in \{\mathbb{A}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$ and every choice of positive integers k_1, k_2. This holds even when the input is restricted to strongly connected digraphs.*

reductions : $\mathcal{F} \longrightarrow$ digraph   polynomial

- $(\mathbb{A}, \mathbb{A})$- patition problem

NAE - 3- SAT $\longrightarrow$ $(\mathbb{A}, \mathbb{A})$-patition

instance $\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$

variables x_1, x_2, x_3, x_4

clauses C_1, C_2, C_3

instance $\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$ $q_1 = 2$

x_1

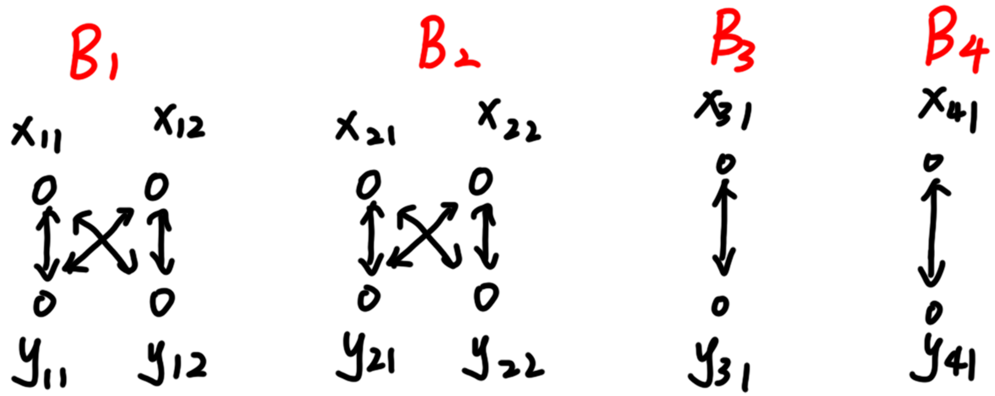
$\downarrow$

$B(\mathcal{F})$:

In contrast, we now prove that the $(\mathbb{A}, \mathbb{P})$-$[k_1, k_2]$-partition problems are NP-complete for $\mathbb{P} \in \{\mathbb{A}, \mathbb{I}, \mathbb{O}, \mathbb{S}, \mathbb{T}, \mathbb{Z}\}$. All our reductions use superdigraphs of the digraph $B(\mathcal{F})$ which is obtained from a given 3-SAT instance $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over the set of n boolean variables $x_1, \ldots, x_n$. The digraph $B(\mathcal{F})$ is defined from $\mathcal{F}$ as follows. Let q_i denote the maximum of the number of times x_i occurs in the clauses and the number of times $\bar{x}_i$ occurs in the clauses. The vertex set of $B(\mathcal{F})$ is $(\bigcup_{i \in [n]} \{x_{i,j} | j \in [q_i]\}) \cup (\bigcup_{i \in [n]} \{y_{i,j} | j \in [q_i]\})$ and the arc set of $B(\mathcal{F})$ is the union of the arc sets of the n complete bipartite digraphs $B_1, B_2, \ldots, B_n$ where B_i has vertex set $\{x_{i,j} | j \in [q_i]\} \cup \{y_{i,j} | j \in [q_i]\}$.

$\downarrow$

$D(\mathcal{F})$



$\cdots\!\to x_i$

$\cdots\!\to \bar{x}_i = 1 \Leftrightarrow x_i = 0$

$C_j \leftrightarrow D(\mathcal{F})$

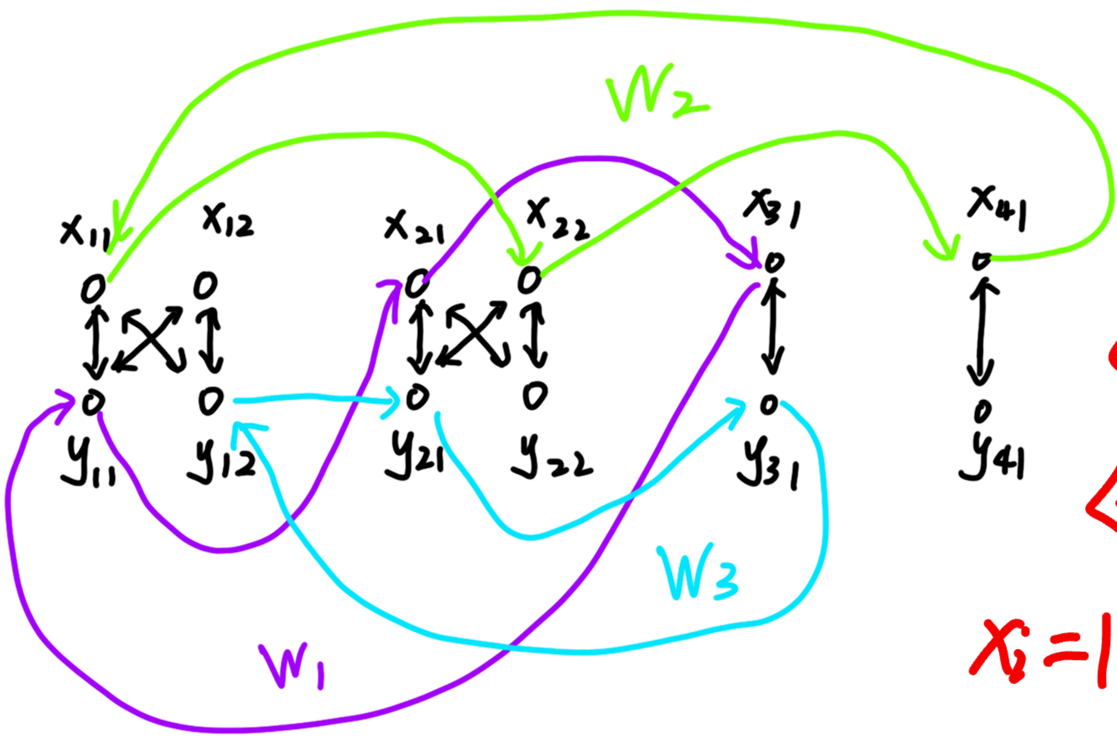
$C_1 \to W_1 = \{y_{11}, x_{21}, x_{31}\}$

$C_2 \to W_2 = \{x_{11}, x_{22}, x_{41}\}$

$C_3 \to W_3 = \{y_{12}, y_{21}, y_{31}\}$

$\downarrow$

$D(\mathcal{F})$



3-cycle in $D(\mathcal{F})$

V_1, V_2 acyclic

$\Longleftrightarrow |W_j \cap V_i| \leq 2$

$\Longleftrightarrow |W_j \cap V_i| \geq 1$

$x_i = 1$ if x_{ij} in V_1

$x_i = 0$ if y_{ij} in V_2

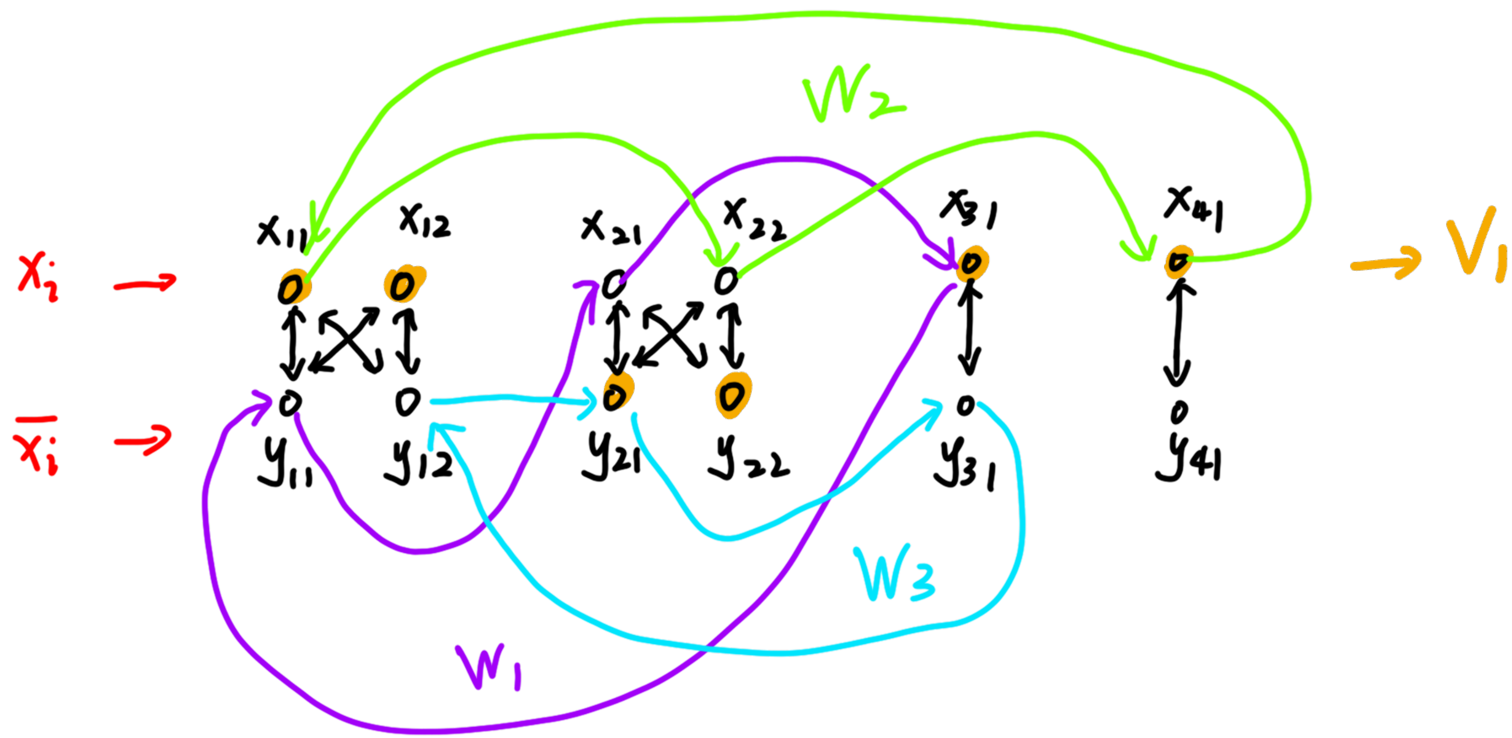
$D(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{A})$-patition $\Longleftrightarrow$ $\mathcal{F}$ is 'Yes'-instance of NAE-3-SAT

$\Rightarrow$

(V_1, V_2) is a $(\mathbb{A}.\mathbb{A})$-patition of $D(\mathcal{F})$ V_1, V_2 acyclic

① $x_{ij}\, y_{ij'}$ one in V_1, other one in V_2

② $W_i \nsubseteq V_j$, $i \in [3]$, $j \in [2]$ $\Rightarrow$ $|V_1 \cap W_i| \geq 1$



Let $V_1 \rightarrow$ true $\qquad x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 1$

$$\mathcal{F} = \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{0}{x_2} \vee \overset{1}{x_3})}_{C_1} \wedge \underbrace{(\overset{1}{x_1} \vee \overset{0}{x_2} \vee \overset{1}{x_4})}_{C_2} \wedge \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{1}{\bar{x}_2} \vee \overset{0}{\bar{x}_3})}_{C_3}$$

$\rightarrow$ NAE-3-SAT true (each C_i has one or two truth literals)

$\Leftarrow$ ϕ: true assignment

$$V_1 = \left(\bigcup_{\{i \mid \phi(x_i) = true\}} \{x_{i,j} \mid j \in [q_i]\} \right) \cup \left(\bigcup_{\{i \mid \phi(x_i) = false\}} \{y_{i,j} \mid j \in [q_i]\} \right), \text{ and } V_2 = V(D(\mathcal{F})) \setminus V_1.$$

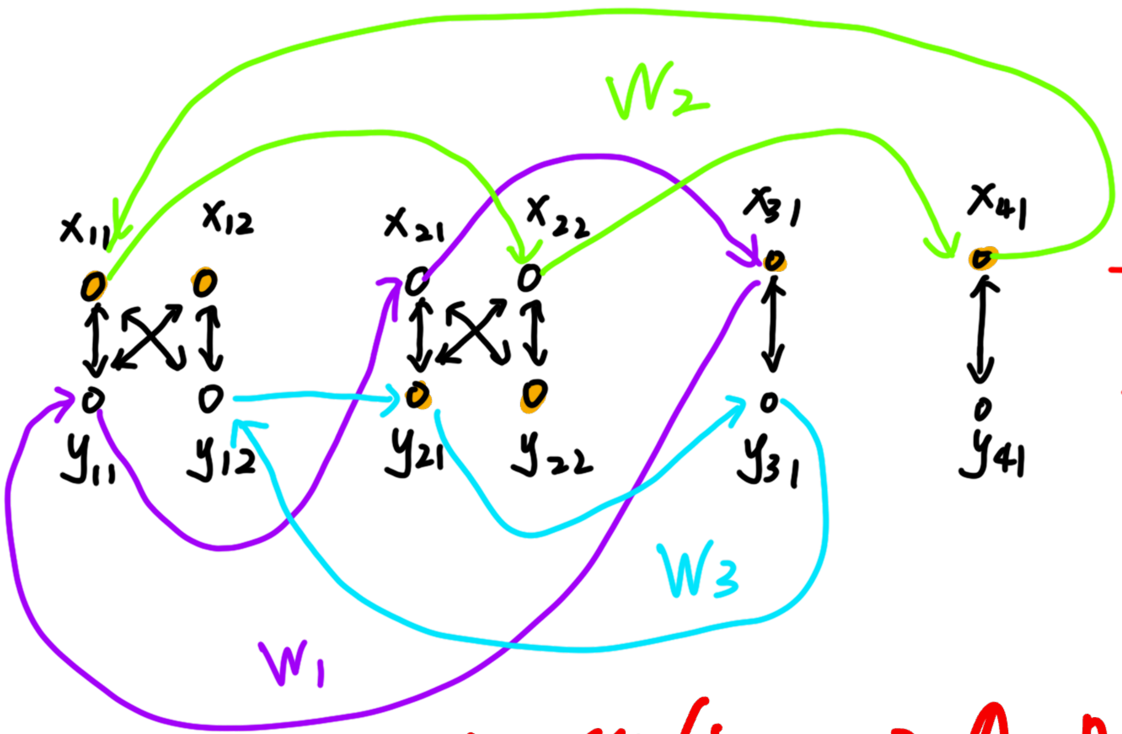
if $x_i = 1$, $\bar{x}_i = 0$ $\rightarrow$ put $x_{i,j}$ into V_1, $y_{i,j}$ into V_2 $\quad j \in [q_i]$

if $x_i = 0$, $\bar{x}_i = 1$ $\rightarrow$ put $y_{i,j}$ into V_1, x_{ij} into V_2

(V_1, V_2) is an (A.A)-partition of $D(\mathcal{F})$

Let $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ ϕ

$$f = \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{0}{x_2} \vee \overset{1}{x_3})}_{C_1} \wedge \underbrace{(\overset{1}{x_1} \vee \overset{0}{x_2} \vee \overset{1}{x_4})}_{C_2} \wedge \underbrace{(\overset{0}{\bar{x}_1} \vee \overset{1}{\bar{x}_2} \vee \overset{0}{\bar{x}_3})}_{C_3}$$



V_1, V_2 no 2-cycle

$\rightarrow x_i$ 3-cycle W_i

$\rightarrow \bar{x}_i$

$\rightarrow$ at least one in each W_i not in V_1, V_2

$V_1, V_2 \rightarrow$ acyclic $\rightarrow$ A.A. partition.

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

- **$\mathcal{C}(\mathbb{A}, \mathbb{I})$-patition problem**

2-IN-3-SAT problem $\longrightarrow$ $\mathcal{C}(\mathbb{A}, \mathbb{I})$-patition problem

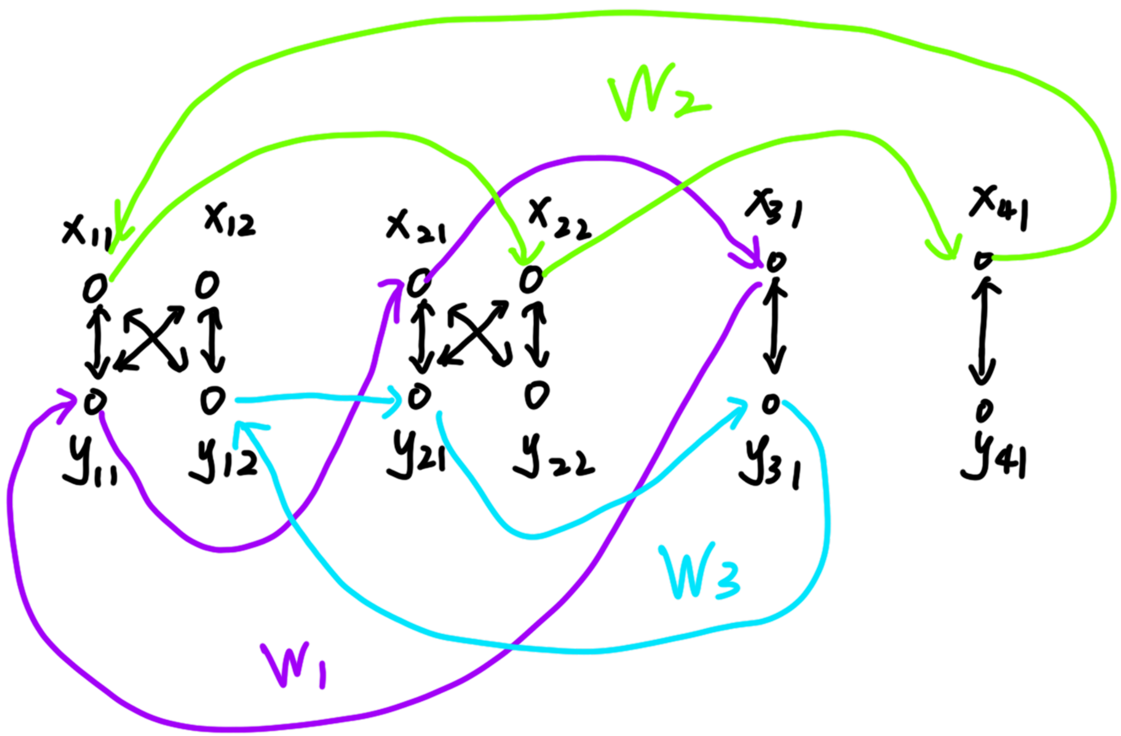
(exactly 2 literals are truth)

$\mathcal{F} \longrightarrow D(\mathcal{F})$, show that

$D(\mathcal{F})$ $\Longleftrightarrow$ $\mathcal{F}$ has a truth assignment

$\exists$ $(\mathbb{A}, \mathbb{I})$-patition which satisfies exactly 2 literals of each clause.

$D(\mathcal{F})$



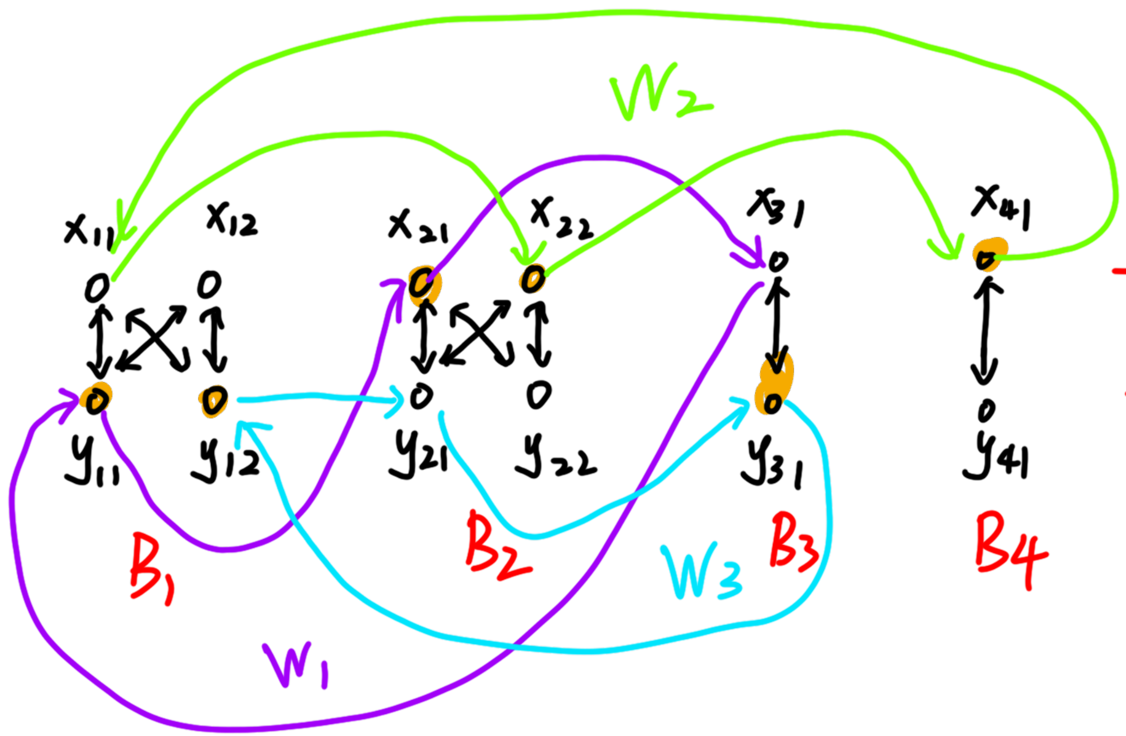
$\Leftarrow$ ϕ : a truth assignment of $\mathcal{F}$.

$$V_1 = \left(\bigcup_{\{i \mid \phi(x_i) = true\}} \{x_{i,j} \mid j \in [q_i]\} \right) \cup \left(\bigcup_{\{i \mid \phi(x_i) = false\}} \{y_{i,j} \mid j \in [q_i]\} \right), \text{ and } V_2 = V(D(\mathcal{F})) \setminus V_1.$$

ϕ: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

$$F = \underbrace{(\overset{1}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3})}_{C_1} \wedge \underbrace{(\overset{0}{x_1} \vee \overset{1}{x_2} \vee \overset{1}{x_4})}_{C_2} \wedge \underbrace{(\overset{1}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})}_{C_3}$$

V_1



V_1

$\rightarrow x_i$

$\rightarrow \bar{x}_i$

Note that, no edge in $D\langle B_i \cap V_j \rangle$

and

(i) $D\langle V_2 \rangle$ has one vertex in each 3-cycle $D\langle W_i \rangle$

$\Rightarrow$ no edge in $D\langle V_2 \rangle$

V_2 is an independent set

(ii) $D\langle V_1 \rangle$ does not contain 3-cycle

$\Rightarrow$ V_1 acyclic

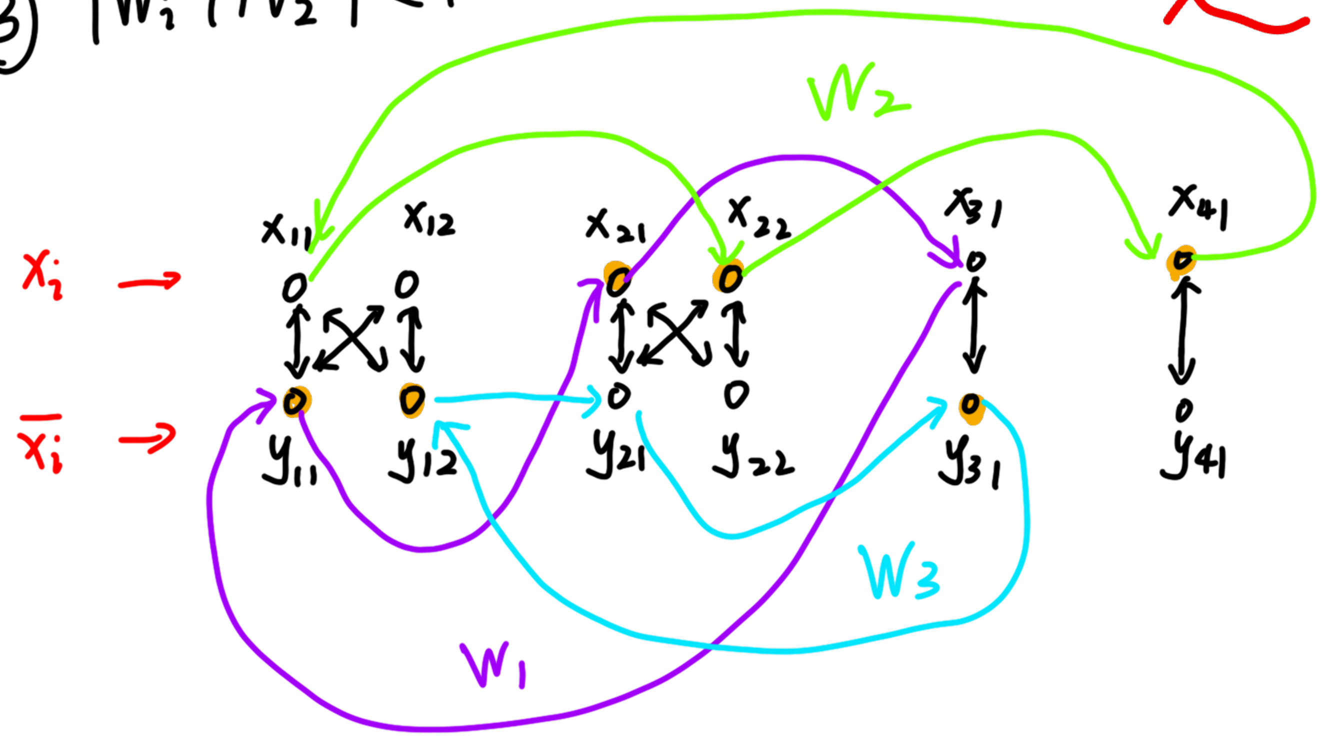
$\Rightarrow$

(V_1, V_2) is an (A.II)-patition of $D(\mathcal{F})$ no 2-cycle in V_1, V_2

① $x_{ij}\, y_{ij'}$ one in V_1, other one in V_2 no cycle in V_1

② $|W_i \cap V_1| \leq 2$ $\Rightarrow$ $|W_i \cap V_1| = 2$ no edge in $D\langle V_2 \rangle$

③ $|W_i \cap V_2| \leq 1$ $|W_i \cap V_2| = 1$ $\leftarrow$ V_1, V_2



$V_1 \rightarrow$ true

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

$$\mathcal{F} = (\underbrace{\overset{1}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}}_{C_1}) \wedge (\underbrace{\overset{0}{x_1} \vee \overset{1}{x_2} \vee \overset{1}{x_4}}_{C_2}) \wedge (\underbrace{\overset{1}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3}}_{C_3})$$

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

- $C(\mathbb{A}, \mathbb{I}\mathbb{O})$-patition problem

3-SAT problem $\longrightarrow$ $C(\mathbb{A}, \mathbb{I}\mathbb{O})$-patition problem

$\mathcal{F} \longrightarrow D(\mathcal{F})$, show that $D(\mathcal{F})$ $\exists$ $(\mathbb{A}, \mathbb{I}\mathbb{O})$-patition $\Longleftrightarrow$ $\mathcal{F}$ has a truth assignment

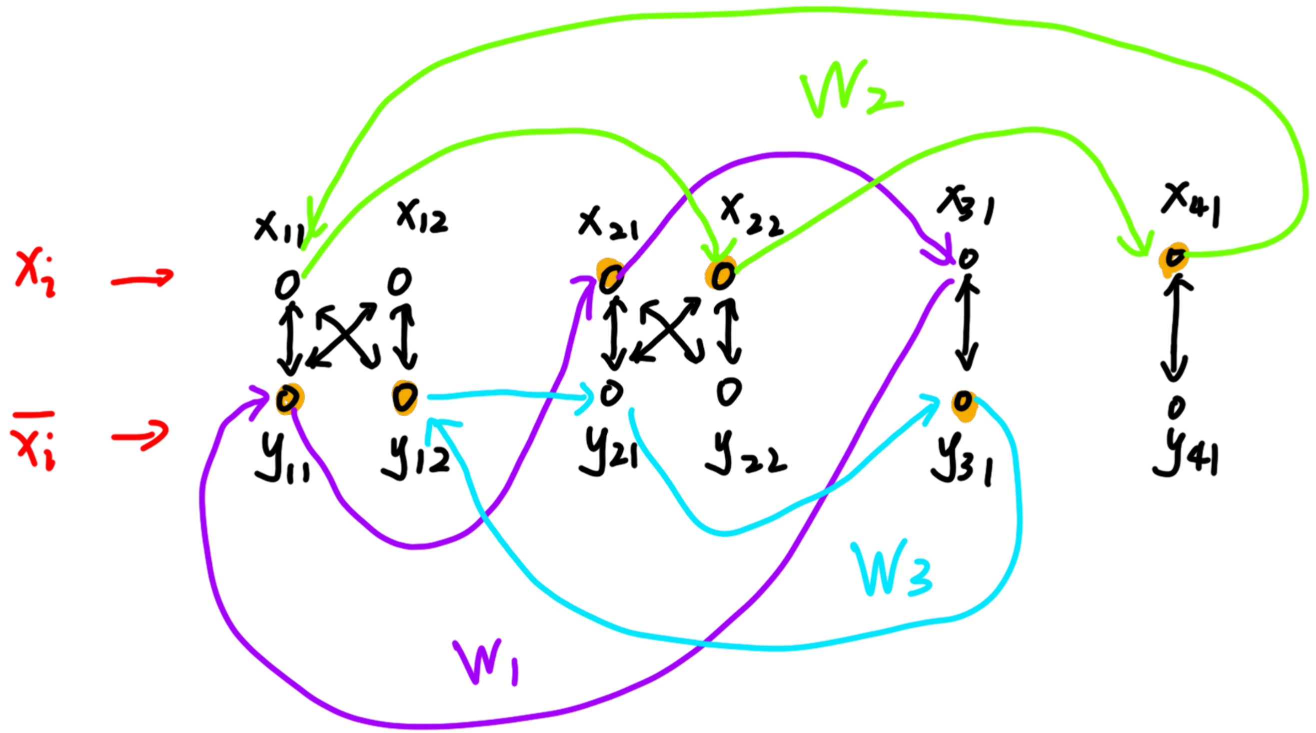
oriented

$\Rightarrow$

(V_1, V_2) is an (A.10)-patition of $D(\mathcal{F})$ oriented

① $x_{ij}\, y_{ij}$, one in V_1, other one in V_2 (2-cycle $\not\to V_1, V_2$)

② $|W_i \cap V_1| \leq 2 \Rightarrow |W_i \cap V_2| \geq 1$



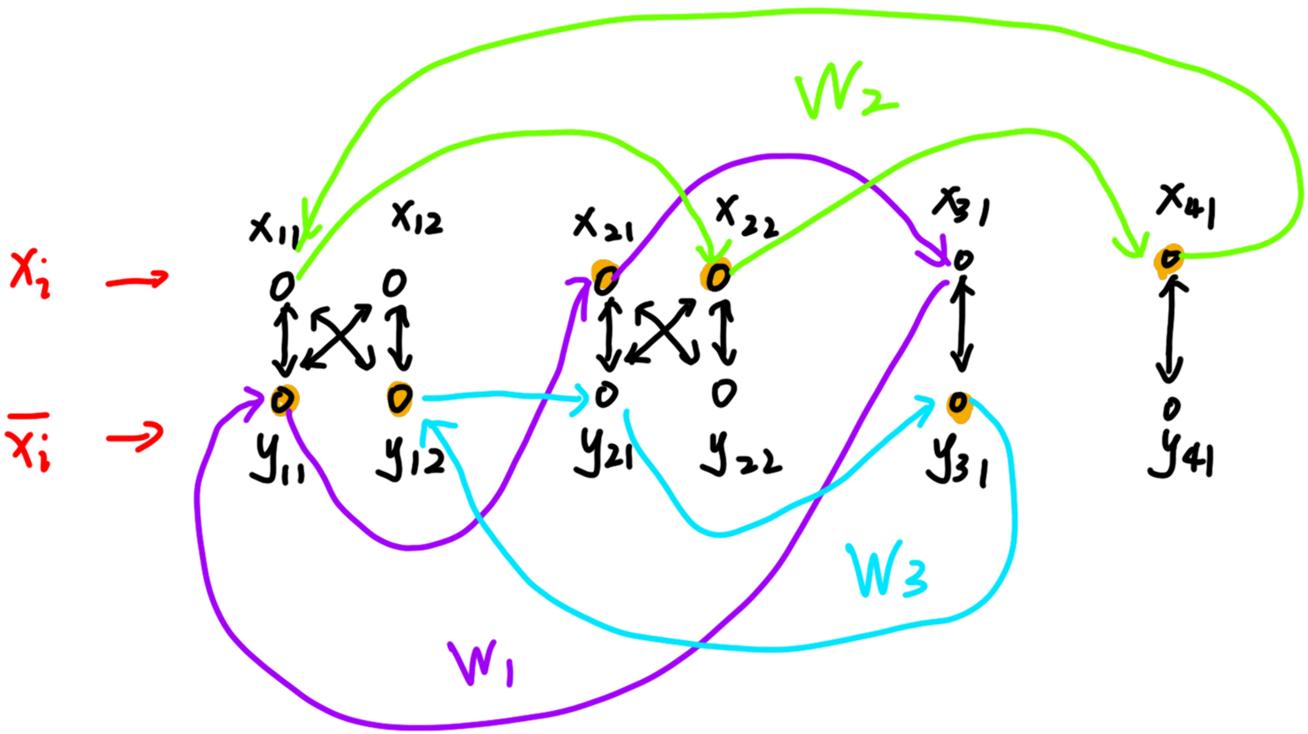
$V_2 \to$ true $\longrightarrow$ truth assignment of $\mathcal{F}$.

$\Leftarrow$ ϕ : a truth assignment of $\mathcal{F}$.

$$V_2 = \left(\bigcup_{\{i \mid \phi(x_i) = true\}} \{x_{i,j} \mid j \in [q_i]\} \right) \cup \left(\bigcup_{\{i \mid \phi(x_i) = false\}} \{y_{i,j} \mid j \in [q_i]\} \right), \text{ and } V_1 = V(D(\mathcal{F})) \setminus V_2.$$

① $x_{ij}\, y_{ij}$: one in V_1, other one in V_2 $\Rightarrow$ $D\langle V_2 \rangle$ oriented

② $|W_i \cap V_2| \geq 1$ $\Rightarrow$ $|W_i \cap V_1| \leq 2$ $\Rightarrow$ $D\langle V_1 \rangle$ acyclic

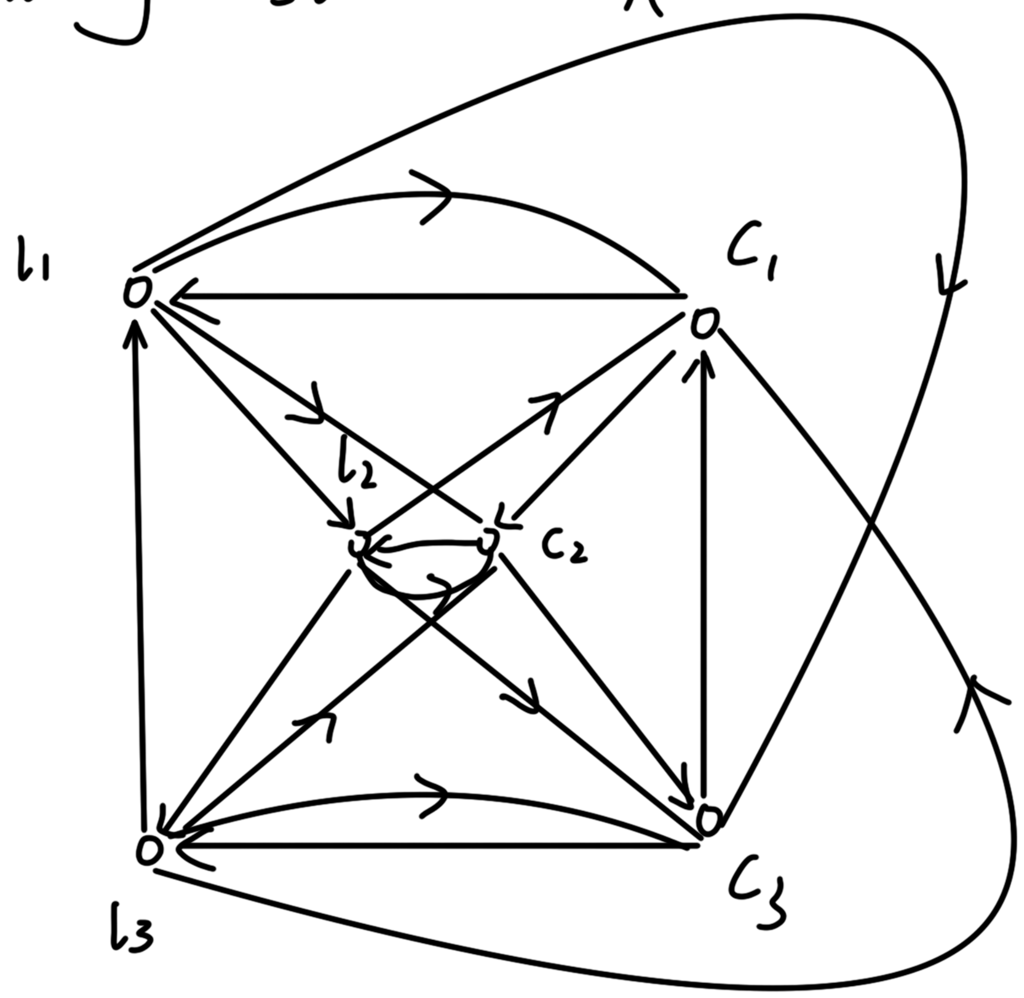


$\Downarrow$

(A, O)-partition of $D(\mathcal{F})$

- $C(A,IT)$-patition problem

auxiliary structure R



$|V_1|=|V_2|=3$ ←

$V(R)=\{\ell_1,\ell_2,\ell_3,c_1,c_2,c_3\}$

$E(R)=\{\ell_1\ell_2,\ell_2\ell_3,\ell_3\ell_1,c_1c_2,c_2c_3,c_3c_1\}\cup\{c_1\ell_1,c_2\ell_2,c_3\ell_3\}\cup\{\ell_ic_j|i,j\in[3]\}$

$C(A,IT)$-patition of R

(V_1,V_2) satisfies

V_2: one of $\{c_1,c_2,c_3\}$ + two $\{l_1,l_2,l_3\}$

or

one of $\{l_1,l_2,l_3\}$ + two $\{c_1,c_2,c_3\}$

$R\langle V_2\rangle$ is a tournament

if $|V_2|\geq 4 \Rightarrow$ 2-cycle

if $|V_2|\leq 2 \Rightarrow$ cycle in $R\langle V_1\rangle$

if $V_2=\{l_1,l_2,l_3\}$ or $\{c_1,c_2,c_3\}$ $\Rightarrow$ 3-cycle in $R\langle V_1\rangle$

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- C(A,IT)- patition problem

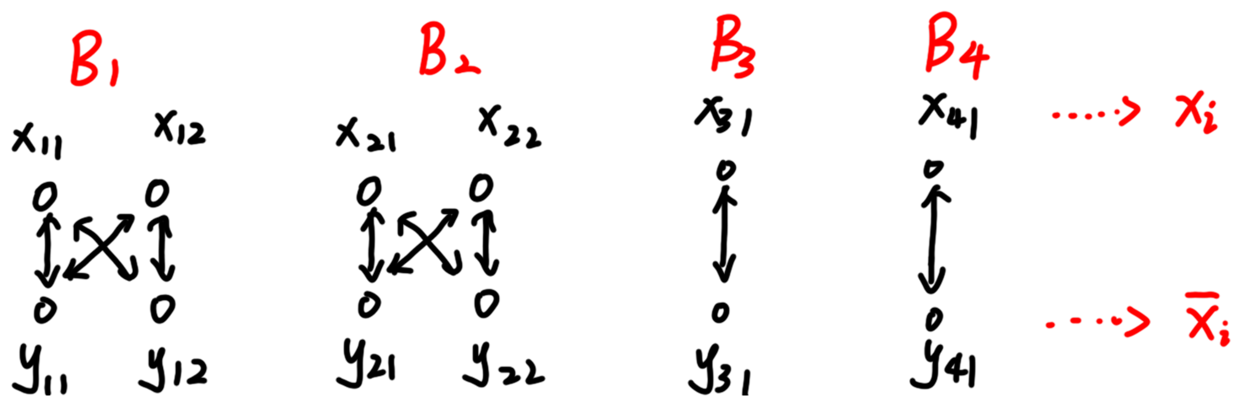
NAE - 3-SAT problem $\longrightarrow$ C(A,IT)- patition problem

instance $\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$

$\downarrow$

$B(\mathcal{F})$:

In contrast, we now prove that the $(\mathbb{A},\mathbb{P})$-$[k_1,k_2]$-partition problems are NP-complete for $\mathbb{P} \in \{\mathbb{A},\mathbb{I},\mathbb{O},\mathbb{S},\mathbb{T},\mathbb{Z}\}$. All our reductions use superdigraphs of the digraph $B(\mathcal{F})$ which is obtained from a given 3-SAT instance $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over the set of n boolean variables $x_1,\ldots,x_n$. The digraph $B(\mathcal{F})$ is defined from $\mathcal{F}$ as follows. Let q_i denote the maximum of the number of times x_i occurs in the clauses and the number of times $\bar{x}_i$ occurs in the clauses. The vertex set of $B(\mathcal{F})$ is $(\bigcup_{i\in[n]}\{x_{i,j} | j \in [q_i]\}) \cup (\bigcup_{i\in[n]}\{y_{i,j} | j \in [q_i]\})$ and the arc set of $B(\mathcal{F})$ is the union of the arc sets of the n complete bipartite digraphs $B_1, B_2, \ldots, B_n$ where B_i has vertex set $\{x_{i,j} | j \in [q_i]\} \cup \{y_{i,j} | j \in [q_i]\}$.



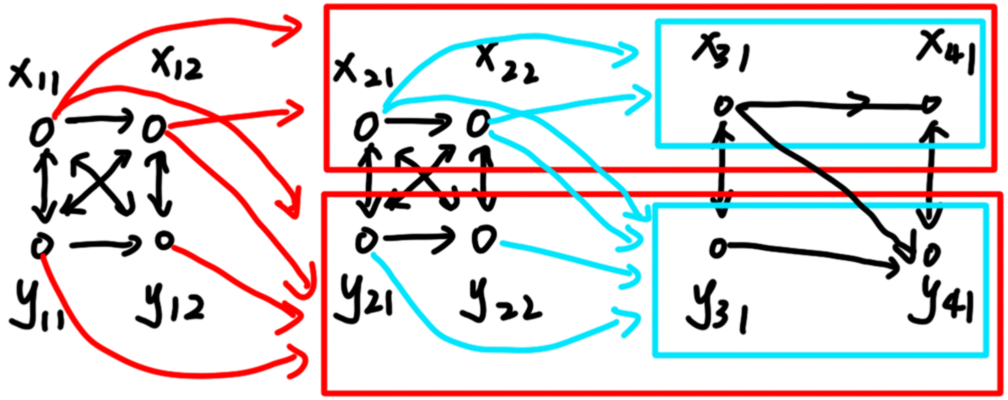
$\downarrow H(\mathcal{F})$

add c_{i1}, c_{i2}, c_{i3} for $i \in [3]$

add edges A_1, A_2

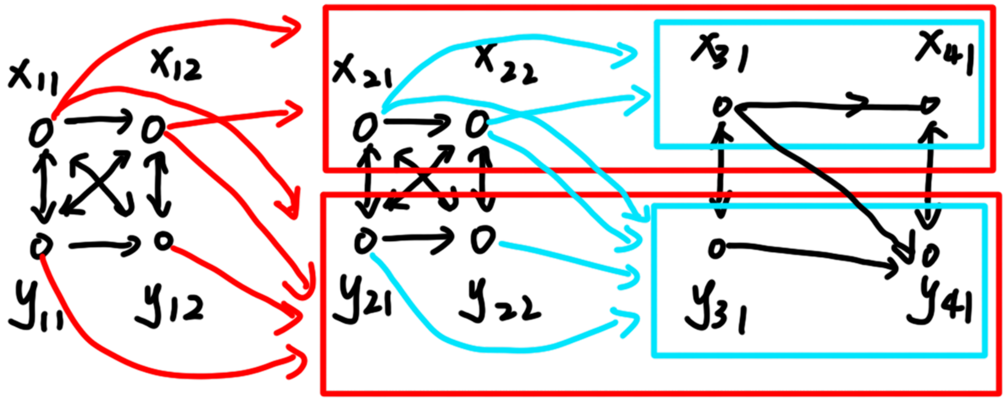
- A_1 consists of the arcs of the m copies R_j, $j \in [m]$ where R_j is obtained by using the 3 vertices in W_j corresponding to the literals of C_j as the vertices $\{\ell_1, \ell_2, \ell_3\}$ and letting $\{c_{j,1}, c_{j_2}, c_{j,3}\}$ correspond to c_1, c_2, c_3.
- A_2 consists of the union of
  * all arcs of the form $x_{i,j}x_{i',j'}$, $i, i' \in [n]$, $j \in [q_i]$, $j' \in [q_{i'}]$, where $i < i'$ or $i = i'$ and $j < j'$,
  * all arcs of the form $y_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i]$, $j' \in [q_{i'}]$, where $i < i'$ or $i = i'$ and $j < j'$,
  * all arcs of the form $x_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i]$, $j' \in [q_{i'}]$, where $i < i'$,
  * all arcs of the form $x_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i]$, $r \in [m]$, $s \in [3]$,
  * all arcs of the form $y_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i]$, $r \in [m]$, $s \in [3]$,
  * all arcs of the form $c_{r,s}c_{r',s'}$, $r, r' \in [m]$, $s, s' \in [3]$, where $r < r'$.

tourn

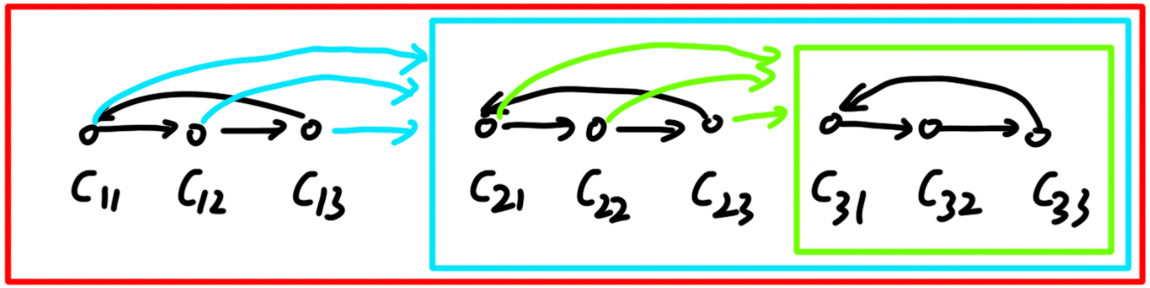


$\downarrow$

$H(\mathcal{F})$



all arcs



A_1: $W_i \to \{l_1, l_2, l_3\}$ $\{c_{i1}, c_{i2}, c_{i3}\} \to \{c_1, c_2, c_3\}$

$\Downarrow$

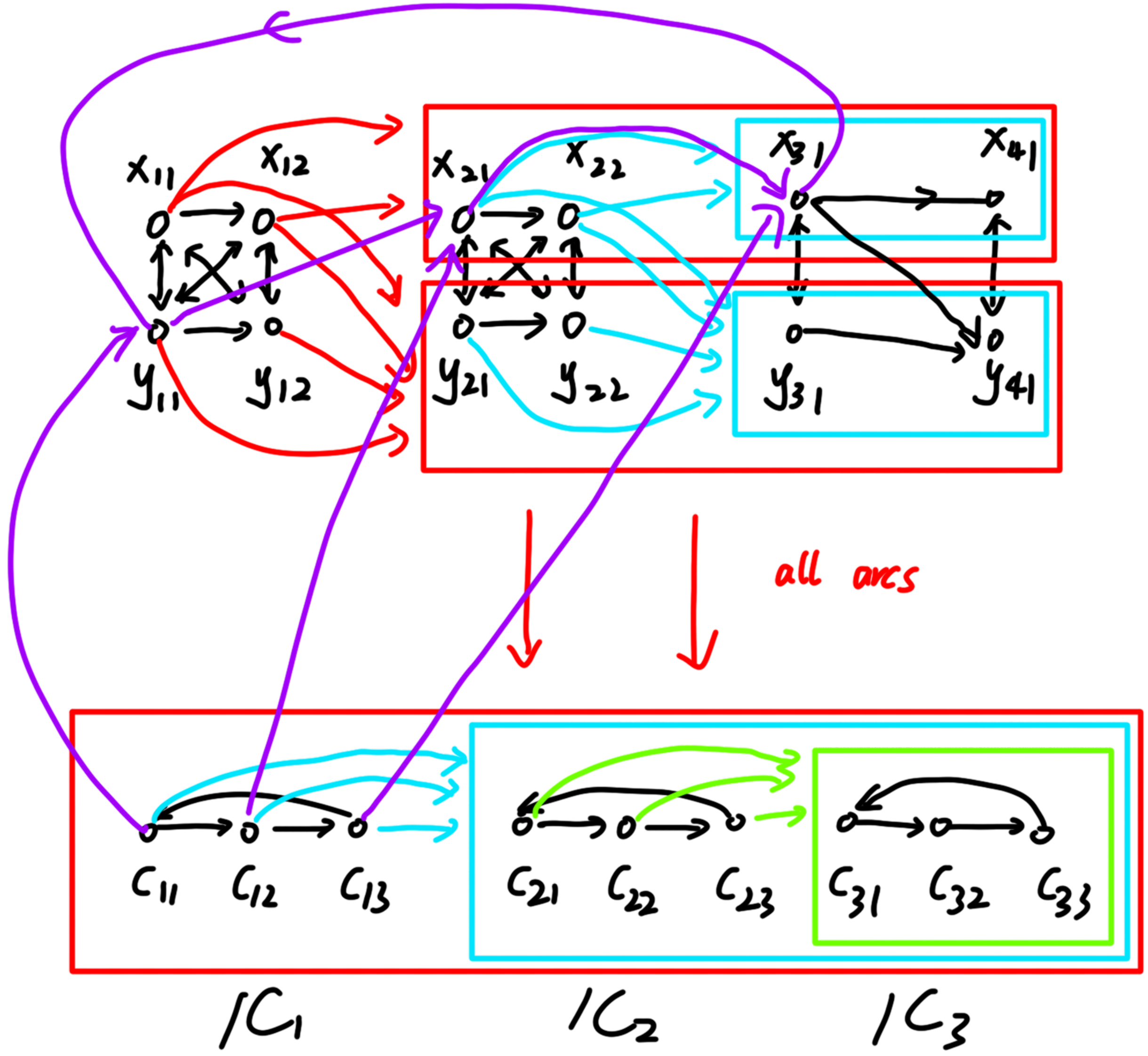
R-copy

$$\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$$

$C_1 \to W_1 = \{y_{11}, x_{21}, x_{31}\}$

$C_2 \to W_2 = \{x_{11}, x_{22}, x_{41}\}$

$C_3 \to W_3 = \{y_{12}, y_{21}, y_{31}\}$



see $/C_1$ and $W_1 \to R_1$

$R_i : /C_i \cup W_i$

$\to H(\mathcal{F})$ is a semicomplete digraph

(no non-adjacent vertices)

$\Leftarrow$ ϕ : such a truth assignment

$$V_2' = \left(\bigcup_{\{i \mid \phi(x_i) = true\}} \{x_{i,j} \mid j \in [q_i]\} \right) \cup \left(\bigcup_{\{i \mid \phi(x_i) = false\}} \{y_{i,j} \mid j \in [q_i]\} \right)$$

$V_2 = V_2' \cup \{ c_{ij} \mid c_{ij}\, x_{i'j'},\ c_{ij}\, y_{i'j'}$ does not 2-cycle for any $x_{i'j'}, y_{i'j'} \in V_2' \}$

NAE - 3-SAT problem $\Rightarrow 1 \le |W_i \cap V_2| \le 2$ ($\Rightarrow |W_i \cap W_1| \ge 1$)

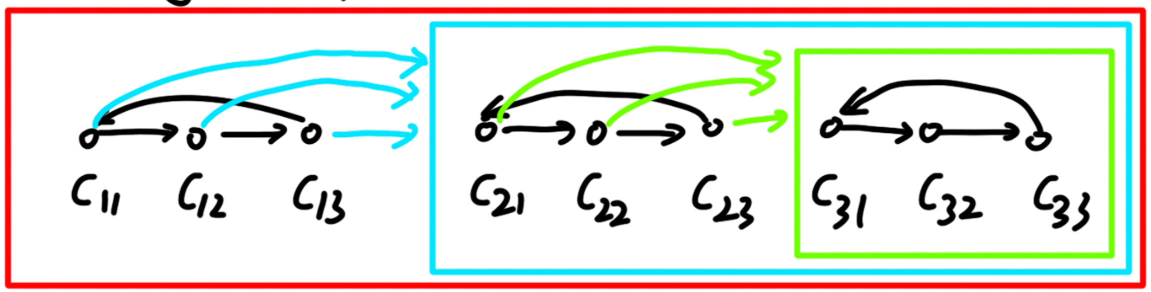
$\Rightarrow$ add c_{ij} $\Rightarrow$ $|V(R_i) \cap V_2| = 3$

By $H(\mathcal{F})$ is a semicomplete digraph $\Rightarrow H(\mathcal{F})\langle V_2 \rangle$ is a semicomi. di.

For each R_i, $V(R_i) \cap V_2$ induced a tour..

def. $\Rightarrow$ $c_{ij}\, x_{i'j'}$ $\Rightarrow$ no 2-cycle

$c_{ij}\, c_{i'j'}$ $\Rightarrow$



$\Rightarrow H(F)\langle V_2 \rangle$ is a tour.

By $V_2 \Rightarrow 1 \le |W_i \cap V_1| \le 2$ $\Rightarrow$ no 3-cycle

$1 \le |\mathcal{C}_i \cap V_1| \le 2$

By R_i-copy $\Rightarrow$ x_{ij} $c_{i'j'}$

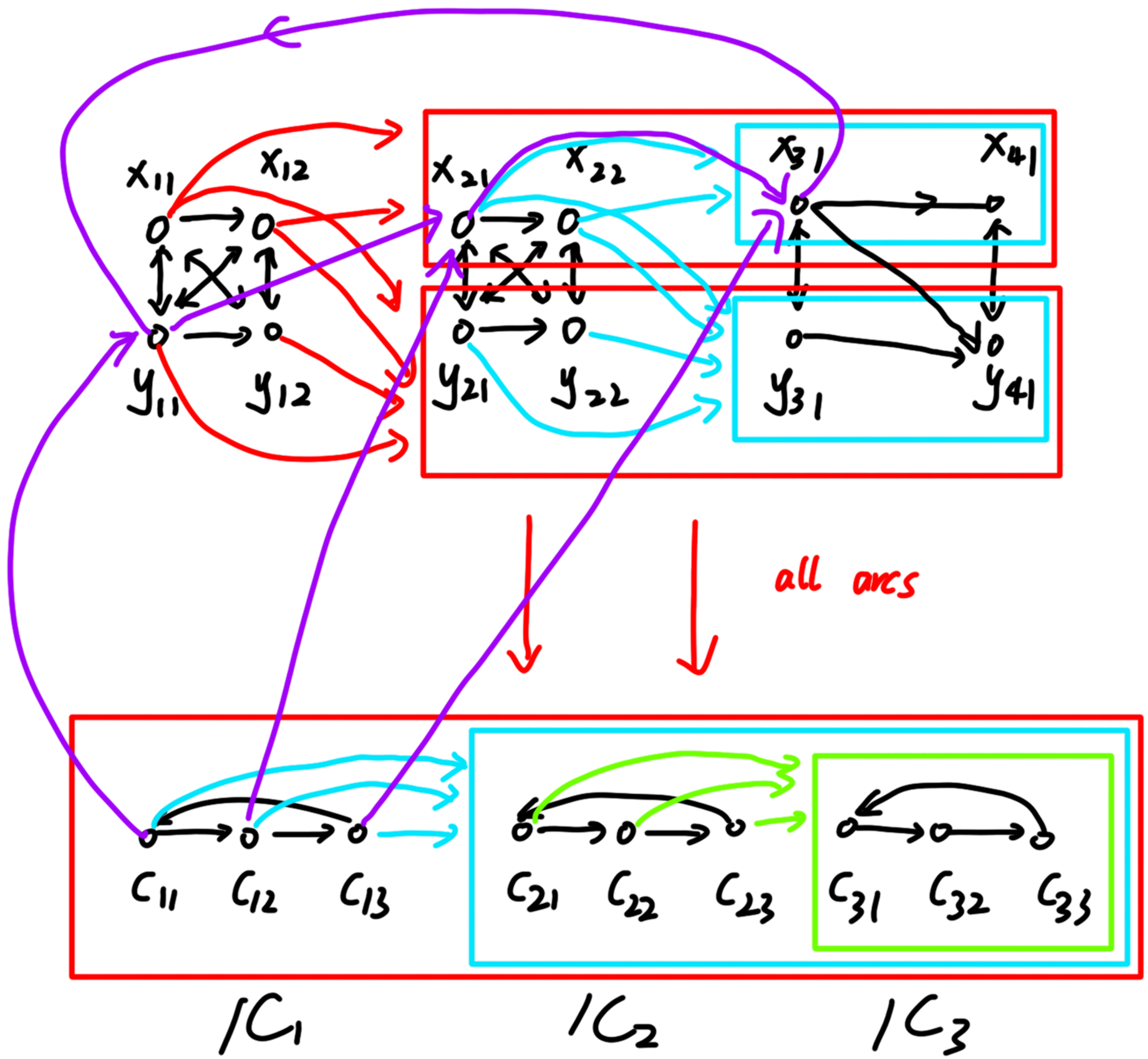
y_{ij} $c_{i'j'}$

$(c \to x, y$ only in $R)$

no 2-cycle

$x_{ij}, y_{ij'}$ only one in V_1

$\Rightarrow H(F)\langle V_1 \rangle$ is acyclic



$\Rightarrow$ (V_1, V_2) is an $(/A, /T)$-partition of $H(F)$

By construction

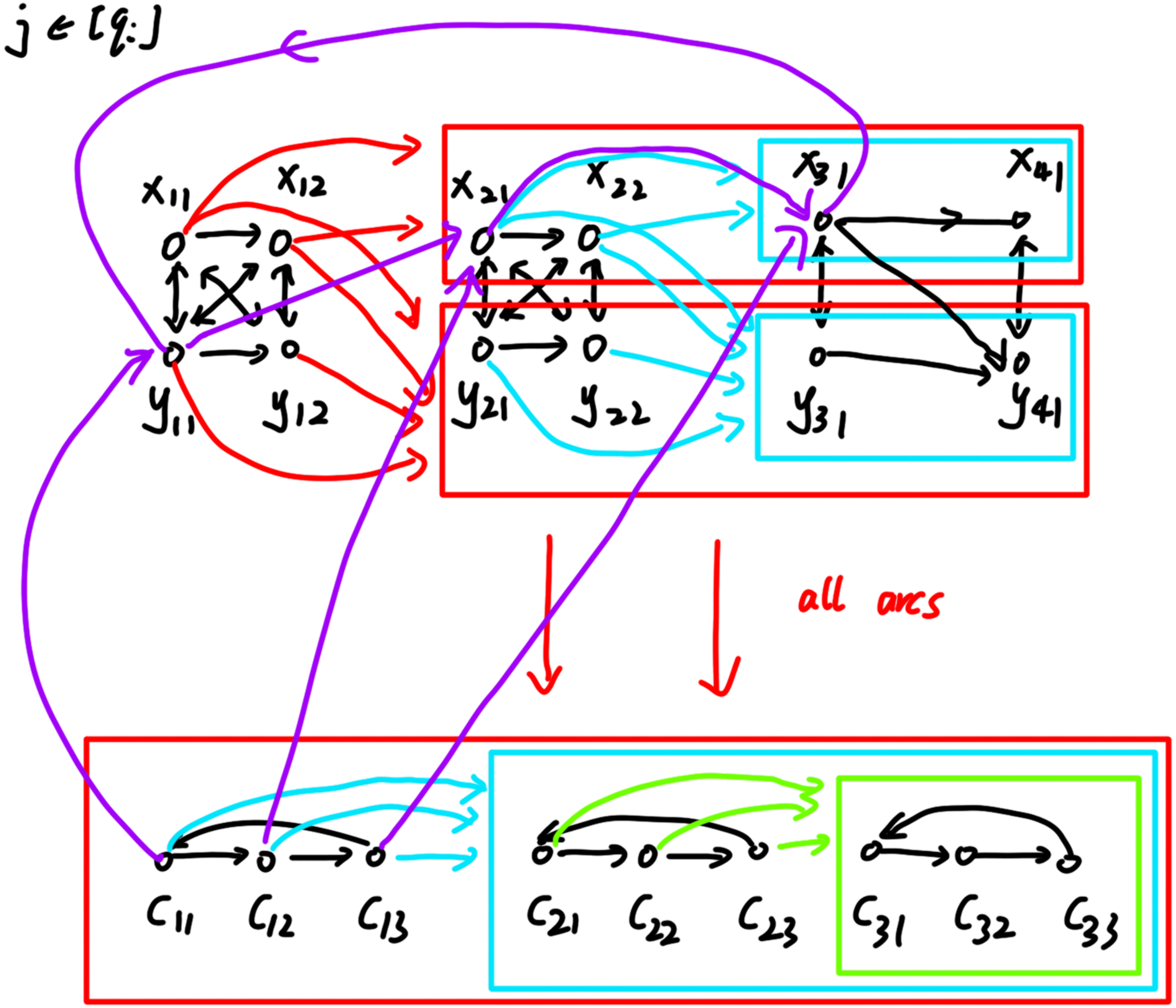
$\forall x_i$, $x_{ij} \subseteq V_1$ $y_{ij} \subseteq V_2$ $j \in [q_i]$

or $x_{ij} \subseteq V_2$, $y_{ij} \subseteq V_1$ $j \in [q_i]$

Let $V_2 \to$ true, by R_i

$\Rightarrow$ $1 \le |V_2 \cap W_i| \le 2$

$\Rightarrow$ ϕ is a truth assignment



Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

- $\mathcal{C}(\mathbb{A}, \mathbb{S})$-patition problem

$D\langle V_2 \rangle$ can 2-cycle.

2-IN-3-SAT problem $\longrightarrow$ $\mathcal{C}(\mathbb{A}, \mathbb{S})$-patition problem

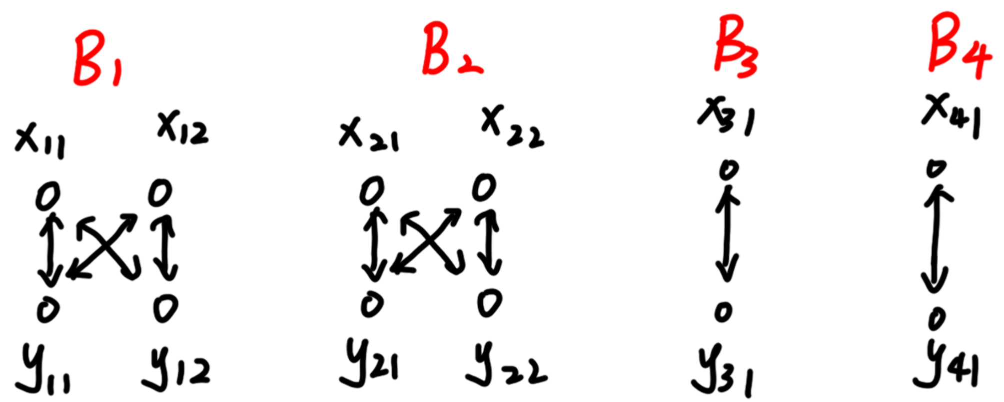
$x_i = 1 \Leftrightarrow$ all x_{ij} in V_2

instance $\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$

$x_j = 0 \Leftrightarrow$ all y_{ij} in V_2

$\downarrow$

$B(\mathcal{F})$



$\cdots\rightarrow x_i$ $|V_i \cap V_2| = 2$

all.

$\cdots\rightarrow \bar{x}_i$

$\downarrow$

$G(\mathcal{F})$

Form the digraph $G(\mathcal{F})$ by adding the following vertices and arcs to $B(\mathcal{F})$: add vertices $\{x_{1,q_1+1}, y_{1,q_1+1}, \ldots, x_{n,q_n+1}, y_{n,q_n+1}\} \cup (\bigcup_{j\in[m]}\{c_{j,1}, c_{j,2}, c_{j,3}\})$ and new arcs formed by the union of A_1, A_2, A_3 defined as follows:

- $A_1 = \{x_{i,q_i+1}y_{i,q_i+1}, y_{i,q_i+1}x_{i,q_i+1} | i \in [n]\}$.
- A_2 consists of the arcs of the m directed 3-cycles $Q_j = c_{j,1}c_{j,2}c_{j,3}c_{j,1}$, $j \in [m]$ and the arcs of the m vertex-disjoint complete digraphs M_j, $j \in [m]$ on three vertices where $V(M_j) = W_j$ for $j \in [m]$. Finally, for each clause C_j, $j \in [m]$, A_2 contains six arcs from W_j to $V(Q_j)$ such that each vertex in $V(Q_j)$ receives exactly two arcs from W_j and each vertex of W_j sends exactly two arcs to Q_j.
- A_3 consists of the union of
  * all arcs of the form $x_{i,j}x_{i',j'}$, $i, i' \in [n]$, $j \in [q_i+1]$, $j' \in [q_{i'}+1]$, where $i < i'$ or $i = i'$ and $j < j'$,
  * all arcs of the form $y_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i+1]$, $j' \in [q_{i'}+1]$, where $i < i'$ or $i = i'$ and $j < j'$,
  * all arcs of the form $x_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i+1]$, $j' \in [q_{i'}+1]$, where $i < i'$,
  * all arcs of the form $x_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i+1]$, $r \in [m]$, $s \in [3]$, except those where $x_{i,j} \in W_r$,
  * all arcs of the form $y_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i+1]$, $r \in [m]$, $s \in [3]$, except those where $y_{i,j} \in W_r$,
  * all arcs of the form $c_{r,s}c_{r',s'}$, $r, r' \in [m]$, $s, s' \in [3]$, where $r < r'$.

$|Q_1 \cap V_1| \leq 2$

$\Rightarrow |Q_1 \cap V_2| \geq 1$

$|W_1 \cap V_1| \leq 1$

$\Rightarrow |W_1 \cap V_2| \geq 2$

$\Rightarrow |W_1 \cap V_2| = 2$

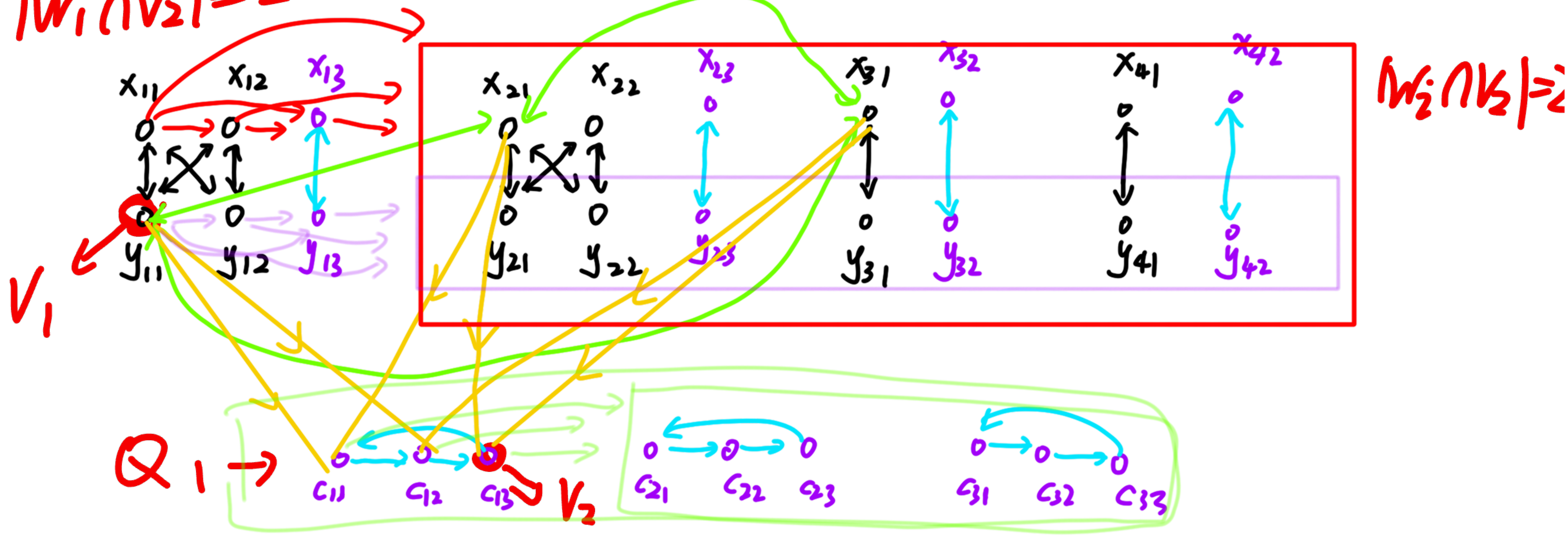
$y_{11} \notin V_2 \Rightarrow |W_1 \cap V_2| = 2$

$W = W_1 \cup W_2 \cup W_3$

$C_1 \to W_1 = \{y_{11}, x_{21}, x_{31}\}$

$C_2 \to W_2 = \{x_{11}, x_{22}, x_{41}\}$

$C_3 \to W_3 = \{y_{12}, y_{21}, y_{31}\}$



V_2 : semi com. condition , 2-cycle $\subseteq G(\mathcal{F})\langle V_2\rangle$

$S_0 \to G(\mathcal{F})\langle W_1\rangle$ use complete digraph.

$\Leftarrow$ $\phi \to V_2 \to K\langle V_2\rangle$ is a sym. digraph

$V_1 = V \setminus V_2 \to K\langle V_1\rangle$ is an acyclic digraph

$\Rightarrow$ V_2 is truth $\Rightarrow |V_2 \cap W_2| \geq 2$

Below the letters $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$, $\mathbb{O}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{Z}$ are shorthand for 'acyclic', 'complete', 'independent', 'oriented', 'semicomplete', 'tournament' and 'symmetric', respectively.

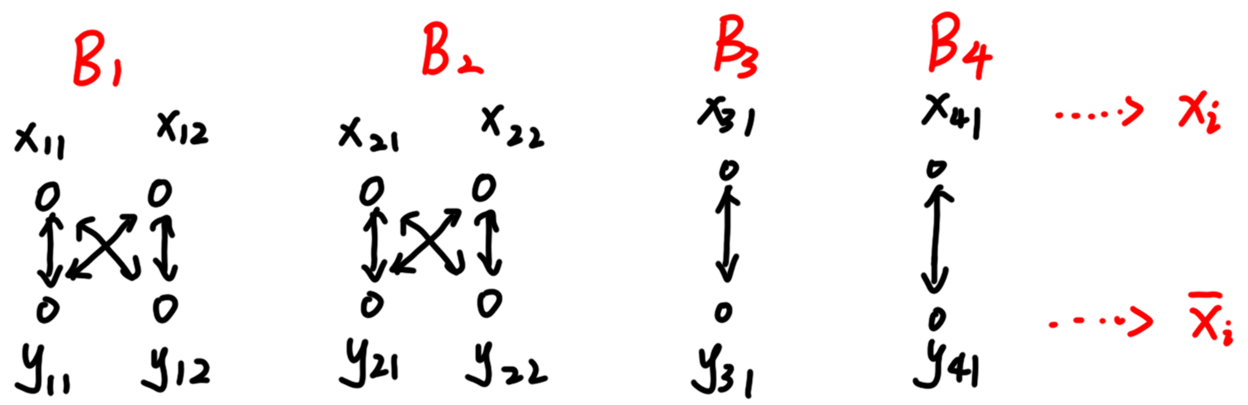
- **$C(\mathbb{A},\mathbb{Z})$-patition problem**

2-IN-3-SAT problem $\longrightarrow$ $C(\mathbb{A},\mathbb{Z})$-patition problem

instance $\quad \mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$

$\downarrow$

$B(\mathcal{F})$



i=1

adding the following vertices and arcs to $B(\mathcal{F})$: add new vertices $\{d_{i,p} | i \in [n], p \in [4]\} \cup (\bigcup_{j \in [m]} \{v_{j,1}, v_{j,2}, v_{j,3}\})$ and the arc sets A_1, A_2, A_3 defined below.

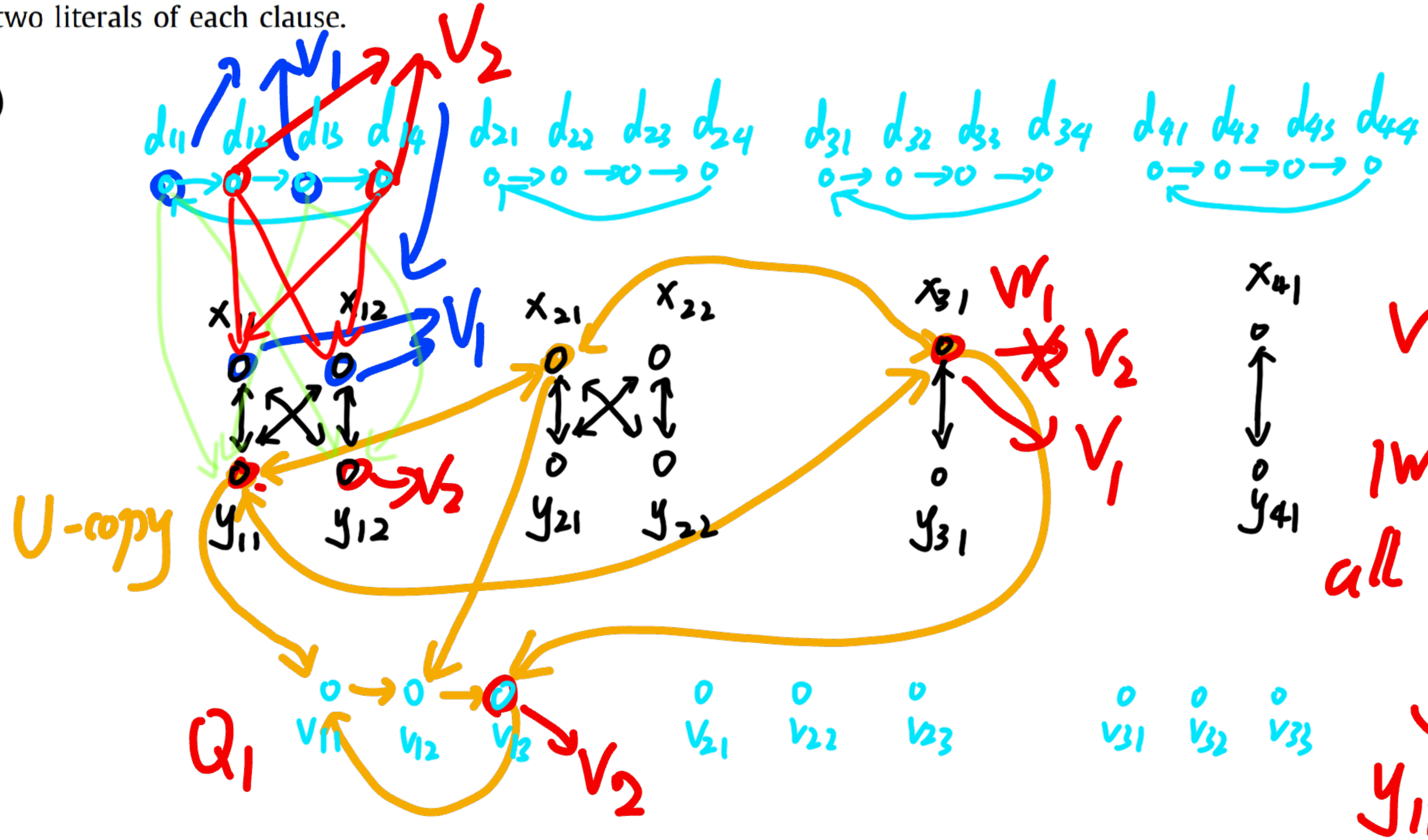
- A_1 is the sets of arcs of the n disjoint directed 4-cycles $d_{i,1}d_{i,2}d_{i,3}d_{i,4}d_{i,1}$, $i \in [n]$.
- A_2 is the arc-disjoint union of the arcs of m copies $U_1, \dots, U_m$ of U where we identify the vertices $u_{j,1}, u_{j,2}, u_{j,3}$ of the j'th copy of U with the vertices of W_j (the $v_{j,i}$-vertices are all distinct).
- $A_3 = \bigcup_{i \in [n], j \in [q_i]} \{d_{i,1}y_{i,j}, d_{i,3}y_{i,j}, d_{i,2}x_{i,j}, d_{i,4}x_{i,j}\}$.

X1

We claim that $K(\mathcal{F})$ has an $(\mathbb{A}, \mathbb{Z})$-partition (V_1, V_2) if and only if $\mathcal{F}$ has a truth assignment which satisfies exactly two literals of each clause.

K(F)

and



$V_1 \to Q_1$

$|W_1 \cap V_2| = 2$

all $y_{11} \to V_2$

$y_{12} \to V_1$

$y_{11}, y_{12} \to V_2$

$$\mathcal{F} = \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_3}$$

$|W_1 \cap V_2| = 2$

$x_{i,j}x_{i',j'}$, $i, i' \in [n]$, $j \in [q_i + 1]$, $j' \in [q_{i'} + 1]$, where $i < i'$ or $i = i'$ and $j < j'$,
$y_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i + 1]$, $j' \in [q_{i'} + 1]$, where $i < i'$ or $i = i'$ and $j < j'$,
$x_{i,j}y_{i',j'}$, $i, i' \in [n]$, $j \in [q_i + 1]$, $j' \in [q_{i'} + 1]$, where $i < i'$,
$x_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i + 1]$, $r \in [m]$, $s \in [3]$, except those where $x_{i,j} \in W_r$,
$y_{i,j}c_{r,s}$, $i \in [n]$, $j \in [q_i + 1]$, $r \in [m]$, $s \in [3]$, except those where $y_{i,j} \in W_r$,
$c_{r,s}c_{r',s'}$, $r, r' \in [m]$, $s, s' \in [3]$, where $r < r'$.

$x_1 = 0$

$W = W_1 \cup W_2 \cup W_3$

$C_1 \to W_1 = \{y_{11}, x_{21}, x_{31}\}$

$C_2 \to W_2 = \{x_{11}, x_{22}, x_{41}\}$

$C_3 \to W_3 = \{y_{12}, y_{21}, y_{31}\}$

$\Leftarrow \phi \to V_2 \to K\langle V_2 \rangle$ is a sym. digraph

$V_1 = V \setminus V_2 \to K\langle V_1 \rangle$ is an acyclic digraph

$\Rightarrow V_2$ is truth $\Rightarrow |V_2 \cap W_2| \geq 2$