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Complexity of some arc-partition problems for digraphs ☆

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3. Polynomially solvable arc-partition problems

3.2.6. *(acyclic, acyclic)-arc-partition*

3.5. *(acyclic spanning, acyclic (spanning))-arc-partition*

4. Some NP-complete arc-partition problems

4.3.3. *(bipartite, cycle factor)-arc-partition*

(P_1, P_2)-arc-partition of a digraph $D=(V,A)$:

History

1973 **Theorem 1** *(Edmonds). [16] Let $D=(V,A)$ be a digraph, let $s \in V$ be a fixed vertex and let $k \geq 2$ be an integer. Then D has k arc-disjoint out-branchings rooted at s if and only if there are k arc-disjoint (s,t)-paths in D for every choice of $t \in V - s$.*

1991 Thomassen proved that the (*having* B^+, *having* B^-)-arc-partition problem is NP-complete

2004 The more constrained (*strong, strong*)-arc-partition problem is NP-complete [11].

2012 The (*strong*, *connected*)-arc-partition problem is NP-c [12]

2-regular   2-arc-disjoint H-cycle

2009 **Theorem 3.** *[6, Theorem 6.1.2] It is NP-complete to decide whether a 2-regular digraph has a hamiltonian cycle*

k (k≥2)   H-cycle

H-path

- *bipartite*: Being bipartite,
- *connected*: The underlying graph is connected,
- *strong*: Being strongly connected,
- *acyclic*: Being acyclic,
- *acyclic spanning* : Being acyclic and each vertex is spanned (minimum degree at least 1).
- *having* B^+: Containing an out-branching rooted in some vertex
- *having* B^-: Containing an in-branching rooted in some vertex
- *cycle factor*: Each vertex has in-degree and out-degree exactly 1,
- $\delta^- \geq k$: Each vertex has in-degree at least k,
- $\delta^+ \geq k$: Each vertex has out-degree at least k,
- $\geq k$ *arcs*: Having at least k arcs,
- $\leq k$ *arcs*: Having at most k arcs,
- *balanced*: $d^+(v) = d^-(v)$ for every vertex v,
- *eulerian*: Being balanced and strongly connected,
- *cycle*: Being an oriented cycle (with possibly some vertices not spanned).

Results

Complexity of arc-partitioning problems.

	Bipartite	Connected	Strong	Acyclic	Acyclic spanning	Having B^+	Having B^-
Bipartite	NP-c (4.3.1)	P (3.1)	P (3.1)	NP-c (4.2.2)	NP-c (4.2.3)	P (3.1)	P (3.1)
Connected	×	P [18,24]	NP-c [12]	P (3.1)	NP-c (4.2.4)	NP-c [12]	NP-c [12]
Strong	×	×	NP-c [11]	P (3.1)	NP-c (4.2.7)	NP-cT (Th. 10)	NP-cT (Th. 10)
Acyclic	×	×	×	P (3.2.6)	P (3.5)	P (3.1)	P (3.1)
Acyclic spanning	×	×	×	×	P (3.5)	NP-c (4.2.6)	NP-c (4.2.6)
Having B^+	×	×	×	×	×	P [16]	NP-c [2]
Having B^-	×	×	×	×	×	×	P [16]

	$\delta^- \geq k$	$\delta^+ \geq k$	Cycle Factor	$\leq k$ arcs	$\geq k$ arcs	Balanced	Eulerian	Cycle
Bipartite	P (3.1)	P (3.1)	NP-c (4.3.3)	NP-c (4.3.2)	P (3.1)	NP-c (4.3.4)	NP-c (4.3.5)	NP-c (4.3.6)
Connected	P (3.4.2)	P (3.4.2)	NP-c (Th. 8)	P (3.1)	P (3.2.1)	P (3.1)	NP-c (Th. 10)	NP-c [12]
Strong	NP-c (Corollary 9)	NP-c (Corollary 9)	NP-c (Th. 8)	P (3.1)	NP-c (Th. 8)	P (3.1)	NP-c (Th. 10)	NP-c [12]
Acyclic	P (3.1)	P (3.1)	NP-cT (Th. 8)	NP-c (4.2.1)	P (3.1)	P (3.2.5)	NP-c (4.2.8)	NP-cT (Th. 10)
Acyclic spanning	P (3.3.1)	P (3.3.1)	NP-cT (Th. 8)	NP-c (4.2.1)	P (3.2.1)	NP-cT (Th. 8)	NP-cT (Th. 10)	NP-cT (Th. 10)
Having B^+	P (3.2.3)	NP-c (4.2.5)	NP-c (Th. 8)	P (3.1)	P (3.2.1)	P (3.1)	NP-c (Th. 10)	NP-c [12]
Having B^-	NP-c (4.2.5)	P (3.2.3)	NP-c (Th. 8)	P (3.1)	P (3.2.1)	P (3.1)	NP-c (Th. 10)	NP-c [12]

	$\delta^- \geq k$	$\delta^+ \geq k$	Cycle Factor	$\leq k$ arcs	$\geq k$ arcs	Balanced	Eulerian	Cycle
$\delta^- \geq k$	P (3.2.3)	P (3.3.2)	P (3.2.2)	P (3.1)	P (3.2.1)	P (3.1)	NP-c (Corollary 9)	P (3.2.3)
$\delta^+ \geq k$	×	P (3.2.3)	P (3.2.2)	P (3.1)	P (3.2.1)	P (3.1)	NP-c (Corollary 9)	P (3.2.3)
Cycle Factor	×	×	P (3.2.2)	P (3.2.1)	P (3.2.1)	P (3.2.2)	NP-c (Th. 8)	NP-c (Th. 8)
$\leq k$ arcs	×	×	×	P (3.2.1)	P (3.1)	P [14]	NP-c [14]	NP-c (Th. 8)
$\geq k$ arcs	×	×	×	×	P (3.2.1)	P (3.1)	NP-c (Th. 8)	P (3.2.1)
Balanced	×	×	×	×	×	P (3.2.4)	P (3.2.4)	P (3.2.4)
Eulerian	×	×	×	×	×	×	NP-c (Th. 10)	NP-c [12]
Cycle	×	×	×	×	×	×	×	NP-c (Th. 10)

P Problems:

3.2.6. (acyclic, acyclic)-arc-partition

3.5. (acyclic spanning, acyclic (spanning))-arc-partition

1° (acyclic spanning, acyclic)-arc-partition

2° (acyclic spanning, acyclic spanning)-arc-partition

Proof: Let D be connected

Claim: D admits an (acyclic spanning, acyclic spanning)-arc-partition $\Leftrightarrow$ $\delta(D) \geq 2$, UG(D) is not a odd cycle

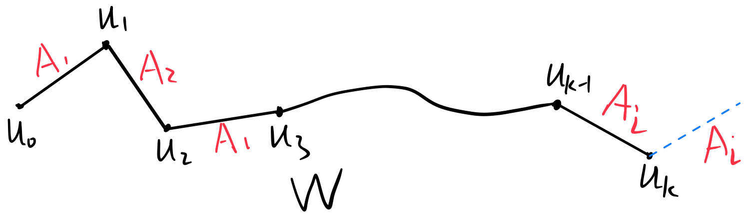
$\Rightarrow$:

$\Leftarrow$: 不妨设UG(D)不是偶圈

取(acyclic spanning, acyclic)-arc-partition (A_1, A_2), 使A_2覆盖尽可能多的点.

设v未被A_2覆盖

Claim 7.1. *$UG(D)$ does not contain a walk $W = u_0u_1 \ldots u_k$ alternating between A_1 and A_2 from $v = u_0$, and such that if $u_{k-1}u_k \in A_i$ the vertex u_k has an incident edge in $A_i \setminus E(W)$.*



$\delta(D) \geq 2 \Rightarrow UG(D)$含圈

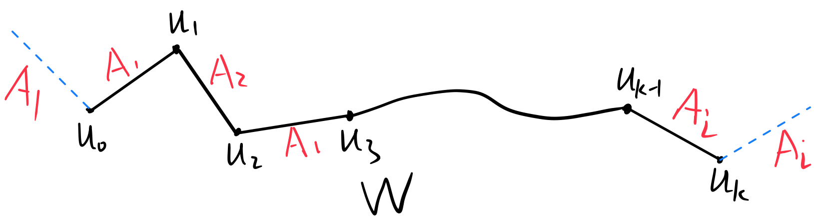
$UG(D)$中所有圈都含v

$d(v) = 2$

$\Delta(D) \leq 2$

Claim 7.1. *$UG(D)$ does not contain a walk $W = u_0u_1\dots u_k$ alternating between A_1 and A_2 from $v = u_0$, and such that if $u_{k-1}u_k \in A_i$ the vertex u_k has an incident edge in $A_i \setminus E(W)$.*

Proof： 取极短 W



- $u_0u_1 \in A_1$ and $d_{A_1}(u_0) > 1$,
- $d_{A_1}(u_j) = 1$ (this edge being $u_{j-1}u_j$) for any odd $j < k$, and
- $d_{A_2}(u_j) = 1$ (this edge being $u_{j-1}u_j$) for any even $0 < j < k$.

NP-C Problems:

Theorem 8. *Let $D = (V, A)$ be a 2-regular digraph on n vertices. The following properties are equivalent:*

0. *D admits a hamiltonian cycle,*
1. *D admits a (strong, $\delta^+ \geq 1$)-arc-partition,*
2. *D admits an (eulerian, $\delta^+ \geq 1$)-arc-partition,*
3. *D admits a (connected, cycle factor)-arc-partition,*
4. *D admits a (strong, cycle factor)-arc-partition,*
5. *D admits a (having B^+, cycle factor)-arc-partition,*
6. *D admits a (strong, $\geq n$ arcs)-arc-partition,*
7. *D admits a (eulerian, $\geq n$ arcs)-arc-partition,*
8. *D admits an (eulerian, cycle factor)-arc-partition,*
9. *D admits a (cycle, cycle factor)-arc-partition,*
10. *D admits a (cycle, $\leq n$ arcs)-arc-partition,*
11. *there is an arc $a \in A$ such that $D - \{a\}$ admits an (acyclic, cycle factor)-arc-partition,*
12. *there is an arc $a \in A$ such that $D - \{a\}$ admits an (acyclic spanning, cycle factor)-arc-partition,*
13. *there is an arc $a \in A$ such that $D - \{a\}$ admits an (acyclic spanning, balanced)-arc-partition,*

$\Longrightarrow$

$\Longleftarrow$ 10:

1、2、4、6、7、8:

3、5、9、11、12:

13:

4.3.3. *(bipartite, cycle factor)-arc-partition*

3-SAT $\longrightarrow$ (bipartite, cycle factor)-arc-partition

Proof: $W[u,v,p,q]$: vertices $\{u, v, y_1, \cdots, y_{3p}, z_1, \cdots, z_{3q}\}$

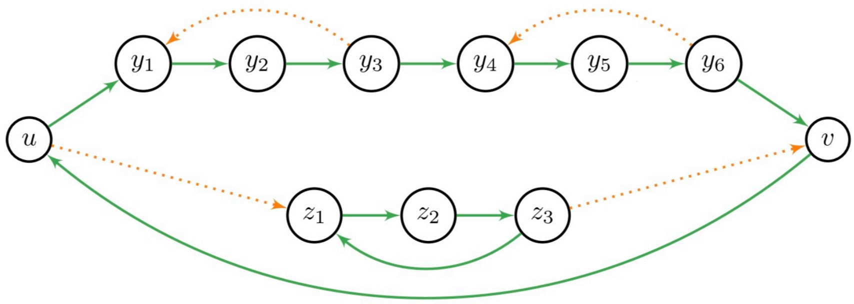
(u,v) paths $uy_1 \cdots y_{3p}v$, $uz_1 \cdots z_{3q}v$.

arcs $y_{3i}y_{3i-2}$, $i \in [p]$

arcs $z_{3j}z_{3j-2}$, $j \in [q]$

arc vu

$W[u,v,2,1]$:

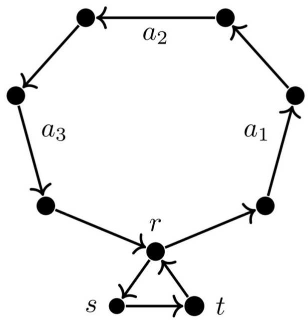


Note that if (A_1, A_2) is a (*bipartite*, *cycle factor*)-arc-partition of $W[u, v, p, q]$, then the following holds:

A_1 A_2

- The arc vu is necessarily in A_2 (because of the cycle in A_2 containing u).
- Either uy_1 or uz_1 belongs to A_2 (same reason).
- Each arc $y_{3i}y_{3i+1}$ (respectively $z_{3i}z_{3i+1}$) belongs to the same part as uy_1 and as $y_{3p}v$ (respectively as uz_1 and as $z_{3q}v$). This follows from the fact that all the vertices y_{3i+2} and z_{3i+2} are incident to two arcs in A_2, and that then among the arcs entering y_{3i+1} and z_{3i+1} (resp. leaving y_{3i} and z_{3i}) exactly one should be in A_2.

$W'[r, s, t, a_1, a_2, a_3]$:



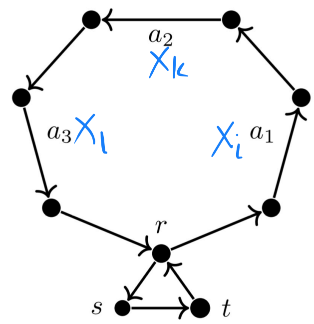
Let l with varibles $x_1, \cdots, x_n$, clauses $C_1, \cdots, C_m$.

Construct D_L:

1° $x_i \longrightarrow W[u_i, v_i, 2p_i, 2q_i]$

2° $C_j \longrightarrow W'[r_j, s_j, t_j, a_{j,1}, a_{j,2}, a_{j,3}]$

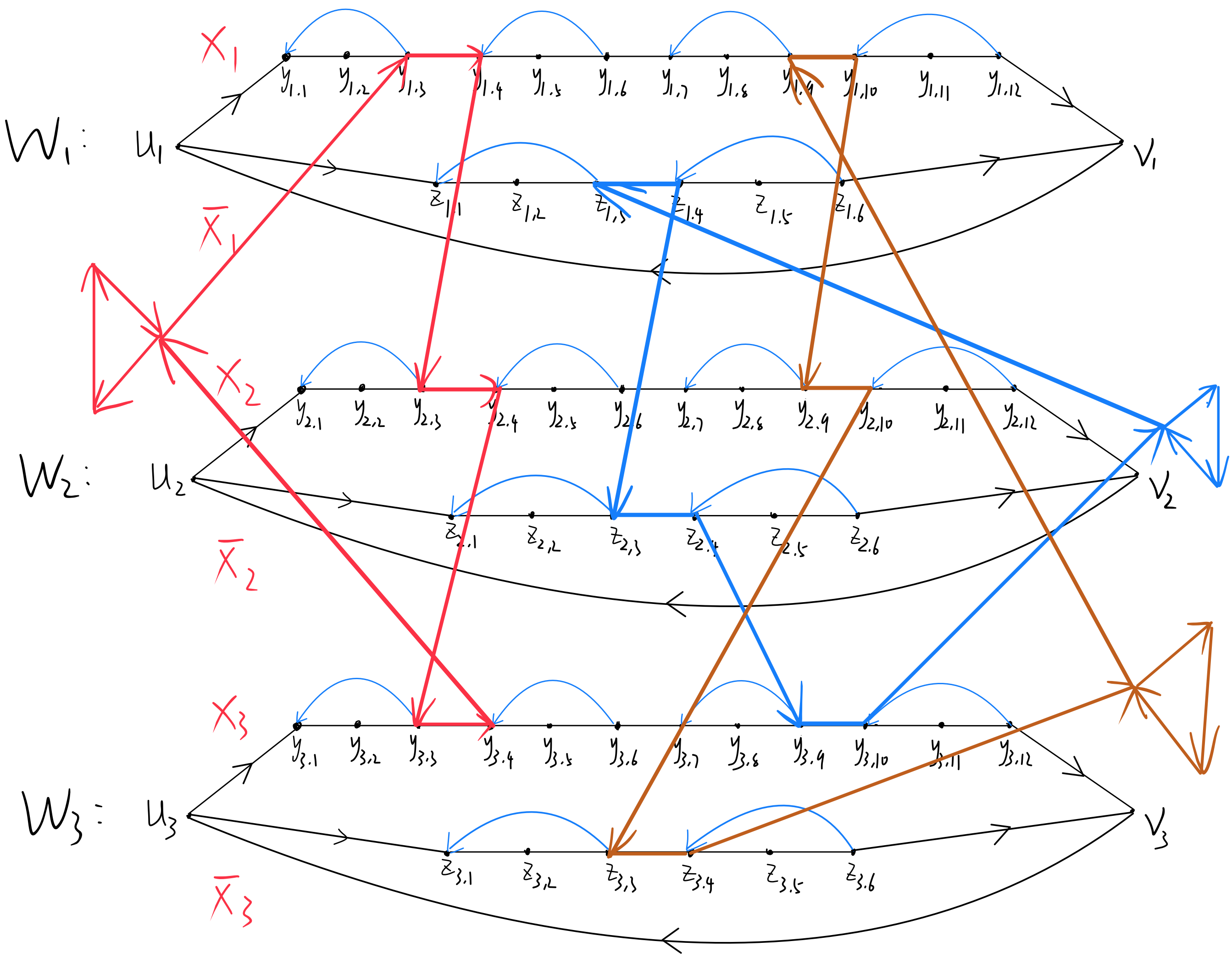
$a_{j,1} \updownarrow x_i$, $a_{j,2} \updownarrow x_k$, $a_{j,3} \updownarrow x_l$



第r个 $\rightarrow y_{6r-3}\, y_{6r-2}$

第r个 $\rightarrow z_{6r-3}\, z_{6r-2}$

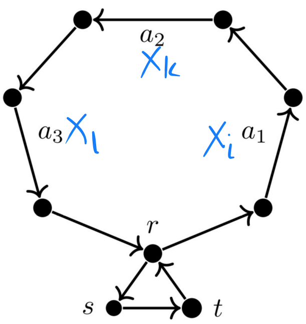
$$(x_1, x_2, x_3) \wedge (\bar{x}_1, \bar{x}_2, x_3) \wedge (x_1, x_2, \bar{x}_3)$$



Claim: I is satisfiable $\Longleftrightarrow$ D_I admit a (bipartite, cycle factor)-arc-partition.

$\Longleftarrow$: define $\begin{cases} x_i \text{ true}, & u_i y_{i,1} \in A_2 \\ x_i \text{ false}, & u_i z_{i,1} \in A_2 . \end{cases}$

W_j':



C_j 变量 x_i, x_k, x_l

$x_i \updownarrow a_{i,1}$

$\Longrightarrow$: 若x_i为真，令 $E_i := \begin{cases} \text{圈 } u y_1 y_2 \cdots y_{6p_i} v \\ \text{圈 } z_{i,3k-2} z_{i,3k-1} z_{i,3k}, \quad 1 \le k \le 2q_i \end{cases}$

若x_i为假，令 $E_i := \begin{cases} \text{圈 } u z_1 z_2 \cdots z_{6q_i} v \\ \text{圈 } y_{i,3k-2} y_{i,3k-1} y_{i,3k}, \quad 1 \le k \le 2p_i \end{cases}$

对 $j \in [m]$，令 $\tilde{E}_j := \{ \text{圈 } r_j s_j t_j$

令 $A_2 := \left(\bigcup_{i \in [n]} E_i \right) \cup \left(\bigcup_{j \in [m]} \tilde{E}_j \right)$

Problems

Other properties: being a spanning bipartite subdigraph

having a perfect matching

being planar

containing a cycle factor

collection of disjoint cycles

having no odd directed cycle

Problem 12. What is the complexity of the following arc-partition problems?

- The (*cycle factor, having no odd directed cycle*)-arc-partition problem
- The (*perfect matching, having no odd directed cycle*)-arc-partition problem
- The (*perfect matching, strong*)-arc-partition problem
- The (*bipartite*, $\Delta^+ \leq k$)-arc-partition problem
- The (*perfect matching, having* B^+)-arc-partition problem