V_1 V_2

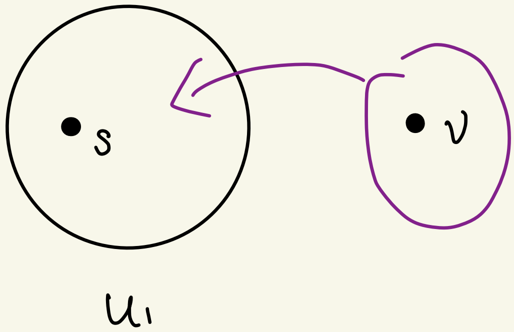
Theorem 3.9. *Deciding whether a digraph has an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition ((in-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition) is polynomial-time solvable.*

Proof.

① at most 1 vertex has no in-neighbor. "s"

② construct U_1 with $|U_1| = k_1$ and $\boxed{s \in U_1}$

$O(n^{k_1})$



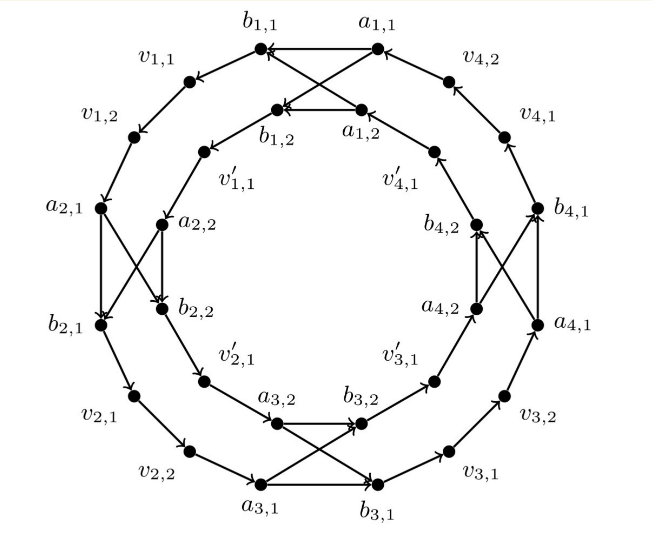
③ add vertices into U_1 $O(n)$

$\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

$\{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots \bar{x}_n\}$

$$V = \{a_{1,1}, \ldots a_{n,1}, a_{1,2} \ldots, a_{n,2}\} \cup$$
$$\{b_{1,1}, \ldots, b_{n,1}, b_{1,2}, \ldots, b_{n,2}\} \cup$$
$$\bigcup_{i=1}^{n} \{v_{i,0}, v_{i,1}, \ldots, v_{i,q_i+1}, v'_{i,0}, v'_{i,1}, \ldots, v'_{i,p_i+1}\}$$

- $\bigcup_{i=1}^{n} \{a_{i,1}b_{i,1}, a_{i,1}b_{i,2}, a_{i,2}b_{i,1}, a_{i,2}b_{i,2}\}$ (the directed complete bipartite graphs)
- the arcs of the paths $P_{i,1}, P_{i,2}$, $i \in [n]$ where $P_{i,1} = b_{i,1}v_{i,0}v_{i,1} \ldots v_{i,q_i+1}a_{i+1,1}$ and $P_{i,2} = b_{i,2}v'_{i,0}v'_{i,1} \ldots v'_{i,p_i+1}a_{i+1,2}$, where $a_{n+1,j} = a_{1,j}$ for $j = 1, 2$.



$\cdot V_{1,?}$
$\cdot V'_{2,2}$
$\cdot V \ldots$

W_j

$x_i = 1.$

a clause C_j $\sim$ a set $W_j = \{v_? \; v_? \; v_?\}$

Theorem 4.1. *Let $\mathcal{F}$ be a 3-SAT formula and let $R(\mathcal{F})$ be the corresponding ring digraph. Then the following holds:*

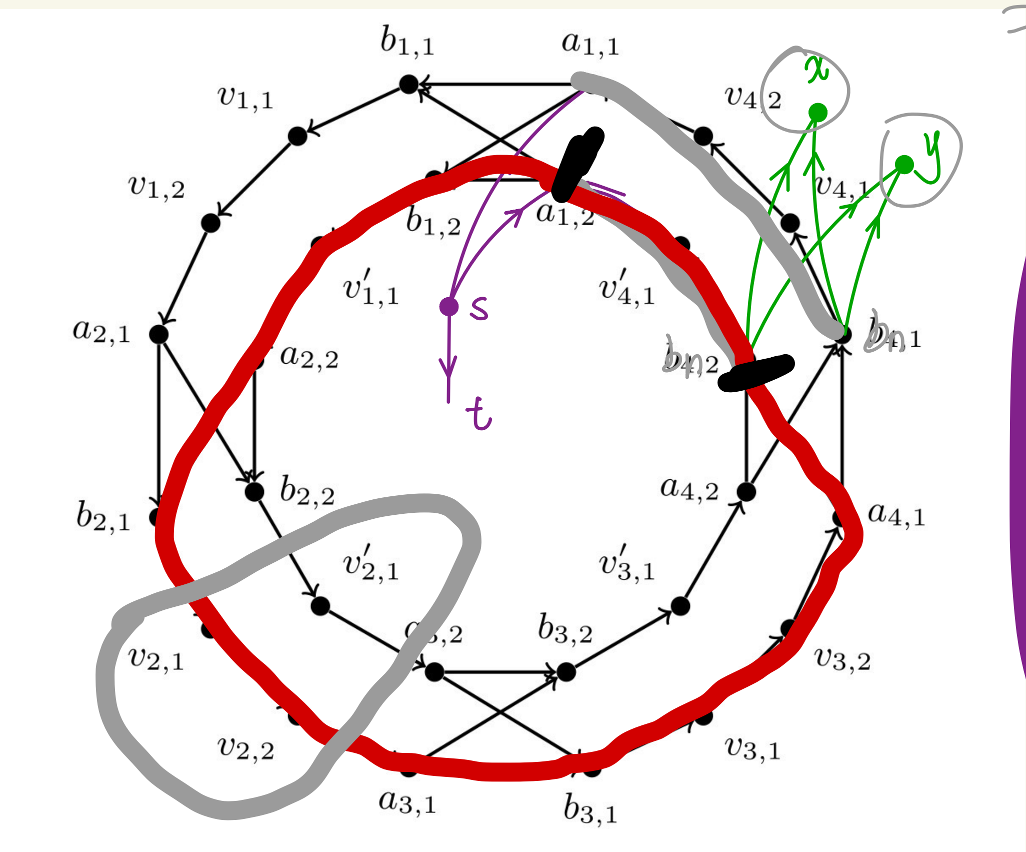
- *$R(\mathcal{F})$ contains a directed cycle which intersects all the sets $W_1, \ldots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of 3-SAT.*
- *$R(\mathcal{F})$ contains two disjoint directed cycles R_1, R_2, each of which intersects all the sets $W_1, \ldots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.*

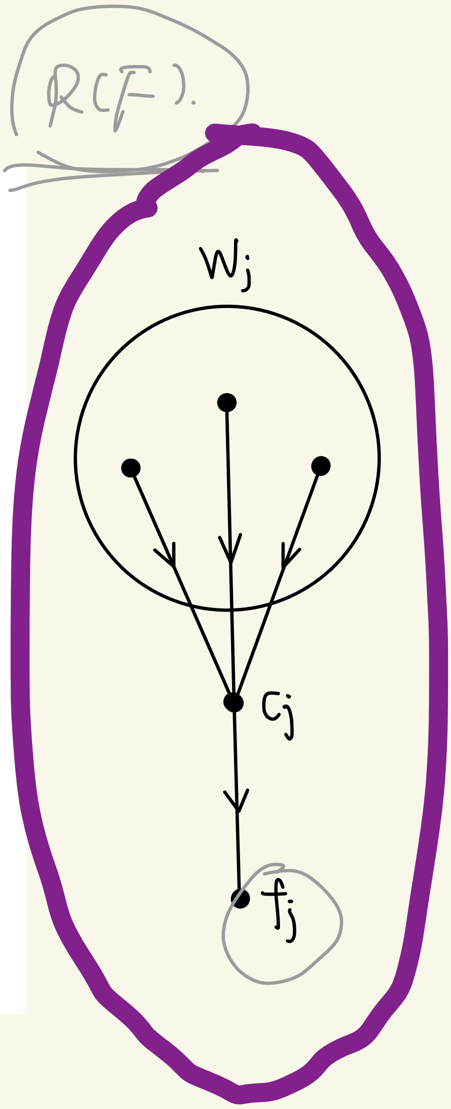
Theorem 4.3. *The following 2-partition problems are NP-complete.*

V_1 V_2

(a) (strong, out-branchable)-$[k_1, k_2]$-partition and (strong, in-branchable)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$,
(b) ($\delta^0 \geq 1$, out-branchable)-$[k_1, k_2]$-partition and ($\delta^0 \geq 1$, in-branchable)-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$.

- Add four vertices s, t, x, y and arcs $\{st, sa_{1,1}, sa_{1,2}, b_{n,1}x, b_{n,1}y, b_{n,2}x, b_{n,2}y\}$.
- For all $j \in [m]$, add a new vertex c_j and all arcs from W_j to c_j.
- For all $j \in [m]$, add a vertex f_j and an arc c_jf_j.





$\boxed{R_2(F)}$ contains such a partition

$\Leftrightarrow$ R(F) contains a cycle intersects all W_j

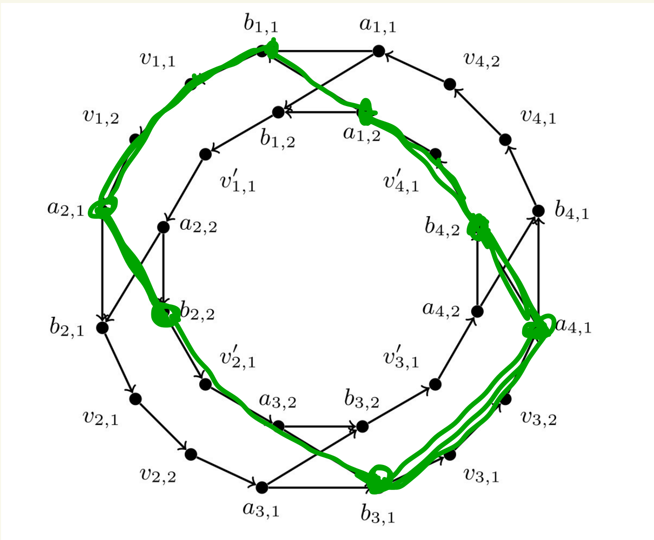
★ $\{s, t, x, y\} \cup \{c_1, \ldots, c_m, f_1, \ldots, f_m\} \in V_2$

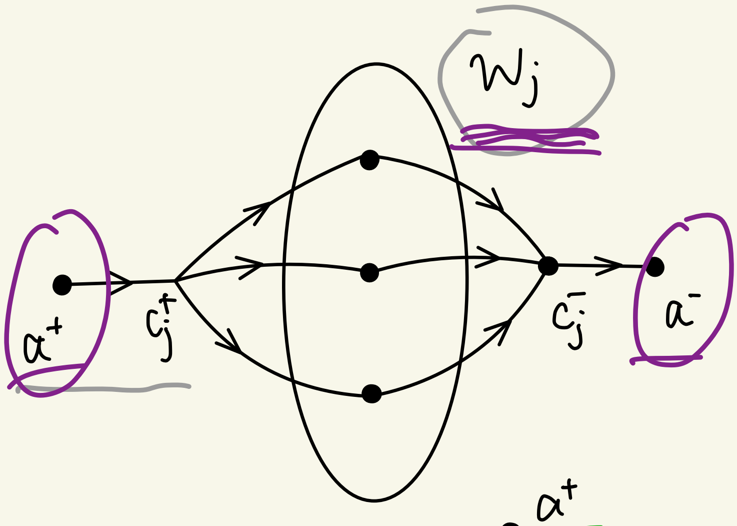
Theorem 4.7. *The following 2-partition problems are NP-complete.*

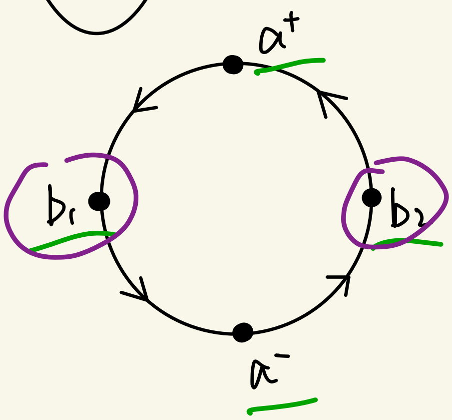
V_1 V_2

(i) $(\delta^+ \geq 1, \delta^- \geq 1)$*-*$[k_1, k_2]$*-partition for* $k_1, k_2 \geq 1$*;*
(ii) (strong, $\delta^+ \geq 1$*)-*$[k_1, k_2]$*-partition and (strong,* $\delta^- \geq 1$*)-*$[k_1, k_2]$*-partition for* $k_1 \geq 2$ *and* $k_2 \geq 1$*;*
(iii) $(\delta^0 \geq 1, \delta^+ \geq 1)$*-*$[k_1, k_2]$*-partition and* $(\delta^0 \geq 1, \delta^- \geq 1)$*-*$[k_1, k_2]$*-partition for* $k_1, k_2 \geq 1$*.*

The digraph $R_6(\mathcal{F})$ is obtained from $R(\mathcal{F})$ by adding new vertices $\{c_1^-, \dots, c_m^-\} \cup \{c_1^+, \dots, c_m^+\} \cup \{a^+, a^-, b_1, b_2\}$ and arcs $(\bigcup_{j \in [m]} \{c_j^+ v \mid v \in W_j\}) \cup (\bigcup_{j \in [m]} \{v c_j^- \mid v \in W_j\}) \cup \{c_j^- a^- \mid j \in [m]\} \cup \{a^+ c_j^+ \mid j \in [m]\} \cup \{a^+ b_1, b_1 a^-, a^- b_2, b_2 a^+\}$.







$R_6(F)$ contains such a partition

$\Leftrightarrow$ $R(F)$ contains a cycle intersects all W_j

☆ $\{a^+, a^-, c_1^+, \dots, c_m^+\} \in V_1$

(or $\{a^+, a^-, c_1^-, \dots, c_m^-\} \in V_2$)