Results and open problems on dichromatic numbers of digraphs and linkages in (di)graphs

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Dichromatic number

The dichromatic number $\vec{\chi}(D)$ of a digraph $D = (V, A)$ is the minimum number of sets in a partition $V_1, \ldots, V_k$ of V into k subsets so that the induced subdigraph $D[V_i]$ is acyclic for each $i \in [k]$.

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This is a generalization of the chromatic number for undirected graphs as a graph has chromatic number at most k if and only if the complete biorientation $\overleftrightarrow{G}$ of G (replace each edge by a directed 2-cycle) has dichromatic number at most k.

The notions of dicolouring and dichromatic number were introduced by Erdős and Neumann-Lara in the late 1970s, rediscovered independently by Mohar in the 2000s, and gained a lot of attention since then.

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(2) It is already NP-complete to decide whether a tournament has dichromatic number 2
(3) Every bipartite digraph has dichromatic number 2.

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This is a refinement of dichromatic number: $\vec{\chi}_{\mathrm{a}}(D) \geq \vec{\chi}(D)$ and it may become much larger.

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Since an acyclic dicolouring is more restrictive than a dicolouring, the complexity may change for some digraphs.

Oriented graphs with high acyclic dichromatic number

For any undirected graph $G = (V, E)$ let $D(G)$ be the oriented graph obtained from $\overleftrightarrow{G}$ by performing the operation called **vertex splitting** on $\overleftrightarrow{G}$ that is, each vertex v is replaced by two vertices v_{in}, v_{out} and the arc $v_{in}v_{out}$ and every arc uv is replaced by the arc $u_{out}v_{in}$.

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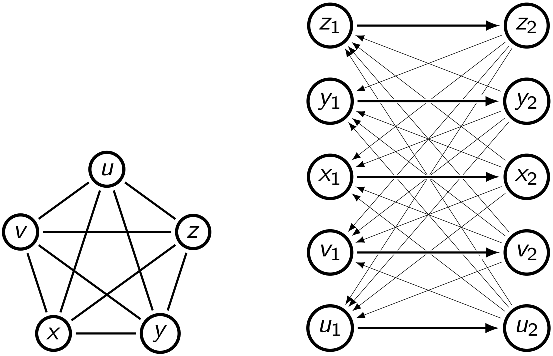
The digraph $D(G)$ is bipartite with bipartition (V_{in}, V_{out}), where $V_{in} = \{v_{in} : v \in V\}$ and $V_{out} = \{v_{out} : v \in V\}$, and the only arcs from V_{in} to V_{out} is the perfect matching $\{v_{in}v_{out} : v \in V\}$.

Observe that every edge uv of G corresponds to the directed 4-cycle $(u_{in}, u_{out}, v_{in}, v_{out})$ and that the only induced cycles of $D(G)$ are 4-cycles of the form above.



(a) (b)

Figure 1: Figure (b) illustrates the digraph $D(G)$ for the graph in Figures (a).

Proposition

For every graph G, $\vec{\chi}_{\mathrm{a}}(D(G)) = \left\lceil \sqrt{\chi(G)} \right\rceil$.

Proof: Let $k = \vec{\chi}_{\mathrm{a}}(D(G))$. We first prove that $k \geq \lceil \sqrt{\chi(G)} \rceil$.

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Let $\phi \colon V(D(G)) \to [k]$ be an acyclic dicolouring of $D(G)$ and assign each vertex $v \in V(G)$ the colour $\psi(v) = (\phi(v_{in}), \phi(v_{out}))$. We claim that ψ is a proper colouring of G.

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If $uv \in E$ is an edge of G such that $\psi(u) = \psi(v)$, then either all vertices of the 4-cycle $(u_{in}, u_{out}, v_{in}, v_{out})$ received the same colour $i \in [k]$ by ϕ or the vertices u_{in}, v_{in} received one colour i under ϕ and u_{out}, v_{out} received another colour j under ϕ.

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In the first case $\phi^{-1}(i)$ is not acyclic and in the second case the 4-cycle $(u_{in}, u_{out}, v_{in}, v_{out})$ alternates between the sets $\phi^{-1}(i)$ and $\phi^{-1}(j)$ and both cases contradict that ϕ is an acyclic dicolouring of $D(G)$.

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Take an arbitrary bijection ρ between the set $\{1, 2, \ldots, \ell\}$ and the set $\{(1,1), \ldots, (1, \sqrt{\ell}), (2,1), \ldots, (2, \sqrt{\ell}), \ldots (\sqrt{\ell}, \sqrt{\ell})\}$, and assign the colours i, j to the vertices v_{in}, v_{out} if $\rho(\psi(v)) = (i, j)$.

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To see that ϕ is an acyclic dicolouring of $D(G)$, it suffices to consider induced directed cycles of $D(G)$ which, by the remark above, are all 4-cycles of the form $C = (u_{in}, u_{out}, v_{in}, v_{out})$ for some pair $u, v \in V$, and such a cycle corresponds to the edge uv of G.

As ψ is a proper colouring of G, it follows from the definition of ϕ above that ϕ cannot assign the same colour to all four vertices of C, so each colour class is indeed acyclic.

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Suppose finally that C alternates between $\phi^{-1}(i)$ and $\phi^{-1}(j)$. Then the definition of ϕ implies that we have $\psi(u) = \psi(v)$, contradicting that ψ is a proper colouring of G.

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Corollary

For every square integer $k = p^2$ there exists a bipartite tournament B with $\vec{\chi}_{\mathrm{a}}(B) = p$.

Proposition

For every integer k there exists a tournament T with $\vec{\chi}(T) = 2$, $\vec{\chi}_{\mathrm{a}}(T) \geq k$ and such that T has a spanning acyclic bipartite subdigraph.

Proof.

Let $B = (V_1, V_2, A)$ be a bipartite tournament with $\vec{\chi}_{\mathrm{a}}(B) \geq k$, (take for instance $D(K_{k^2})$)

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Form the tournament T from two copies B_1, B_2 of B with vertex sets $V_{1,1} \cup V_{1,2}$, $V_{2,1} \cup V_{2,2}$ respectively by adding the arcs of a transitive tournament inside each $V_{i,j}$ and all arcs from $V_{1,1} \cup V_{1,2}$ to $V_{2,1} \cup V_{2,2}$.

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The partition $W_1 = V_{1,1} \cup V_{2,1}$, $W_2 = V_{2,1} \cup V_{2,2}$ shows that $\vec{\chi}(T) = 2$ and the fact that B is a subdigraph of T shows that $\vec{\chi}_{\mathrm{a}}(T) \geq k$. Moreover, T has a spanning acyclic bipartite tournament induced by the arcs between $V_{1,1} \cup V_{1,2}$ and $V_{2,1} \cup V_{2,2}$. □

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Since, for every integer $k \geq 2$, it is NP-complete to decide whether a graph is k^2-colourable, we directly obtain the following.

Corollary

For every integer $k \geq 2$, Acyclic k-Dicolourability *is NP-complete even when restricted to bipartite digraphs.*

A digraph $D = (V, A)$ is a **split digraph** if we can partition its vertices into two sets V_1, V_2 so that $D[V_1]$ has no arcs and $D[V_2]$ is a semicomplete digraph. We call V_1, V_2 a **split partition of** $V(D)$. A split digraph $D = (V, A)$ is **semicomplete** if every vertex in V_1 is adjacent to every vertex in V_2.

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For every fixed $k \geq 3$, Acyclic k-Dicolouring *is NP-complete even when restricted to split digraphs.*

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Our proof is inspired by a proof due to Klingelhoefer and Newman that a 10-dicolouring of a tournament T can be computed in polynomial time when T is guaranteed to have dichromatic number 2.

Precisely, we make use of a decomposition of 2-dicolourable light tournaments that they obtained. A tournament T is **light** if, for every arc $uv \in A(T)$, the subtournament $T[N^+(v) \cap N^-(u)]$ is acyclic.

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Proposition

If $\vec{\chi}_{\mathrm{a}}(T) \leq 2$ then T is light.

Proof:

Assume that $\vec{\chi}_{\mathrm{a}}(T) \leq 2$ and let ϕ be an acyclic 2-dicolouring of T.

Assume for a contradiction that T is not light, so there exists an arc $uv \in A(T)$ such that $T[N^+(v) \cap N^-(u)]$ contains a directed triangle (x, y, z).

Assume first that $\phi(u) = \phi(v)$. Since (x, y, z) is a directed triangle, one of these three vertices, say x, uses colour $\phi(u)$. Therefore, (v, x, u) is a monochromatic directed triangle, a contradiction.

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Assume now, without loss of generality, that $\phi(u) = 1$ and $\phi(v) = 2$. Since ϕ uses both colours 1 and 2 on $\{x, y, z\}$ (otherwise (x, y, z) is monochromatic), there exists an arc, say xy, such that $\phi(x) = 1$ and $\phi(y) = 2$.

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Therefore, (v, x, y, u) is a directed cycle alternating between colours 1 and 2, a contradiction to ϕ being an acyclic dicolouring. $\square$

The cases of more than two colours remain open.

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Acyclic k-Dicolouring, *when restricted to tournaments, is solvable in polynomial time for every fixed* k.

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If the conjecture above is true, then in particular there is a trivial polynomial time algorithm for deciding $\vec{\chi}_{\mathrm{a}}(T) \leq k$, consisting of checking whether T contains one of the $(k+1)$-critical tournaments. The conjecture remains open for every $k \geq 2$.

Degenerate digraphs

A graph G is d-**degenerate** if every non-empty subgraph H of G contains a vertex of degree at most d. A digraph is d-degenerate if its underlying graph is d-degenerate

It is well-known that 2-degenerate graphs have unbounded acyclic chromatic number, while 1-degenerate graphs (*i.e.* forests) trivially admit an acyclic 2-colouring.

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In particular, we derive the following, as outerplanar graphs are 2-degenerate.

Corollary

Every oriented outerplanar digraph D satisfies $\vec{\chi}_{\mathrm{a}}(D) \leq 2$.

Theorem

For every $k \in \mathbb{N}$, there exists a 3-degenerate oriented graph D with $\vec{\chi}_{\mathrm{a}}(D) \geq k$.

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Theorem (Borodin)

Every planar graph admits a proper 5*-colouring such that the union of any two colour classes induces a forest.*

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The same bound directly follows for the acyclic dichromatic number.

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Every planar oriented graph D satisfies $\vec{\chi}_{\mathrm{a}}(D) \leq 5$.

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Every planar oriented graph D satisfies $\vec{\chi}_a(D) \leq 5$.

Proposition

There exists a planar oriented graph D with $\vec{\chi}_a(D) \geq 3$.

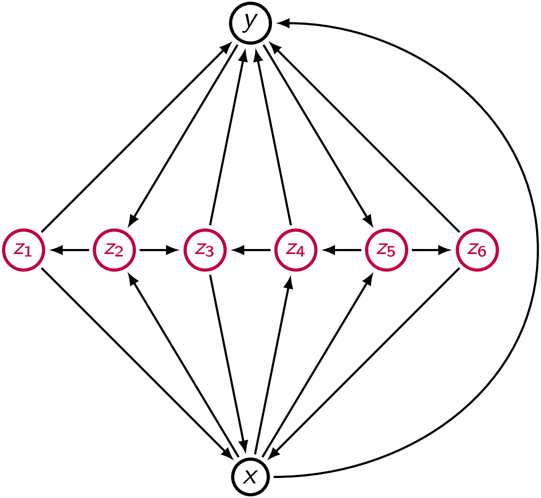

Figure 2: The digraph $W[x, y]$ together with one of its acyclic 2-dicolourings. In all such colourings x and y receive the same colour

Proof.

Consider the digraph D obtained from a directed triangle (u, v, w) by further gluing $W[u, v]$, $W[v, w]$, and $W[w, u]$, where $W[x, y]$ is the planar oriented graph given in Figure 2. The statement follows from the observation that, in every 2-dicolouring of $W[x, y]$, both x and y receive the same colour. □

We do not believe that the bound of Corollary 11 is tight and pose the following.

Conjecture

Every oriented planar graph D satisfies $\vec{\chi}_{\mathrm{a}}(D) \leq 3$.

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A digraph D is a **locally tournament digraph** if $D[N^+[v]]$ and $D[N^-[v]]$ are tournaments for every vertex v of D.

Problem

What is the complexity of deciding $\vec{\chi}_{\mathrm{a}}(D) \leq 2$ when D is a locally tournament digraph?

Note that the definition of light still works for locally tournament digraphs and every local tournament D with $\vec{\chi}_{\mathrm{a}}(D) = 2$ is light.

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This is open already for bipartite tournaments, but polynomial for the special class of multipartite tournaments called **extended tournaments**. These oriented graphs are of the form $D = T[I_{k_1}, I_{k_2}, \ldots, I_{k_t}]$ where T is a tournament and D is obtained from T by substituting an independent set of vertices I_{k_j} for the jth vertex of T for $j \in [|V(T)|]$.

It is easy to check that if D is an extension of T then $\vec{\chi}_{\mathrm{a}}(D) = \vec{\chi}_{\mathrm{a}}(T)$.

Yet another generalization of tournaments is the class of **quasi-transitive digraphs**. These are digraphs containing no induced directed path of length 2.

They have a very useful recursive structure.

Theorem

Let D be a quasi-transitive digraph.

(a) If D is not strongly connected, then there exists a transitive oriented graph T such that $D = T[H_1, H_2, \dots, H_t]$, where each H_i is a strongly connected quasi-transitive digraph.

(b) If D is strongly connected, then there exists a strong semicomplete digraph S such that $D = S[Q_1, Q_2, \dots, Q_s]$ where each Q_i is either a single vertex or a non-strong quasi-transitive digraph.

Note that a strong quasi-transitive digraph $D = S[Q_1, Q_2, \ldots, Q_s]$ contains the semicomplete digraph S as an induced subdigraph (just take an arbitrary vertex from each Q_i).

Thus, $\vec{\chi}_{\mathrm{a}}(D) \leq 2$ only if we also have $\vec{\chi}_{\mathrm{a}}(S) \leq 2$. The other direction does not always hold, e.g. in the case when each Q_i contains a directed cycle. In fact, the following holds.

Theorem

A strong quasi-transitive digraph $D = S[Q_1, Q_2, \ldots, Q_s]$ has $\vec{\chi}_{\mathrm{a}}(D) = 2$ if and only if $\vec{\chi}_{\mathrm{a}}(S) = 2$ and each Q_i is acyclic.

Proof:

Let $D = S[Q_1, Q_2, \ldots, Q_s]$ be a strong quasi-transitive digraph without 2-cycles.

By the remark above, we may assume that $\vec{\chi}_{\mathrm{a}}(S) = 2$.

Suppose first that some Q_i contains a cycle C and that $\vec{\chi}_{\mathrm{a}}(D) = 2$.

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Let ϕ be an acyclic 2-dicolouring of D. Then C contains an arc uv such that $\phi(u) = 1$ and $\phi(v) = 2$.

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Since S is strong and hence vertex pancyclic by Moon's theorem, there are indices j, k distinct from i such that D contains the subdigraph $\vec{C}_3[Q_i, Q_j, Q_k]$.

If both of Q_j, Q_k contain a vertex coloured $a \in \{1,2\}$ by ϕ then D contains a monochromatic triangle so we may assume that ϕ uses one colour a on all vertices of Q_j and the other colour $b \neq a$ on all vertices of Q_k.

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Combining this with the fact that ϕ uses both colours of C we obtain a contradiction to ϕ being an acyclic 2-dicolouring of D.

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Now let ψ be an acyclic 2-dicolouring of (this copy of) S. Then we can extend ψ to an acyclic 2-dicolouring of D by giving all vertices of Q_i the colour $\psi(s_i)$. □

Notice that above we can extend any acyclic 2-dicolouring of S to all of D when each Q_i is acyclic so we obtain the following corollary of Theorem 6.

Corollary

There is a polynomial-time algorithm for checking whether a quasi-transitive digraph D has $\vec{\chi}_{\mathrm{a}}(D) \leq 2$.

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There is a polynomial-time algorithm for checking whether a quasi-transitive digraph D has $\vec{\chi}_{\mathrm{a}}(D) \leq 2$.

Problem

Is there a polynomial-time algorithm for deciding whether a semicomplete digraph $D = (V, A)$ contains a spanning tournament $T = (V, A')$ with $\vec{\chi}_{\mathrm{a}}(T) \leq 2$?

Linkages in digraphs

A (di)graph G is **k-linked** if it has disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for every choice of $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.

k-linkage problem
Input: G and distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G.
Question: Does G contain disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for $i \in [k]$?

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(3) Does there exists a K such that every K-strong digraph D is unilaterally k-linked?

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Question: are we guaranteed one more?

undirected variants

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Let $s_1, s_2, s_3, t_1, t_2, t_3$ be distinct vertices in an undirected graph G.

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5-connected is not enough as there are planar counterexamples.

Tournaments

Let T be a tournament and let $s_1, s_2, s_3, t_1, t_2, t_3$ be distinct vertices of T.

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