

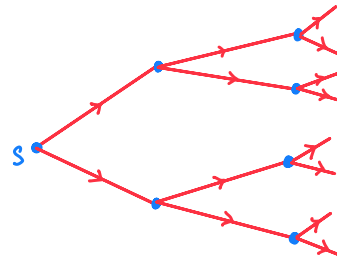
**Theorem 1.5.** It is  $\mathcal{NP}$ -complete to decide for a given digraph  $D = (V, A)$  and a vertex  $s \in V$  whether  $D$  contains an out-branching  $B_s^+$  such that  $UG(D - A(B_s^+))$  is connected.

**Theorem 3.3.** The following problems are all NP-complete

- (i) Given a digraph  $D$  and  $s, t \in V(D)$ ; does  $D$  have an  $(s, t)$ -path  $P$  such that  $D - A(P)$  is connected?
- (ii) Given a digraph  $D$  and  $s, t \in V(D)$ ; does  $UG(D)$  have an  $(s, t)$ -path  $P$  such that  $D - A(P)$  contains an out-branching rooted in  $s$ ?
- (iii) Given a strong digraph  $D$ ; does  $D$  contain a cycle  $C$  such that  $D - A(C)$  is connected?

$UG(D)$ : the underlying graph of  $D$

out-branching  $B_s^+$  a connected spanning subdigraph of  $D=(V, A)$  each vertex  $x \neq s$  has only one arc entering it,  $s$  has no arcs entering it.



3-CNF : a boolean formula express as an AND of clauses , each of which is an OR of exactly 3 distinct literals.

$$\mathcal{F} = (\underbrace{\bar{x}_1 \vee x_2 \vee x_3}_{C_1}) \wedge (\underbrace{x_1 \vee x_2 \vee x_4}_{C_2}) \wedge (\underbrace{\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3}_{C_3})$$

3-SAT . deciding whether a boolean 3-CNF formula  $\mathcal{F}$  is satisfiable.

Not-All-Equal 3-SAT (NAE-3-SAT).  $\mathcal{F}$  and its negation can be satisfied by the same truth assignment  $t$

We have the conclusion:

both 3-SAT and NAE-3-SAT are NP-complete problems

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Proof: the clause gadget  $H(r)$ :

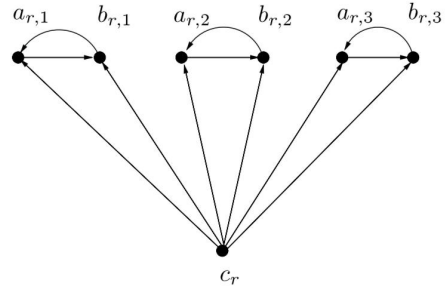


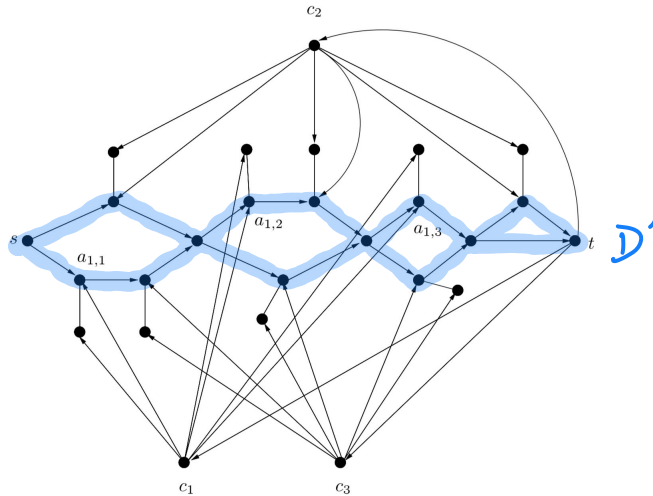
Fig. 1. The clause gadget  $H(r)$ .

$W[u, v, p, q]$ : vertices  $\{u, v, y_1, \dots, y_p, z_1, \dots, z_q\}$

two  $(u, v)$ -paths  $u, y_1, y_2, \dots, y_p, v$ ,  $u, z_1, z_2, \dots, z_q, v$

$p+q \geq 1$

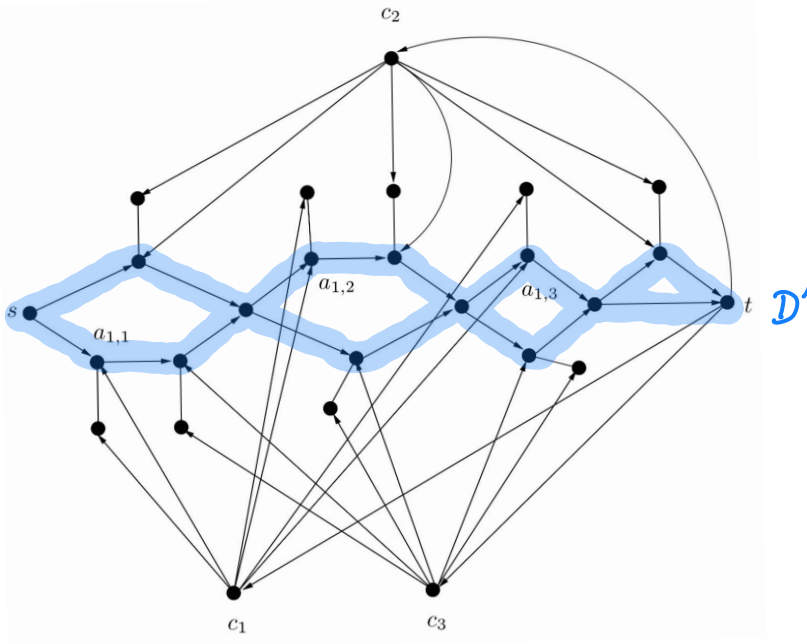
# The Construction of $D_{\mathcal{F}}$



$\mathcal{F}$  has variables  $x_1, x_2, x_3, x_4$  and clauses  $C_1 = (\bar{x}_1 \vee x_2 \vee x_3)$ ,  $C_2 = (x_1 \vee x_2 \vee x_4)$ ,  $C_3 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$D'$  induced by the union of all vertices from  $W[u_i, v_i, p_i, q_i]$ ,  $i \in [n]$ .

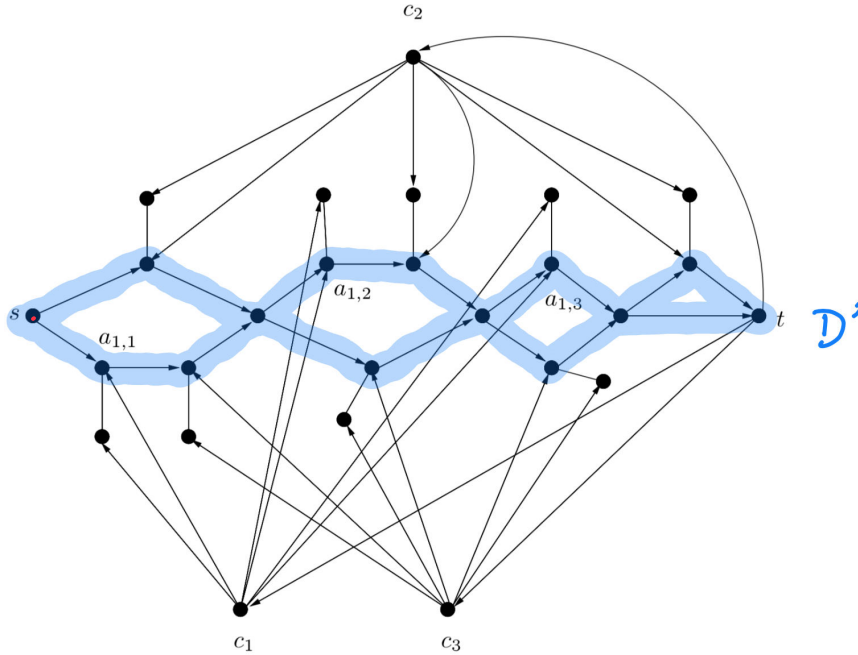
**Claim 1.**  $D'$  contains an  $(s, t)$ -path  $P$  which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$  if and only if  $\mathcal{F}$  is satisfiable.



$$\mathcal{F} = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

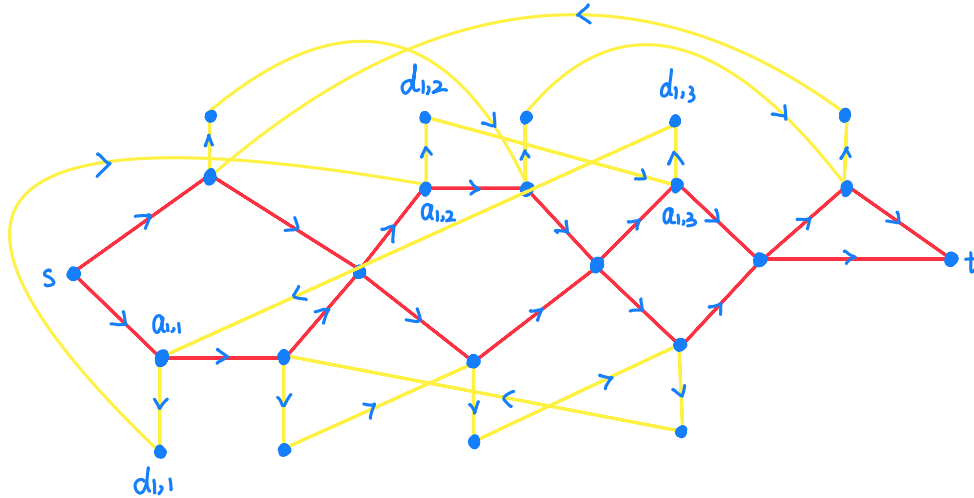
$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ C_1 & C_2 & C_3 \end{array}$$

**Claim 2.**  $D_{\mathcal{F}}$  has an out-branching  $B_s^+$  such that  $D_{\mathcal{F}} - A(B_s^+)$  is connected if and only if  $D'$  contains an  $(s, t)$ -path  $P$  which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$ .



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