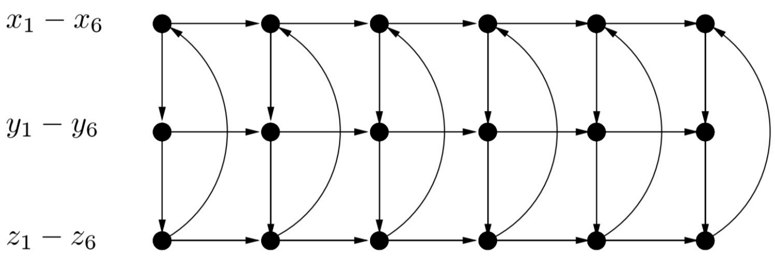
Theorem 2.1. *It is $\mathcal{NP}$-complete to decide whether a 2-regular digraph D contains a pair of arc-disjoint hamiltonian cycles.*

We will reduce the Not-All-Equal 3-SAT problem to this problem.

I 'yes' $\Longleftrightarrow$ D_I 'yes'

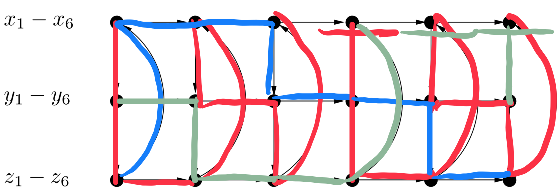
$H(x,y,z)$

$$\begin{aligned} V(H(x,y,z)) &= \{x_i, y_i, z_i : i = 1,2,3,4,5,6\}, \\ A(H(x,y,z)) &= \{x_iy_i, y_iz_i, z_ix_i : i = 1,2,3,4,5,6\} \cup \\ &\quad \{x_jx_{j+1}, y_jy_{j+1}, z_jz_{j+1} : j = 1,2,3,4,5\} \end{aligned}$$



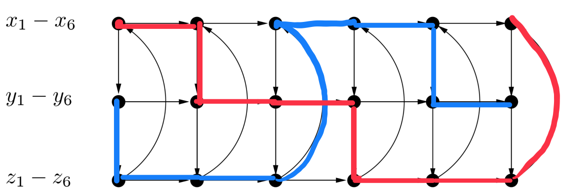
- H-path P start at $X_1 \Longrightarrow P$ end at X_6. $Q: y_1 \to y_6$

  $H(x,y,z) - A(P)$ has a unique 2-path factor $Q \cup R$. $R: z_1 \to z_6$



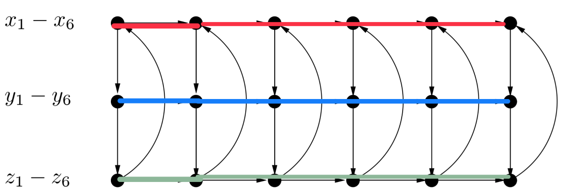
- 2-path factor $P \cup Q$ start at X_1, y_1, end in $\{X_6, y_6, z_6\} \Longrightarrow P, Q$ end at X_6, y_6.

  $H(x,y,z) - A(P \cup Q)$ is a H-path $z_1 \longrightarrow z_6$



- 3-path factor $P \cup Q \cup R$ start at $X_1, y_1, z_1 \Longrightarrow$ end at X_6, y_6, z_6.

  $H(x,y,z) - A(P \cup Q \cup R)$ 6 vertex-disjoint 3-cycles, no arcs between them.



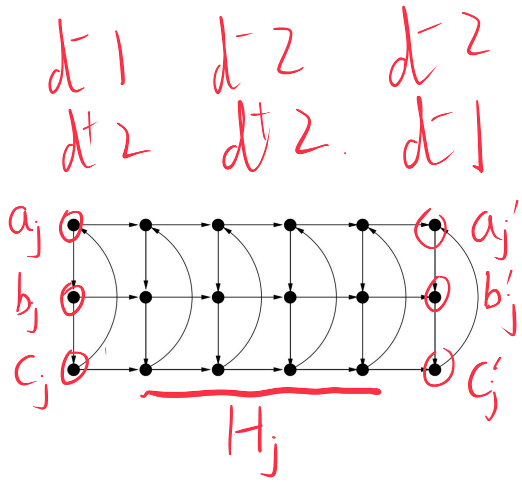
(k-path factor: spanning subdigraph consist of k disjoint path)

Let ℓ with variables $v_1,\cdots,v_k$ and clauses $C_1,\cdots,C_p$.

Assume ℓ contains v_i and $\bar{v}_i$, $i\in[k]$.

Construct D_ℓ:

1° $\forall$ clause C_j, let $H_j = H(a_j, a_j', b_j, b_j', c_j, c_j')$



2° $\forall$ varible v, $C_1,\cdots,C_p \Rightarrow C_{v,1},\cdots,C_{v,p_v}$

$H_{v,1},\cdots,H_{v,p_v}$

$\bar{v}$ $H_{\bar{v},1},\cdots,H_{\bar{v},q_{\bar{v}}}$

$1\le r\le p_v-1$, add an arc from $H_{v,r}$ to $H_{v,r+1}$

$1\le r\le q_{\bar{v}}-1$, $H_{\bar{v},r}$ $H_{\bar{v},r+1}$

for example:

$$v_1\begin{cases} H_{v_1,1} := H_1 \\ H_{v_1,2} := H_2 \\ H_{\bar{v}_1,1} := H_3 \end{cases}$$

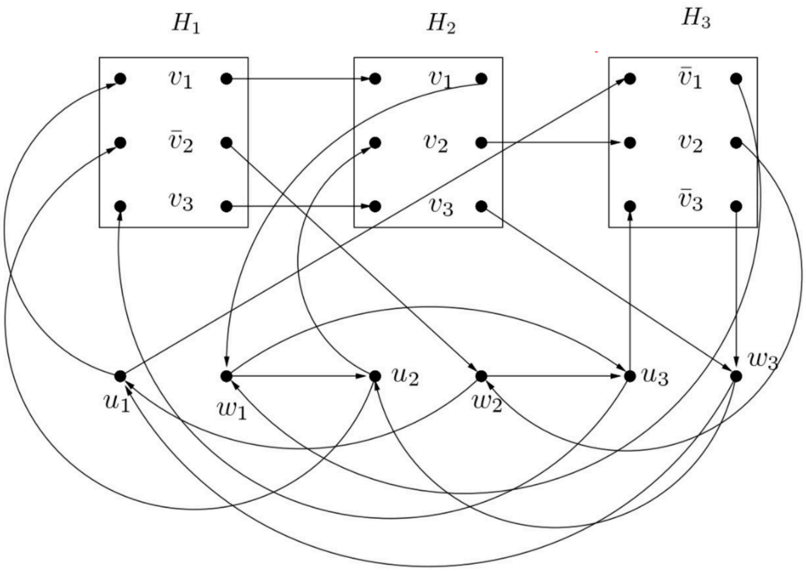


Fig. 4. An illustration of the digraph D_ℓ for the formula $\ell = (v_1 \vee \bar{v}_2 \vee v_3)(v_1 \vee v_2 \vee v_3)(\bar{v}_1 \vee v_2 \vee \bar{v}_3)$. For convenience only the six important vertices of each H_i is shown and in the middle column of each H_i we show the 3 literals in the order they appear in C_i. Thus the top literal corresponds to the two top vertices etc.

3° $\forall$ varible v_i, add u_i, w_i.

$u_i \to H_{v_i,1}$, $u_i \to H_{\bar{v}_i,1}$ $\quad$ $H_{v_i,p_{v_i}} \to w_i$, $H_{\bar{v}_i,q_{\bar{v}_i}} \to w_i$

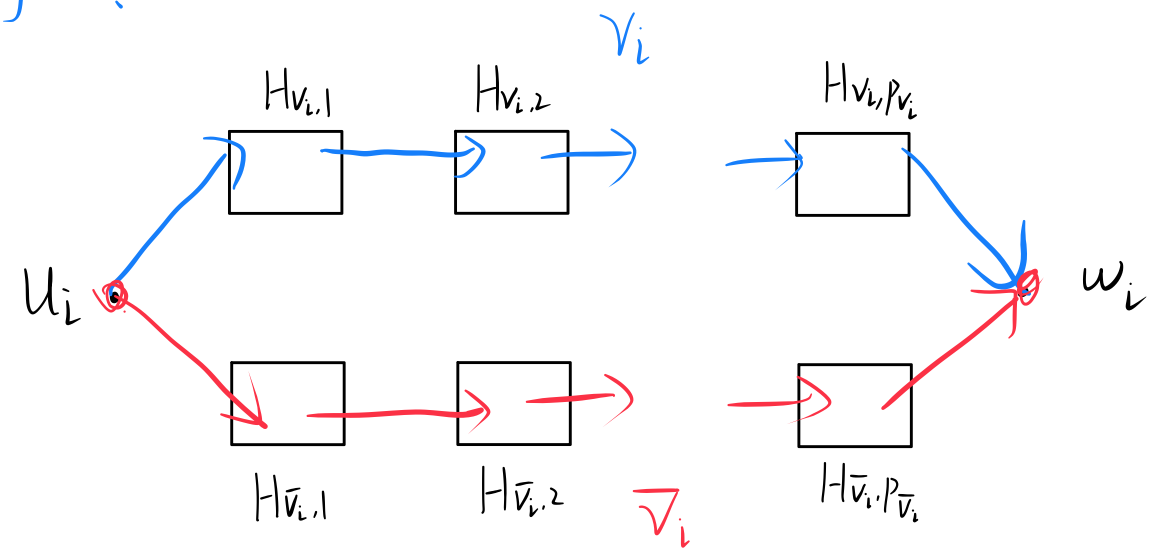
4° $\forall i\in[k]$, add (w_i,u_{i-1}), (w_i,u_{i+1})

d^+2 $\quad$ d^-2

$d^-\phi 2$ $\quad$ $d^+\phi 2$

D_l is 2-regular.

Arcs added in 2° and 3°:



Claim: l has a satisfying truth assignment t if and only if D_l has a pair of arc-disjoint hamiltonian cycle C, C'.

$\Rightarrow$:

For each varible v_i

(u_i, w_i) path through $H_{v_i,1}$

v_i is true: let C contain the arcs corresponding to v_i

C' $\bar{v}_i$

(u_i, w_i) path through $H_{\bar{v}_i,1}$

v_i is false: let C contain the arcs corresponding to $\bar{v}_i$

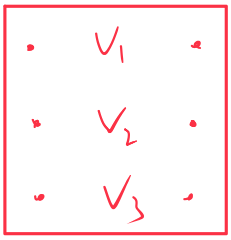
C' v_i

Then let C contain the arc (w_i, u_{i+1})

C' (w_i, u_{i-1})

In fact, C choose the path corresponding to true literal,

C' false

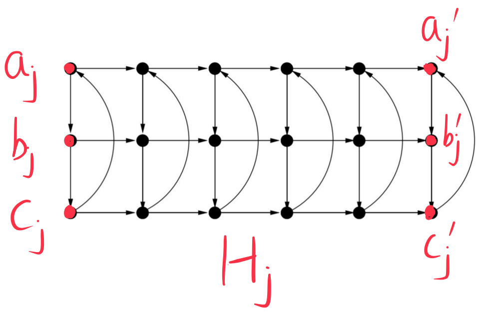


C_j has s true literal, $3-s$ false literal, $1 \le s \le 2$

C choose s-path factor, C' choose $(3-s)$-path factor

C, C' are closed trail $\xRightarrow{\text{Properties of } H(x,y,z)}$ C, C' are arc-disjoint H-cycles

$\Leftarrow$: D_ℓ has a pair of arc-disjoint hamiltonian cycles C, C'.



C and C' through H_j at least once,
at most twice

If C use the arc from u_i to $H_{v_i,1}$, set v_i true.

u_i to $H_{\bar{v}_i,1}$, set v_i false.

A literal is true $\Longleftrightarrow$ C uses the arcs of D_ℓ that correspond to this literal

So we get truth assignment t.