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Digraphs: Theory and Algorithms

[0] This thesis is handed in for partial fullfillment of the requirements for obtaining a danish doctoral degree in natural sciences.

Contents

1. Introduction

The purpose of this survey is to give an overview of some of my most important results obtained in my research on directed graphs. Together with the enclosures (1), (61) and (62) this forms my thesis handed in for partial fulfillment of the conditions for obtaining the Danish doctoral degree in natural sciences. Since most of the results are described and explained in detail in (1) I have chosen not to use a Theorem, Lemma, Proposition style presentation, but rather explain in words the flavour of my results.

I am very interested in algorithmics, theoretical computer science and mathematics and this is reflected in my research. My main research interest are:

1. Directed graphs in general. The main enclosure for the thesis is the monograph (1) which gives a comprehensive account of both algorithmic and theoretical results on directed graphs.
2. The study of the structure and computational complexity of problems related to paths, cycles and connectivity in graphs.
3. Connectivity in graphs and digraphs. In particular problems related to increasing the connectivity by adding or reversing arcs in an optimal way. I have done a considerable amount of work on this type of problems especially for undirected graphs. However, as this thesis concentrates on my work on directed graphs, I will not discuss the results for undirected graphs here but just refer to my list of publications (enclosed as Chapter 14).
4. Various classes of graphs and digraphs (convex-round or concave-round graphs, locally semicomplete digraphs, quasi-transitive digraphs, ...). Perhaps my main contribution to the area of digraphs has been to show that there are many classes of digraphs which are much larger than the class of tournaments for which one can still find a lot of structure, enabling one to solve problems such as the hamiltonian cycle problem efficiently.
5. I have also done a significant amount of work on the so-called H-colouring problem (see Chapter 9). Here one studies the computational complexity of a generalization of the well-known colouring problem.

I would characterize my research as deep, rather than sketchy. Instead of trying to cover many different areas by spreading my research into lots of different directions, I have chosen to spend my efforts on a deep and penetrating understanding of the structure of certain difficult problems (such as paths and cycle in digraphs and edge-connectivity in graphs and digraphs). The reward for this has been several highly non-trivial results and polynomial algorithms for problems that are very complex e.g. the polynomial algorithm for the hamiltonian cycle problem in semicomplete multipartite digraphs in (44), hamiltonian cycles avoiding prescribed arcs in tournaments (39), connectivity of tournaments (61, 62) and the partition constrained augmentation results in (51). Except for the last result which deals with undirected graphs, all these results are outlined in my book (1) and will be discussed in this survey.

The largest single part of my research has been devoted to digraphs, in particular to generalizations of tournaments. My work in this area has stimulated several researchers to work in this area and two entire Ph.D theses [32, 56] have been written on the class of locally semicomplete digraphs which I introduced in 1989. Furthermore, another Ph.D thesis [86] has been written on the class of in-tournament digraphs which I introduced together with Huang and Prisner in (19) (see Chapter 4). I feel that through my work and that by other researchers in the area (stimulated by my work) a significant amount of interesting information has been added to the knowledge about directed graphs. We have shown that the class of tournaments is very far from being the only non-trivial class of digraphs with a lot of structure and for which polynomial algorithms are possible for problems which are NP-complete for general digraphs. Through our investigations we have contributed new methods to the area that may prove useful in other areas too. As an example of this, the reader may consult subsection 11.1, where we show that using the ideas from our generalizations of tournaments, one can obtain a new and more natural sufficient condition for hamiltonicity in directed graphs. Another example is my extensive work with Gutin on generalizations of quasi-transitive digraphs, the so-called totally Φ-decomposable digraphs (see section 6.2). Here our method for solving the hamiltonian cycle problem is based on the interplay between the path and cycle structure of various subclasses of digraphs and a non-trivial application of network flow theory. In the paper (61) with Huang and Yeo we develop a method for finding a small (measured in number of arcs) spanning k-arc-strong digraph of a k-arc-strong tournament. This technique turns out to be useful for other problems on tournaments. In (62) with Yeo we use the method to prove a conjecture (by myself and Gutin) on arc-disjoint in- and out-branchings in tournaments with a high arc-strong connectivity.

Because of my interest in mathematics, I often study the mathematical structure of the problems I am considering. From a mathematical point of view it is nice to have a characterization for the problem one considers (preferably a good one in the sense of Edmonds) whereas, from a computational point of view, having a polynomial algorithm is the most important. We may also consider a polynomial algorithm as a good characterization and hence such an algorithm is also of interest from a mathematical point of view. However, it is very often the case that it is impossible to find a polynomial algorithm for the problem at hand, without first obtaining a detailed understanding of the mathematical structure of the problem (for example the structure of paths and cycles in certain classes of graphs). As an illustration of this it took us three years of research to find enough structure of cycles in semicomplete multipartite digraphs to allow us to find a polynomial algorithm for the hamiltonian cycle problem (44).

In the sections to follow I will outline my own contributions within specific subclasses of directed graphs. Except for the two new papers (61) and (62) all these are mentioned in my book. Due to the way the book is organized (focusing on topics rather than classes of digraphs) I will not go through the results contributed by me chapter by chapter, but using the comprehensive index which I added to the book, it is easy to locate the corresponding topics in the book. Preliminary versions of the papers (3,4,6,9,10,13) formed my Ph.D thesis (2). In the description below I will still mention part of these results since they are relevant for later work extending these.

Except for standard graph theoretic terminology, the relevant terminology is either introduced in the text, or it can be found in section 13. In referring to papers I use the notation that (i) refers to the i'th paper in my list of publications (see Chapter 14), while [i] refers to the i'th reference in the list of additional references.

I would like to thank all my coauthors for many inspiring discussions and stimulating collaborations, in particular I wish to thank Gregory Gutin, Jing Huang and Anders Yeo for countless discussions on directed graphs and Gregory also for smooth collaboration on the book which forms the main enclosure to this thesis.

Most importantly, I wish to thank my wife Lene and my children, Louise, Jonas and Victoria, without whose constant support I would never have gotten this far in science.

Odense, Denmark Jørgen Bang-Jensen

April 2001

2. Digraphs: Theory, Algorithms and Applications

My book (1) forms the main enclosure to this thesis as it documents most of my research on digraphs. Below I will give a short account for our reasons to write the book and outline the structure of the book.

The theory of graphs can be roughly partitioned into two branches: the areas of undirected graphs and directed graphs (digraphs). Even though both areas have numerous important applications, for various reasons, undirected graphs have been studied much more extensively than directed graphs. One of the reasons is that undirected graphs form in a sense a special class of directed graphs (symmetric digraphs) and hence problems that can be formulated for both directed and undirected graphs are often easier for the latter. Another reason is that, unlike for the case of undirected graphs, for which there are several important books covering both classical and recent results, no previous book covers more than a small fraction of the results obtained on digraphs within the last 25 years. Typically, digraphs are considered only in one chapter or by a few elementary results scattered throughout the book.

Despite all this, the theory of directed graphs has developed enormously within the last three decades. There is an extensive literature on digraphs (more than 3000 papers). Many of these papers contain, not only interesting theoretical results, but also important algorithms as well as applications. This clearly indicates a real necessity for a book, covering not only the basics on digraphs, but also deeper, theoretical as well as algorithmic, results and applications.

Our book (1) is an attempt to fill this huge gap in the literature and may be considered as a handbook on the subject. It starts at a level that can be understood by readers with only a basic knowledge in university mathematics and goes all the way up to the latest research results in several areas (including connectivity, orientations of graphs, submodular flows, paths and cycles in digraphs, generalizations of tournaments and generalizations of digraphs). Even though the book should not be seen as an encyclopedia on directed graphs, we included as many interesting results as possible. The book also contains a considerable number of proofs, illustrating various approaches and techniques used in digraph theory and algorithms.

One of the main features of this book is the strong emphasis on algorithms. Many excellent books on graph theory made the error of not focusing enough (or not at all!) on algorithmic aspects. To me algorithmic aspects are the most important and this by no means excludes theoretical results as being interesting since these are often the prerequisites for designing efficient algorithms.

Algorithms on (directed) graphs often play an important role in problems arising in several areas, including computer science and operations research. Secondly, many problems on (directed) graphs are inherently algorithmic. Hence, whenever possible we give constructive proofs of the results in the book. From these proofs one can very often extract an efficient algorithm for the problem studied.

Highlights

We cover the majority of the important topics on digraphs ranging from quite elementary to very advanced ones. Below I give a brief outline of some of the main highlights of this book. I will not try to cover all my own results in the short overview, since this is done in the later sections of this survey.

Chapter 1 gives a unified account of the terminology and notation on directed graphs (as we propose it to be) as well several basic results. These results are not only used frequently in other chapters, but also serve as illustrations of digraph concepts. Furthermore, several applications of directed graphs are based on these elementary results. One such application is provided in the last section of the chapter. Basic concepts on algorithms and complexity can also be found in the chapter. In fact, that chapter alone could form the basis of an elementary course on (di)graphs.

Chapters 2 and 3 cover distances and flows in networks. Although the basic concepts of these two topics are elementary, both theoretical and algorithmic aspects of distances in digraphs as well as flows in networks are of great importance, due to their high applicability to other problems on digraphs and large number of practical applications, in particular, as a powerful modeling tool.

We start with the shortest path problem and a collection of classical algorithms for distances in weighted and unweighted digraphs. The main part of Chapter 2 is devoted to minimization and maximization of distance parameters in digraphs. We conclude the chapter by the following applications: the one-way street problem, the gossip problem and exponential neighbourhood local search, a new approach to find near optimal solutions to combinatorial optimization problems.

In Chapter 3 we cover basic, as well as some more advanced topics on flows in networks. These include several algorithms for the maximum flow problem, feasible flows and circulations, minimum cost flows in networks and applications of flows. We also illustrate the primal-dual algorithm approach for linear programming by applying it to the transportation problem. Although there are several comprehensive books on flows, we believe that our fairly short and yet quite detailed account of the topic will give the majority of readers sufficient knowledge of the area. Furthermore, our account focusses more on applications to graphtheoretic problems than any other text.

Chapter 4 is devoted to describing several important classes of directed graphs including many classes that were pionered by me alone or with coauthors. We concentrate on characterization, recognition and decomposition of these classes. Many properties of these classes are studied in more detail in the rest of the book.

In Chapter 5 we give a detailed account of results concerning the existence of hamiltonian paths and cycles in digraphs as well as some extensions to spanning collections of paths and cycles. Many results of this chapter deal with generalizations of tournaments and hence document my work as mentioned later. We illustrate that several of these much larger classes of digraphs share various nice properties with tournaments. In particular the hamiltonian path and cycle problems are polynomially solvable for most of these classes. The chapter illustrates various methods (such as the multi-insertion technique) for proving hamiltonicity.

In Chapter 6 we describe a number of interesting topics related to hamiltonicity. These include hamiltonian paths with prescribed end-vertices, pancyclicity, orientations of hamiltonian paths and cycles in tournaments and the problem of finding a strong spanning subdigraph of minimum size in a strong digraph. We cover one of the main ingredients in a recent proof by Havet and Thomassé of Rosenfeld's conjecture on orientations of hamiltonian paths in tournaments and outline a polynomial algorithm for finding a hamiltonian path with prescribed end-vertices in a tournament. We discuss the so-called minimum strong spanning subdigraph problem and describe some results obtained by me with coauthors on the problem. We conclude the chapter with a brief introduction of a new approach to approximation algorithms, domination analysis. We illustrate this approach by applying results on hamiltonian cycles in digraphs to the travelling salesman problem.

Connectivity in (di)graphs is a very important topic. It contains numerous deep and beautiful results and has applications to other areas of graph theory and mathematics in general. We give a comprehensive account of connectivity topics in Chapters 7, 8 and 9 which deal with global connectivity issues, orientations of graphs and local connectivities, respectively.

Chapter 7 starts from basic topics such as ear-decompositions and the fundamental Menger's theorem and then moves on to advanced topics such as connectivity augmentation, properties of minimally k-(arc)-strong digraphs, highly connected orientations of digraphs and packing directed cuts in digraphs. We describe the splitting technique due to Mader and Lovász and illustrate its usefulness by giving an algorithm, due to Frank, for finding a minimum cardinality set of new arcs whose addition to a digraph D increases its arc-strong connectivity to a prescribed number. We illustrate a recent application due to Cheriyan and Thurimella of Mader's results on minimally k-(arc)-strong digraphs to the problem of finding a small certificate for k-(arc)-strong connectivity. Many of the proofs in the chapter illustrate the important proof technique based on the submodularity of degree functions in digraphs.

Chapter 8 covers important aspects of orientations of undirected and mixed graphs. These include underlying graphs of certain classes of digraphs. Nowhere zero integer flows, a special case of flows, related to (edge-)colourings of undirected graphs is discussed along with Tutte's 5-flow conjecture, which is one of the main open problems in graph theory. The famous theorem by Nash-Williams on orientations preserving a high degree of arc-strong connectivity is described and the weak version dealing with uniform arc-strong connectivities is proved using splitting techniques. Submodular flows form a powerful generalization of circulations in networks. We introduce submodular flows and illustrate how to use this tool to obtain (algorithmic) proofs of many important results in graph theory (including the Lucchesi-Younger theorem and the uniform version of Nash-Williams' orientation theorem). Finally we describe in detail an application, due to Frank, of submodular flows to the problem of orienting a mixed graph in order to maintain a prescribed degree of arc-strong connectivity.

Chapter 9 deals with problems concerning (arc-)disjoint paths and trees in digraphs. We prove that the 2-path problem is $\mathcal{NP}$-complete for arbitrary digraphs, but polynomially solvable for acyclic digraphs. Linkings in planar digraphs, eulerian digraphs as well as several generalizations of tournaments are discussed. Edmonds' theorem on arc-disjoint branchings is proved and several applications of this important result are described. The problem of finding a minimum cost out-branching in a weighted digraph generalizes the minimum spanning tree problem. We describe an extension, due to Frank, of Fulkerson's two-phase greedy algorithm for finding such a branching.

Chapter 10 describes results on (generally) non-hamiltonian cycles in digraphs. We cover cycle spaces, polynomial algorithms to find paths and cycles of 'logarithmic' length, disjoint cycles and feedback sets, including a scheme of a solution of Younger's conjecture by Reed, Robertson, Seymour and Thomas, applications of cycles in digraphs to Markov chains and the even cycle problem, including Thomassen's even cycle theorem. We also cover short cycles in multipartite tournaments, the girth of a digraph, chords of cycles and Ádám's conjecture. The chapter features various proof techniques including several algebraic, algorithmic, combinatorial and probabilistic methods.

Chapter 11 deals with generalizations of digraphs. Digraphs may be generalized in at least two different ways, by considering edge-coloured graphs or by considering directed hypergraphs. Alternating cycles in 2-edge-coloured graphs generalize the concept of cycles in bipartite digraphs. Certain results on cycles in bipartite digraphs, such as the characterization of hamiltonian bipartite tournaments, are special cases of results for edge-coloured complete graphs. There are useful implications in the other direction as well. In particular, using results on hamiltonian cycles in bipartite tournaments, we prove a characterization of those 2-edge-coloured complete graphs which have an alternating hamiltonian cycle. We describe an application of alternating hamiltonian cycles to a problem in genetics. Generalizations of the classical theorems by Camion, Landau and Rédei to hypertournaments are described.

Chapter 12 contains some topics that were not covered in other chapters. These include: an elementary proof (due to Havet and Thomassé) of Seymour's second neighbourhood conjecture in the case of tournaments, various types of orderings of the vertices of digraphs of paired comparisons, kernels, a recent proof by Galvin of the Dinitz conjecture on list colourings using kernels in digraphs, and homomorphisms (an elegant generalization of colouring and also a useful vehicle for studying the borderline between $\mathcal{P}$ and $\mathcal{NP}$-complete problems). We describe basic concepts on matroids as well as questions related to the efficiency of matroid algorithms. We give a brief account on simulated annealing, a broadly applicable meta-heuristic which can be used to obtain near optimal solutions to optimization problems, in particular, on digraphs. We discuss briefly how to implement and tune simulated annealing algorithms so that they may produce good solutions.

We have also included more than 700 exercises thus enabling students and researchers to check their understanding of the topics. The book is first and foremost a monograph which documents the state of the art of directed graphs. In that sense the book also serves to document my influence on the area (of course one should not deny that personal taste influenced our choice for what to include, but I think every reader will see that we covered much more than just our own work). Being as comprehensive as it is, I believe that through the book I will have a strong influence on research on directed graphs in the future. It is also my hope that many people will now start doing research on directed graphs, using our book as the starting point (either by attempting to solve one of the many open problems listed there or using the book to learn enough about a subarea to be able to start discovering new things). The fact that we have presented difficult topics such as submodular flows and connectivity augmentation in a careful way will help more people to use these powerful results. Hence, even on parts of the book that do not document my own research, I will influence future research simply by making difficult results accessible to a larger audience.

In order to speed up the process of writing the book we divided the work so that we both wrote draft versions (based on a commonly decided guideline for which results should be included) of 6 chapters which were then carefully read and commented by the other author in at least two rounds. I wrote the chapters 3,6,7,8,9 and 12. It should be evident that several of the other chapters also document my research (e.g. 4,5 and 11). It was also me who supplied the majority of the 700 exercises, most of which appear in a book for the first time. Undoubtedly, such large collection of exercises will help many new researchers becoming acquainted with the area.

3. Tournaments and semicomplete digraphs

The papers (4,9,10, 12,13, 33, 39, 40, 53, 61, 66, 62) address this topic.

In [17] Edmonds solved the problem of deciding whether a given digraph D has k arc-disjoint out-branchings rooted at some vertex v. His characterization leads to a polynomial algorithm for deciding the existence of such branchings rooted at v. Furthermore, Lovász [69] gave a constructive proof which can be turned into a polynomial algorithm for finding the desired branchings when they exist. On the other hand, Thomassen showed that if u, v are vertices of a digraph D and we ask whether D has an out-branching F_u^+ rooted at u and an in-branching F_v^- rooted at v, which are arc-disjoint, then the problem is NP-complete, even when $u = v$ (see (9)). Thomassen has conjectured that if D has very high arc-connectivity then such branchings always exist for any choice of u and v. So far very little is known about the truth of this conjecture. We only know that the arc-connectivity must be at least 3. In (9) I prove that the problem is polynomially solvable for tournaments, by completely characterizing those tournaments that do not have such branchings. It follows from this characterization that 2-arc-connectivity is enough for tournaments with respect to the conjecture above (a much shorter proof of this result can be found in my recent paper (62) with Yeo). In the case $u = v$ the characterization is very simple and nice. However, in the case $u \neq v$ the characterization is very complicated and the polynomial algorithm uses polynomial algorithms for two well known problems, which are NP-complete for general digraphs. These are the following problems (stated for tournaments):

(P1) Given a strong tournament T and distinct vertices u, v, x, y of T. Decide whether there exist arc-disjoint $(u, v)-, (x, y)-$paths.

(P2) Given a strong tournament T and distinct vertices x, y, z of T. Decide whether there exist arc-disjoint $(x, y)-, (y, z)-$paths.

In (9) both of these problems are completely solved in terms of good mathematical characterizations (Good in the sense of Edmonds). The characterization is simple in the case of problem (P2), but non-trivial in the case of problem (P1). Both of these characterizations are proved constructively and the proofs imply a polynomial algorithm for the problem.

In the very recent paper (62) with Yeo (enclosed as part of this thesis) we prove a generalization of the branching result from (9) to the case when one wants many arc-disjoint in and out-branchings. In fact, we show that every tournament which is k-arc-strong and has minimum in- and out-degree at least $37k$ contains k spanning arc-disjoint strong subdigraphs. This implies that every $74k$-arc-strong tournament contains $2k$ arc-disjoint branchings $F^+_{u_1}, F^+_{u_2} \ldots, F^+_{u_k}, F^-_{v_1}, F^-_{v_2} \ldots, F^-_{v_k}$, where $F^+_{u_i}$ is an out-branching rooted at u_i and $F^-_{v_i}$ is an in-branching rooted at v_i for $i = 1, 2, \ldots, k$ and the roots can be chosen arbitrarily. This proves a conjecture by Gutin and myself from (45). The main tool for proving this result is the powerful method developed in (61).

In (4) the vertex-disjoint path version (P1') of problem (P1) is studied in order to reveal the structure of tournaments without such paths. It turns out that there exist highly non-trivial tournaments which are not 2-linked, that is there exist some 4 distinct vertices u, v, x, y, such that the tournament has no pair of disjoint $(u, v)-, (x, y)-$paths. I describe an infinite family of 4-connected tournaments with this property and show that every 5-connected tournament is 2-linked. I also show that any 3-connected tournament has a cycle through any pair of disjoint arcs. These results indicate that a complete characterization of those tournaments which are 2-linked could be a very difficult task, due to the existence of the highly non-trivial non 2-linked tournaments. However in (13) we use the results in (4) to derive enough structural information to allow for a polynomial algorithm to decide if a given tournament T has disjoint $(u, v)-, (x, y)-$paths for given distinct vertices u, v, x, y of T. The algorithm is quite complicated, but we where fascinated by the fact that it was highly non-trivial to make the algorithm work, even though the underlying idea is a simple divide and conquer approach. In fact, it turns out that there are certain irreducible (and highly non-trivial!) instances where the algorithm cannot proceed with a new reduction. Hence we had to prove constructively that for all those instances the desired paths exist.

In (13) we also describe two problems for general digraphs which remain NP-complete for tournaments. The first is the problem of finding a minimum vertex set whose deletion destroys all cycles in the remaining digraph. This is the so-called Feedback vertex set problem which for tournaments is equivalent to finding the maximum transitive subtournament of a tournament. The second is the problem of deciding the existence of a cycle through a specified arc set in a tournament. On page 313 of (1) we describe a generalization of that result which shows that the hamiltonian cycle problem is very hard even for digraphs that are obtained from a semicomplete digraph D by adding just a few new vertices and joining these to D in some arbitrary way. In particular, if the number of new vertices is not fixed then the problem is NP-complete.

Since every oriented graph is a subdigraph of some tournament it is, of course, possible to formulate a number of artificial $NP-$complete problems involving (subdigraphs of) tournaments. However, our examples are the first examples of natural problems which remain $NP-$complete for tournaments to appear in the literature. The NP-completeness of the maximum transitive subtournament problem was proved independently by Speckenmeyer [85].

In (10) I show that in the vertex-disjoint analogue (P2') of (P2), i.e. the special case of (P1') when $v = x$, a simpler algorithm can be obtained. I also give a construction method for those tournaments which do not have such paths for given vertices x, y and z.

The question of the existence of a polynomial algorithm for hamiltonian connectedness in tournaments was raised by Soroker in [84]. In (12) we use results from [88] as well as an extension of Thomassens result in [88] to derive enough structural information about hamiltonian paths with specified end vertices to allow for such an algorithm. The algorithm (which is outlined in Section 6.4 of (1)) is highly non-trivial, once again due to the very complicated process of fitting different structural analyses together to a complete algorithm which terminates in polynomial time.

In (39) we prove the following result: Let $T = (V, A)$ be a k-strong tournament on n vertices and let $X_1, X_2, \ldots, X_p$ ($p \geq 1$) be a partition of V such that $|X_1| \leq |X_2| \leq \ldots \leq |X_p|$. Let D be the digraph one obtains from T by deleting all the arcs of the subtournament induced by T_i for each $i = 1, 2, \ldots, p$. If $|X_p| \leq n/2$ and $k \geq |X_p| + \sum_{i=1}^{p-1} \lfloor |X_i|/2 \rfloor$ then D has a hamiltonian cycle. In other words T has a hamiltonian cycle which avoids all arcs with both head and tail in some X_i. This implies the result by P. Fraisse and C. Thomassen [20] that every k-strong tournament has a hamiltonian cycle avoiding any set of $k-1$ arcs. Note that in some cases our result allows the deletion of almost k^2 arcs, provided they have the structure described above. Interestingly our result is obtained using a description of a minimal factor in a multipartite tournament by Yeo [97]. The original proof of Fraisse and Thomassen was long and complicated (and even relied on Thomassens result that every 4-strong tournament is hamiltonian-connected [87]) our proof, once we have the above description of a minimal factor in a multipartite tournament, is only a couple of pages. This shows that even very special cases of the hamiltonian cycle problem for multipartite tournaments can be useful to solve. Such applications of even very special instances of the problem also indicate that the hamiltonian cycle problem for multipartite tournaments is highly non-trivial and quite important. As I shall mention in Chapter 5, Gutin, Yeo and myself recently managed to find a polynomial algorithm for the hamiltonian cycle problem for multipartite tournaments. Using this we obtain a polynomial algorithm for the deciding, in a given tournament $T = (V, A)$, the existence of a hamiltonian cycle which avoids all arcs both of whose end vertices belongs to one of the sets $V_1, V_2, \ldots, V_r$, where $V_1, V_2, \ldots, V_r$ are disjoint subsets of V.

In the end of Chapter 8 we discuss results from (40,66) related to increasing the strong connectivity (arc-strong connectivity) of a semicomplete digraph by either adding new arcs or reversing existing arcs.

In (33) we give a detailed survey on paths, cycles and trees in tournaments and semicomplete digraphs. The survey appeared in a collection of two special issues of Congressus Nummerantium which were devoted to surveys in graph theory and all surveys were written by experts in the respective areas by invitation from the editors.

The famous Kelly conjecture[1] says that every k-regular tournament can be decomposed into arc-disjoint hamiltonian cycles. In (62) we propose a generalization of the Kelly conjecture, namely, we conjecture that every k-arc-strong tournament contains k spanning strong arc-disjoint subdigraphs. We verify the conjecture for $k = 2$ and for the case when the tournament contains a non-trivial k-cut (that is a cut $(S, V - S)$ where both sides have size at least 2 and there are precisely k arcs from S to $V - S$). Our result mentioned above that every tournament which is k-arc-strong and has minimum in- and out-degree at least

[1] R. Häggkvist has announced a proof of this conjecture (for regular tournaments with many vertices) at several conferences more than a decade ago, but to this date no proof of the conjecture has been published.

$37k$ contains k spanning arc-disjoint strong subdigraphs may also be seen as support for the truth of the conjecture.

Further results on tournaments, concerning small spanning subdigraphs which preserve the arc-strong connectivity of the tournaments are described in Chapter 7.

4. Generalisations of Tournaments

The papers (7,8, 14,16 19,21, 22,27,28, 29, 34, 36,38, 41,47,57,56,69, 72, 73) address this topic which is the area where my influence is most clearly demonstrated. In fact, several of the classes of digraphs described below where introduced by me, either alone (7,28,69) or with coauthors (19,27,36,72). Chapter 5 also deals with generalizations of tournaments, but I have chosen to include in the present section only those generalizations in which I have been the principal initiator or one of the principal initiators of investigation of these classes.

The main purpose of my research in this area is to try to identify classes of digraphs that satisfy some of the nice properties which tournaments have. In particular to find classes of digraphs for which problems which are NP-complete for general digraphs (such as the hamiltonian cycle problem) can be solved in polynomial time. Tournaments have a very nice structure:

– Every tournament has a hamiltonian path,
– every strong tournament has a hamiltonian cycle,
– the strong components of every non strong tournament can be ordered uniquely in such a way that the components themselves form an acyclic digraph.

Apart from the large number of papers on generalizations of tournaments in which I have been involved myself, several other authors have also studied the topic and three entire Ph.D. theses have been devoted to the topic. The following list is representative, but far from complete: [7, 10, 11, 16, 32, 33, 34, 35, 36, 37, 38, 46, 50, 55, 56, 57, 100, 102]. I will discuss some of the results obtained by other authors below, but only when it is related to the description of my own results. For comprehensive information on results by other authors as well as results on generalizations of tournaments see my book (1). In particular, Chapter 4 contains descriptions and basic results on the structure of many of these classes of digraphs. As the classes of digraphs generalizing tournaments (that is containing tournaments as a subclass) get further and further away from the tournaments, the question of characterizing the underlying graphs of digraphs in these classes becomes more and more and difficult. Recognition of the underlying graphs of generalizations of tournaments is treated in Chapter 8 of (1).

4.1 Locally semicomplete digraphs

In some sense this is where it all started. In late 1988 when I was visiting Pavol Hell in Vancouver, working on a complexity problem concerning generalized colourings of digraphs (see Chapter 9), I tried to generalize some of the observations we had made on tournaments and semicomplete digraphs in order to study generalized colourings of more general digraphs. This lead me to consider digraphs with the property that for every vertex x, the in-neighbours of x induce a semicomplete digraph and the out-neighbours of x induce a semicomplete digraph. Very soon I discovered that many of the basic results for tournaments could be generalized to this much larger class of digraphs which I named *locally semicomplete digraphs*. I proved in (7) that connected locally semicomplete digraphs share all the three properties above with the tournaments. Furthermore, I established many other nice structural properties of local tournaments. I particular, I proved that the vertices of any local tournament, for which the out-neighbours and the in-neighbours of every vertex induce a transitive tournament, can be labeled $v_1, v_2, \ldots, v_n$ such that the out-neighbours (respectively, in-neighbours) of any vertex v_i occur just after (respectively just before) v_i in this labelling. Also there is a polynomial algorithm to find such a labelling. In [2] this result was shown in the case of tournaments. The existence of such a labelling turns out to be important in connection with recognizing the underlying graphs of locally semicomplete digraphs. Later Huang [57] has generalized my solution to the case when 2-cycles are allowed.

It follows from the proofs in (7) that there are polynomial algorithms to find a hamiltonian path in a given local tournament and a hamiltonian cycle in a given strong local tournament. These algorithms are very similar to the obvious ones for the same problems for tournaments. Hence it might be argued that this paper does not contribute anything significantly new to the theory of algorithms. I claim that this is not true. In fact, as I explained above in section 1 and 3, for some of the problems for tournaments we had to have highly non-trivial information available on the structure of certain tournaments to make our algorithms work. Hence, in the case of local tournaments, properly containing the tournaments, we would likely need even more structural information before we can prove these same problems polynomial, which I believe them to be.

In (14) several additional results on the structure of local tournaments are obtained. In particular I solve a special case of the 2-linkage problem for local tournaments. This is Theorem 4.5 in (14). Using this result I show that a result obtained in (12), as a partial result in the design of a polynomial algorithm for hamiltonian connectedness of tournaments, can be generalized to local tournaments (Theorem 4.9). These results may be seen as a first step towards polynomial algorithms for 2-linkage and hamiltonian connectedness of local tournaments, but they are only very preliminary results on a seemingly long path to such algorithms. Note however, that even the very special case (Theorem 4.5 mentioned above) turned out to be very useful in (29) where we solved the problem of characterizing (constructively) weakly hamiltonian-connected locally semicomplete digraphs.

In (8) we derive fast algorithms for finding hamiltonian paths and cycles in local tournaments. This is done by making more explicit use of the structure of local tournaments, and choosing appropriate data structures for the algorithms. We also give a new proof of the fact (proved in [82]) that the underlying graphs of the local tournaments are precisely the proper circular arc graphs - known for their usefulness in cyclic scheduling and compiler design (a connected graph is a proper circular arc graph if it is the intersection graph of a family of arcs on a circle, none of which properly contains another). From this proof we derive a new algorithm to recognize proper circular arc graphs and to find a representation for such a graph. This algorithm is better than the previously known $O(n^2)$ algorithm by Tucker [91] when the maximum degree of the graph is $o(\sqrt{n})$, where n is the number of vertices in the graph.

In (21) we characterize those chordal graphs which are proper circular arc graphs, and hence are orientable as local tournaments. It turns out that those chordal graphs which are proper circular arc graphs but not proper interval graphs have a very special structure that allows of to characterize them completely.

Locally semicomplete digraphs are interesting, not just because they are a natural generalization of tournaments, but also because of their underlying undirected graphs. As mentioned above, these are exactly the proper circular arc graphs. This fact, together with Huang's deep structure theorem on locally semicomplete digraphs with no directed cycles of length two [55] was used in [16] to develop a fast algorithm for recognizing proper circular arc graphs, in [7] to develop optimal linear algorithms for maximum clique in proper circular arc graphs, and in (22) to develop a fast algorithm to recognize locally semicomplete digraphs.

In [55] J. Huang completely characterized local tournaments. This is a deep and difficult result which involves arc-reversals, something which we used explicitly in the recognition algorithm for locally semicomplete digraphs in (22). However, because the characterizations in [55] uses arc-reversals, it cannot be easily applied to solve other structural or algorithmic problems on locally semicomplete digraphs (such as problems concerning paths and cycles). In (38) we prove another characterization which is very useful in the study of structural and algorithmic properties of locally semicomplete digraphs and we derive several consequences of this reformulated version. The main advantage of this new characterization of locally semicomplete digraphs is that it allows one to prove results for locally semicomplete digraphs without reproving the same statement for tournaments. We use the structure theorem to give complete characterizations of pancyclic and vertex-pancyclic locally semicomplete digraphs. We also show how to derive the preliminary characterization obtained by me in (73) (in an even more precise form) without using Huang's characterization, thus obtaining a direct proof of the useful characterization[1]. We also give a polynomial algorithm to decide if a given locally semicomplete digraph has a kernel.

In (47) I further illustrate the power of our classification of locally semicomplete digraphs from (38) by extending results by Thomassen for tournaments [87] and prove that there exists a natural number $f(k)$ such that every $f(k)$-strong locally semicomplete digraph D

[1] Before the joint work (38) I had already made a preliminary version by observing Huang's characterization could be reformulated in a more useful form see (73).

has k disjoint paths $P_1, \ldots, P_k$ such that P_i starts in x_i and ends in y_i, for every choice of $2k$ distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$ of D.

In (29) we use our structure theorem to give a complete characterization of weakly hamiltonian-connected locally semicomplete digraphs. Again we avoid reproving the tournament case. It is also worth mentioning that in (29) we make use of the very special case of the 2-path problem of locally semicomplete digraphs which was solved in (14).

Finally let us remark that many of the significant structural results obtained for tournaments of path and cycles have been shown to have analogs for local tournaments and locally semicomplete digraphs:

- In his thesis Guo proved that every 4-strong locally semicomplete digraph is hamiltonian-connected [32]. This generalizes Thomassens well-known result [88]. It is still a challenging open problem to find a polynomial algorithm for deciding if a given locally semicomplete digraph has a hamiltonian path from x to y for specified vertices x, y.
- Guo and Volkmann proved that every 2-strong locally semicomplete digraph on n vertices contains a pair of disjoint cycles of lengths k and $n-k$ for each $k = 3, 4, \ldots, n$ [37]. This generalizes a recent result by Song [83].
- Bu and Zhang characterized those arc-3-cyclic local tournaments that are arc-pancyclic [10]. This generalizes the result by Zhang et al. for tournaments [90].
- Bu and Zhang characterized those arc-3-cyclic and arc-3-anti-cyclic local tournaments for which every arc has bypasses of every possible length and is in cycles of all possible lengths [11]. This generalizes the result by Zhang et al. for tournaments in [101].
- In [34] Guo unified both results of Bu and Zhang to obtain a much simpler proof and a more general result.

4.2 In-tournaments

In (19) we consider a further generalization of local tournaments, the so-called in-tournaments. These are defined to be those digraphs for which the inset of every vertex induces a tournament. We show that every strong in-tournament has a hamiltonian cycle and we characterize those in-tournaments that have hamiltonian paths. These results lead to simple polynomial algorithms. The underlying graphs of the in-tournaments seem difficult to characterize. In fact we do give such a characterization in terms of self representability, but such a characterization is not very useful (for another characterization which is also not of use when studying structural properties of in-tournaments see [92]). However, we give an $O(\Delta m)$ algorithm for recognizing those graphs which can be oriented as in-tournaments. Here Δ and m denote the maximum degree, respectively the number of edges of the graph. We show that any graph representable in a unicyclic graph is orientable as an in-tournament. Thus all chordal graphs and all circular arc graphs orientable in this way. We characterize those line-graphs which can be oriented as in-tournaments and we give examples a class of graphs which are orientable as in-tournaments if and only if they are orientable as strong in-tournaments, and hence hamiltonian, by the result mentioned above. Finally we prove that in a certain sense a graph orientable as an in-tournament is either very simple, or it contains many triangles, see Section 4 in (19) for more details.

In (16) we extend the algorithms for hamiltonian paths and cycles in (8) to in-tournaments. We give a fuller treatment of the data structures required to make the algorithm work. It was shown in (19) that an in-tournament has a hamiltonian path if and only if it has an in-branching. Our results in (16) imply that a hamiltonian path in an in-tournament, for which an in-branching is given, can be found in time $O(n \log n)$, where n denotes the number of vertices in the in-tournament.

4.3 Path-mergeable digraphs

In (28) I took a different approach to generalizing tournaments. I had observed that in several proofs, concerning paths and cycles in tournaments, one really only seemed to use the fact that certain paths could be merged together nicely. This lead me to study those digraphs D with the property that for any choice of vertices $x, y \in V(D)$ and any pair of internally disjoint $(x,y)-$paths P, P', there exists an $(x,y)-$path P^* in D, such that $V(P^*) = V(P) \cup V(P')$. I call such digraphs *path-mergeable.* I proved that, given the paths P, P' above, the path P^* can be found in linear time. Furthermore, P^* can be chosen in such a way that vertices from P (resp., P') appear in the same order on P^* as they do on P (resp., P'). The class of path-mergeable digraphs contains all the previously mentioned generalizations of tournaments, but it is much more general. Yet, I could still prove that every strong path-mergeable digraph for which the underlying graph has no cut vertex (an obvious necessary condition for the existence of a hamiltonian cycle), has a hamiltonian cycle. The proof leads to a polynomial algorithm to find such a cycle if it exists. It might seem that recognizing path-mergeable digraphs would be difficult, due to the possible existence of many paths which have to be checked. However, I show how to recognize these digraphs in polynomial time. I also prove that a path-mergeable digraph D has a hamiltonian path from x to y if and only if it contains a spanning acyclic subgraph D' with the property that D' contains a pair of internally disjoint paths from x to z and from z to y for every choice of $z \in V(D) - \{x, y\}$. This generalizes an important result by Thomassen in [88] which has been used several times in the literature. The characterization of (x, y)-hamiltonian paths in path-mergeable digraphs does not imply the existence of a polynomial algorithm to decide the existence of a hamiltonian path in a path-mergeable digraph and this seems to be a difficult problem. The question of which graphs can be oriented as path-mergeable digraphs is also open and seems difficult, also from an algorithmic point of view, since the property is not a local one as were the other generalizations.

4.4 Quasi-transitive digraphs

Having found that in-tournaments also had nice properties in common with tournaments, it was natural for me to investigate whether the same would hold if instead of requirering all in-neighbours of a vertex to be adjacent, one required that all in-neighbours should be adjacent to all out-neighbours. A digraph is *transitive* if whenever x, y and z are distinct vertices such that x dominates y and y dominates z, then x dominates z. This corresponds to the requirement above with the further condition that every in-neighbour must dominate every out-neighbour. By relaxing this to the condition above we obtain the quasi-transive digraphs. A digraph is *quasi-transitive* if whenever x, y and z are distinct vertices such that x dominates y and y dominates z, then x and z are adjacent, i.e., there exists at least one arc from x to y or from y to x.

Quasi-transitive digraphs were introduced by Ghouilà-Houri [28] who proved that an undirected graph G which contains no multiple edges or loops can be oriented as a quasi-transitive digraph if and only if it can be oriented as a transitive digraph, i.e., G is a comparability graph. However, neither Ghouilà-Houri nor anyone else had studied the structural properties of quasi-transitive digraphs until Huang and I did it in (27). We soon discovered that quasi-transitive digraphs have a lot of structure. In particular, many results on connectivity and disjoint paths extend from tournaments to quasi-transitive digraphs. Indeed, we show that every strongly connected quasi-transitive digraph D on at least four vertices has two vertices v_1 and v_2 such that $D - v_i$ is strongly connected for $i = 1, 2$. We study the problem of finding arc-disjoint in- and out-branchings rooted at the same vertex. We show that, in fact, my characterization in (9) for tournaments also holds for general digraphs, provided that $u = v$ and v is adjacent to all other vertices. Using this result we solve the problem completely for quasi-transitive digraphs. Our result implies the existence of a polynomial algorithm for the problem.

Structural properties of quasi-transitive digraphs have been studied in (27,41, 57) and [46, 50, 56, 82]. In (27) we prove that quasi-transitive digraphs can be decomposed in the following way:

Theorem 4.4.1 *Let D be a quasi-transitive digraph on n vertices.*

(1) If D is not strong, then there exist a natural number h, a transitive digraph H on h vertices and strong quasi-transitive digraphs $S_1, S_2, \dots, S_h$ such that $D = H[S_1, S_2, \dots, S_h]$ (this notation is defined in section 13).

(2) If D is strong, then there exist a natural number t, a semicomplete digraph T on t vertices and quasi-transitive digraphs $Q_1, Q_2, \dots, Q_t$ such that each Q_i is either a single vertex or is not strongly connected and $D = T[Q_1, Q_2, \dots, Q_t]$. Furthermore, if T has a cycle of length two induced by vertices v_i and v_j, then the corresponding digraphs Q_i and Q_j are trivial, i.e., each of them has only one vertex.

As will become evident from the rest of this survey, this decomposition result is very useful from an algorithmic as well as a structural point of view.

Some properties of tournaments, like hamiltonicity, can not be extended directly to quasi-transitive digraphs. In (27) we show the existence of highly connected quasi-transitive digraphs D with a factor (a collection of disjoint cycles covering the vertex set of D), which have a cycle of every length $3 \leq k \leq |V(D)| - 1$ through every vertex and yet they are not hamiltonian. Using Theorem 4.4.1 above, we characterize those quasi-transitive digraphs that have a hamiltonian cycle, respectively hamiltonian path. Unfortunately, our characterizations do not imply polynomial algorithms for the problems. In [46] Gutin found such algorithms by showing how the problem could be reduced to that of finding the path covering number of a quasi-transitive digraph. His method is a clever combination of existing optimization techniques, such as flow algorithms, our characterization of hamiltonian quasi-transitive digraphs and Theorem 4.4.1 (another indication of the usefulness of this result). Finally, in (27) we give a complete characterization of those hamiltonian quasi-transitive digraphs that are pancyclic respectively, vertex-pancyclic. Our proofs lead to polynomial algorithms to recognize these classes of quasi-transitive digraphs. In the proofs we use Gutin's characterizations of pancyclic and vertex-pancyclic extended tournaments [42]. This is but one illustration of the close interplay between various classes of generalizations of tournaments.

In (47) I show that using the structure theorem for quasi-transitive digraphs from (27) (Theorem 4.4.1 above), it is possible to reduce the 2-path problem (i.e., problem P2' mentioned in section 3) for quasi-transitive digraphs to the 2-path problem for semicomplete digraphs and hence obtain a polynomial algorithm using the algorithm of (13). In particular there is a polynomial algorithm for the 2-path problem for every digraph which is of the form $D = S[D_1, D_2, \ldots, D_{|S|}]$, where S is a strong semicomplete digraph and $D_1, D_2, \ldots, D_{|S|}$ are arbitrary digraphs. I also show that results by Thomassen on highly connected tournaments [87] extend to quasi-transitive digraphs.

4.5 Arc-local tournament digraphs

In (69) I introduce the class of arc-local tournaments which is a common generalization of tournaments and bipartite tournaments. These are defined as those digraphs for which every in-neighbour (out-neighbour) of x is adjacent to every in-neighbour (out-neighbour) for every arc $x \rightarrow y$. I show that the characterizations in [49] of those bipartite tournaments that have a hamiltonian path respectively hamiltonian cycle are still valid for this larger class of digraphs. The proofs imply polynomial algorithms to determine the existence of such a path or cycle and find one if it exists. My results generalizes results by Las Vergnas in [67]. After finishing the work (69) I discovered that arc-local tournaments which are strong and neither semicomplete digraphs nor semicomplete bipartite digraphs seem to be rather restricted in structure and hence I chose to leave the paper as a preprint. However, in the paper I used a technique which we have subsequently used successfully in many other papers. This is what we call the multi-insertion technique in (1). See Section 5.6 of my book. Using that technique one can obtain elegant proofs of many results on hamiltonian cycles in digraphs.

5. Multipartite tournaments and semicomplete multipartite digraphs

The papers (20,23,25, 31,35,32, 44,74, 52, 57,56) address this topic.

This class of generalizations of tournaments has been known for more than 30 years (see e.g. Moon's book on tournaments [74]). Until the early 90'es it was mostly bipartite tournaments which received much attention, but this has changed after I and my coauthors started working on the full class and inspired many others to do so also. As I will explain below, also for the class of semicomplete multipartite digraphs I have contributed to pushing our knowledge of these digraphs much further. As my results with coauthors have stimulated lots of further work in the area, the impact is even larger. In particular, Yeo's Ph.D. thesis [98] (under my supervision) contains a wealth of deep results on the cycle structure of semicomplete multipartite digraphs, several of which build on my results as described below.

The following is an obvious necessary condition for the existence of a hamiltonian path with end vertices x and y in a digraph D:

(*) D contains a path P with end vertices in $\{x,y\}$ such that $D-P$ has a factor.

Thomassen proved in [88] that every 4-connected tournament is hamiltonian connected. Such a result cannot be obtained for bipartite tournaments, since there exist arbitrarily highly connected bipartite tournaments which do not satisfy (*) for any choice of vertices x and y and hence do not have a hamiltonian path [49]. Hence it is natural to expect that problems regarding hamiltonian paths for bipartite tournaments will be at least as hard as the corresponding problems for tournaments. In (25) we give a complete characterization of those bipartite tournaments that are weakly hamiltonian connected, i.e. for any two distinct vertices x, y there exists a hamiltonian path with end vertices x, y. Our result has some similarities to the corresponding result for tournaments by Thomassen [88]. However, contrary to the tournament case, our characterization is quite complicated and involves several infinite families of exceptions, even when the trivial condition (*) is satisfied. We also give give a polynomial algorithm to decide the existence of such a hamiltonian path with end vertices in the set $\{x,y\}$ and find one if it exists.

In (23) we characterize weakly hamiltonian-connected *extended tournaments*. These are digraphs obtained from tournaments by substituting an independent set of vertices for each vertex of the tournament and hence form a subclass of the multipartite tournaments. We show that apart from the condition (*), the characterization of those extended tournaments that have a hamiltonian path with end vertices x and y is very similar to the corresponding result for tournaments by Thomassen [88]. Our proofs are constructive and they imply a polynomial algorithm to find a hamiltonian path with (unordered) end vertices x, y in an extended tournament, or show that no such path exists.

In (20) we prove that every k-strong bipartite tournament has a cycle through any set of k distinct vertices and give a polynomial algorithm to find such a cycle in a k-strong bipartite tournament. This was conjectured in [49], where it was shown that k-strong is best possible. Recently this result has been extended to the whole class of semicomplete multipartite digraphs by Yeo [97].

In (35) we prove the existence of an efficient parallel algorithm (i.e one belonging to the complexity class NC) to find a hamiltonian cycle in a semicomplete bipartite digraph provided that a factor is found in advance. Together with the existence of an efficient randomized parallel algorithm for the problem of finding a perfect matching in a bipartite graph [75] this implies that the hamiltonian cycle problem belongs to the complexity class RNC. We also show how to obtain an RNC algorithm for the hamiltonian path problem.

In (74) we show how to modify the algorithm in (35) to obtain RNC algorithms for the following problems: Hamiltonian path and cycles in extended semicomplete digraphs and weakly-hamiltonian-connectedness in semicomplete bipartite digraphs as well as in extended tournaments. All of these algorithms will belong to the class NC provided that the maximum matching problem in bipartite graphs belongs to NC. The result on hamiltonian paths with prescribed (unordered) end vertices in tournaments provides an affirmative answer to a question by Soroker [84].

In (31) we describe a sufficient condition for a semicomplete multipartite digraph to be hamiltonian. Unfortunately this condition does not imply the existence of a polynomial algorithm for the hamiltonian cycle problem in semicomplete multipartite digraphs. However, using this condition and the proof-technique used, we obtain simple proof of many well-known results on multipartite tournaments, one of them being Gutin's characterizations of multipartite tournaments with a hamiltonian path [41]. Furthermore, extending our ideas, Yeo has extended our condition to a more detailed form. This last form, although still not easily implying the existence of a polynomial algorithm, has a number of very important implications, the most prominent probably being that every regular multipartite tournament is hamiltonian (this was conjectured by C.Q.Zhang [99]).

We also show in (31) that there exist arbitrarily highly connected multipartite tournaments which have a factor but are not hamiltonian. This is another indication that the hamiltonian cycle problem is highly non-trivial for multipartite tournaments. Finally, in (44) we managed to use the results from (31) and [97] along with a number of new results to prove (constructively) the existence of a polynomial algorithm for the hamiltonian cycle problem in semicomplete multipartite digraphs. **I regard this result as one of my main results, it is the result of more than 3 years research on the problem.** In order to solve it, we needed a very detailed understanding of the cycle structure in semicomplete multipartite digraphs. Even with this detailed knowledge, the algorithm is still highly non-trivial and it involves several non-trivial components (such as a detailed analysis of a last cycle in certain cycle factors, special highly non-trivial algorithms for converting one cycle-factor into a better one etc).

The class of *extended semicomplete digraphs* (previously called ordinary complete multipartite digraphs by Gutin) has been studied extensively, see e.g. [42, 43, 45] by Gutin. In these papers it was shown that an extended semicomplete multipartite digraph has a hamiltonian path (cycle) if and only if it has a a path with a cofactor (is strong and has a factor). It was also shown that there exist polynomial algorithms to find a longest cycle and a longest path in an extended semicomplete digraph. Furthermore pancyclic and vertex-pancyclic extended semicomplete digraphs were characterized. Analogous characterizations of hamiltonian semicomplete bipartite digraphs and of semicomplete bipartite digraphs with a hamiltonian path, as well as polynomial algorithms for the longest path and cycle problems, were obtained in [45, 49, 72]. As mentioned earlier, the results above, on extended semicomplete digraphs, were used in (27) to characterize hamiltonian, pancyclic and vertex pancyclic quasi-transitive digraphs and in [46] to prove the existence of polynomial algorithms for the hamiltonian cycle and hamiltonian path problems for quasi-transitive digraphs.

These applications of results on extended semicomplete digraphs show that, although – at first sight– extended semicomplete digraphs seem like just a small inconsequential generalization of semicomplete digraphs, there is actually some important structure here. In fact, it was shown in (27) (see also (57) and [46]), that in order to understand the structure and algorithmic properties of quasi-transitive digraphs, one needs to understand the structure and algorithmic properties of extended semicomplete digraphs.

In (57) we proved a very nice characterization of the longest cycle in an extended semicomplete digraph. We proved that if $I_1, I_2, \ldots, I_s$ is the partition of the vertices an extended semicomplete digraph $D = (V, A)$ into maximal independent sets and if m_i is the maximum number of vertices of I_i which can be covered by any cycle subgraph of D, then there is a cycle C in D which covers exactly m_i vertices from I_i for $i = 1, 2, \ldots, s$. Obviously C is a longest cycle in D. Again this result has implications on the structure of longest cycles in quasi-transitive digraphs (see e.g. (1) Theorem 5.9.3) as well as on the problem of finding a minimum strong spanning subdigraph of as quasi-transitive digraph (see Chapter 7)

Further results on extended semicomplete graphs can be found in Chapter 7.

6. Longest paths and cycles and cycles through specified vertices in decomposable digraphs

We mentioned earlier how the structure theorem for quasi-transitive digraphs allows one to design recursive algorithms for many problems concerning this class of digraphs. This inspired Gutin and myself to try to generalize the results on longest paths and cycles in quasi-transitive digraphs to other classes of digraphs with similar structure. Our results from the papers (32,34,36, 49,72) are outlined in Sections 5.8-5.10 of (1).

6.1 Extended locally semicomplete digraphs

The purpose of (36) is to study an extension of locally semicomplete digraphs, where we allow substitution of independents sets of vertices for each vertex in the locally semicomplete digraph. We call the digraphs *extended locally semicomplete digraphs* (extended lsd for short). Our hope was to obtain results on longest paths and cycles for this class which could be of use for larger classes of digraphs with a structure related to locally semicomplete digraphs in the same way as quasi-transitive digraphs are related to semicomplete digraphs. In the light of the observations made above on the interplay between quasi-transitive digraphs and extended semicomplete digraphs, there is reason to believe that understanding the structure of extended lsd's will play an important role in understanding the structure of other, more general classes of digraphs. Indeed, we make extensive use the results of this paper in (72) (see below and Section 5.10 of (1)).

We show that the characterizations above for hamiltonian paths and cycles in extended semicomplete digraphs hold for extended lsd's as well. We prove the existence of $O(n^2)$ algorithms for finding a hamiltonian path and a hamiltonian cycle in an extended lsd D with n vertices (if D contains one). The algorithm for constructing a hamiltonian cycle in case of extended lsd's is more difficult than that in case of extended semicomplete digraphs. Using the algorithms above as well as some additional results we construct polynomial algorithms for finding a longest path and a longest cycle in an extended lsd.

We show that there is a polynomial algorithm which decides whether a given digraph is an extended locally semicomplete digraph.

6.2 Totally Φ-decomposable digraphs

As mentioned earlier Gutin [46] described polynomial algorithms for finding a Hamiltonian cycle and a Hamiltonian path (if one exists) in a quasi-transitive digraph. The approach taken in that paper is based on finding an optimal path covering and cannot be used to find a longest path or cycle in polynomial time. In (34) we present a principally new approach that leads to polynomial algorithms for finding maximum weight paths and cycles in quasi-transitive digraphs with non-negative weights on the vertices (the weight of a cycle (path) is the sum of the weights of its vertices). This, in particular, provides an answer to a question by N. Alon on longest paths and cycles in quasi-transitive digraphs. In (72) we show that the conclusions of (34) are valid for a much larger class of graphs. Our results are described below.

Let Φ be some class of digraphs. A digraph D is *Φ-decomposable* if there is a decomposition $D = H[D_1, \ldots, D_k]$, $k \geq 2$, where $H \in \Phi$ and $D_1, \ldots, D_k$ are arbitrary digraphs. D is *totally Φ-decomposable* if, either $D \in \Phi_i$, or $D = H[D_1, \ldots, D_k]$, $k \geq 2$, where $H \in \Phi$ and each D_i is totally Φ-decomposable.

Let Φ_0 be the union of all semicomplete multipartite, extended locally semicomplete and acyclic digraphs, Φ_1 be the union of all extended locally semicomplete and acyclic digraphs, and Φ_2 be the union of all semicomplete bipartite, extended locally semicomplete and acyclic digraphs.

In (72), we prove the following theorems.

Theorem 6.2.1 *Let D be a digraph of order n with nonnegative integer weights on the vertices. One can check whether D is totally Φ_i-decomposable for $i = 0, 1, 2$ in time $O(n^4)$. Moreover,*

(1) If D is totally Φ_0-decomposable, then for all $k = 1, ..., n$, some maximum weight k-path subgraph of D can be found in time $O(n^5)$;
(2) If D is totally Φ_1-decomposable, then a maximum weight cycle of D can be found in time $O(n^5)$;
(3) If D is totally Φ_1-decomposable and $X \subset V(D)$, then a cycle of D, containing all vertices of X, can be found in time $O(n^5)$ (if one exists);
(4) If D is totally Φ_2-decomposable, then a longest cycle of D can be found in time $O(n^5)$.

Theorem 6.2.2 *Suppose a connected digraph D has a decomposition $D = H[M_1, ..., M_h]$, $h \geq 2$, where $H \in \Phi_0$. Then D has a hamiltonian path if and only if D contains a spanning 1-path-cycle subgraph F such that no cycle or path of F is completely contained is any M_i for $i = 1, ..., h$.*

Theorem 6.2.3 *Suppose a strong digraph D has a decomposition $D = H[M_1, ..., M_h]$, $h \geq 2$, where $H \in \Phi_2$. Then D has a hamiltonian cycle if and only if D contains a spanning cycle subgraph F such that no cycle of F is completely contained is any M_i for $i = 1, ..., h$.*

An *extended in-semicomplete digraph* is defined analogously to extended locally semicomplete digraphs. Combining the results of (19) and some of the proofs given in (72), one can show that the characterization of hamiltonian extended lsds holds for extended in-semicomplete digraphs as well. We also point out in (36), that the characterization for extended lsds with a hamiltonian path does not extend to extended in-semicomplete digraphs.

Let Φ_3 be the union of all symmetric digraphs (i.e. digraphs obtained from undirected graphs by replacing each edge by a directed cycle of length two), semicomplete bipartite digraphs, extended locally semicomplete digraphs and acyclic digraphs.

Thomassen proved in [89] that there is no degree of strong connectivity which guarantees a cycle through two given vertices in a digraph. In (32) we consider totally Φ_3-decomposable digraphs. We prove that for every natural number k, every k-strong totally Φ_3-decomposable digraph D is k-cyclic, i.e. for every set X of k vertices of D, there exists a cycle of D containing all the vertices of X. In particular, this implies that every k-strong quasi-transitive digraph is k-cyclic.

We prove that if X is a set of vertices in a k-strong digraph D such that the maximum size of an independent set in the digraph induced by X is at most k, then D has a collection of disjoint cycles (possibly one) covering all the vertices of X. This generalizes a previous result by Jackson and Ordaz [60].

Finally, we consider k-cyclic semicomplete multipartite digraphs (SMD). We conjecture that every k-strong SMD is k-cyclic and provide some support for this conjecture. Subsequently, Yeo has proved this conjecture in [97]. His proof uses the above result on disjoint cycles covering all the vertices of a specified set X as well as the results from (31). As mentioned earlier Yeo's result has several implications. We also use the structural information he derives in (39).

In the paper (49) we survey results on hamiltonian paths and cycles in generalizations of tournaments. We also prove the existence of a polynomial algorithm which finds a path-factor of D starting a a predescribed vertex x (i.e., a minimum collection of disjoint paths covering $V(D)$ such that one of these start in x) for any Φ_2-decomposable digraph. In particular, this implies that the hamiltonian path problem with one fixed end vertex can be solved in polynomial time for quasi-transitive digraphs.

7. Small certificates for arc-strong connectivity

The papers (56,57,61) address this topic. In those papers we consider the following problem, which we denote by MSSS (Minimum Spanning Strong Subgraph): given a strongly connected digraph D, find a strongly connected spanning subgraph D' of D such that D' has as few arcs as possible. This problem, which generalizes the hamiltonian cycle problem and hence is NP-hard, is of practical interest and has been considered several times in the literature, see e.g. [1, 29, 53, 65, 66, 81]. The MSSS problem is an essential subproblem of the so-called *minimum equivalent digraph problem* (in fact, these two problems can be reduced to each other in polynomial time). Here one is seeking a spanning subgraph with the minimum number of arcs in which the reachability relation is the same as in the original graph (i.e. there is a path from x to y if and only if the original digraph has such a path). Since the MSSS problem is NP-hard, it is natural to study the problem under certain extra assumptions. In order to find classes of digraphs for which we can solve the MSSS problem in polynomial time, we must consider classes of digraphs for which we can solve the hamiltonian cycle problem in polynomial time. This follows from the fact that the hamiltonian cycle problem can be solved if we can solve the MSSS problem.

In [66] the MSSS problem was considered for digraphs whose longest cycle has length r for some r. It was shown that if $r \leq 3$, then the problem is polynomial and that it is NP-hard already when $r = 5$.

In (56) we give a lower bound for the number of arcs in a minimum spanning strong subgraph of an arbitrary strong digraph. This bound can be calculated in polynomial time using any polynomial algorithm for finding a maximum matching in a bipartite graph. We prove that this lower bound is attainable for extended semicomplete digraphs and for semicomplete bipartite digraphs. Both of these are subclasses of the class of semicomplete multipartite digraphs for which we remark that the lower bound is not always attainable.

In order to prove the main result for semicomplete bipartite digraphs, we derive a number of new results on semicomplete bipartite digraphs which may be of independent interest. In particular, results on path-covers of acyclic semicomplete bipartite digraphs.

In (57) we give another lower bound for the number of arcs in any minimum spanning strong subgraph of an arbitrary given strong digraph. This lower bound is strongly related to the path-covering number and hence is NP-hard to calculate for general digraphs. For quasi-transitive digraphs this bound can be calculated in polynomial time using Gutin's algorithm for finding a minimum path-cover in a quasi-transitive digraph. We prove that this lower bound is also attainable for quasi-transitive digraphs [46]. The proof of this uses the new characterization of a longest cycle in an extended semicomplete digraph which was mentioned in the end of Chapter 5.

A common feature of the results above on the MSSS problem for semicomplete bipartite digraphs, extended semicomplete digraphs and quasi-transitive digraphs is that there always exist an optimal spanning strong subdigraph which contains a longest cycle of the starting digraph. This is easily seen not to be the case for arbitrary digraphs, but it is an interesting open problem whether it is still true for semicomplete multipartite digraphs. Note that it is also an open problem whether there exists a polynomial algorithm for finding a longest cycle in a semicomplete multipartite digraph.

In (57) we also point out the our methods imply that the MSSS problem can be solved efficiently for a much larger superclass of semicomplete digraphs than just quasi-transitive digraphs.

Khuller, Raghavachari and Young [65] gave a 1.65-approximation algorithm for the size of a minimum strongly connected subgraph of any strongly connected digraph. This was later improved to about 1.61 using results from [66] and to 1.5 by Vetta [93]. In (61) we prove that every strong digraph D with $\bar{\Delta}(D) \leq \frac{n}{r}$ (the maximum degree in the complement) contains a spanning strong digraph with at most $(1+\frac{1}{r})n$ arcs. Furthermore, such a subdigraph can be found in polynomial time. Note that whenever $r \geq 2$ our result gives a better approximation guarantee.

In (61), which is enclosed with this thesis, we study the equivalent of the MSSS problem for higher of arc-strong connectivities. For an arbitrary k-arc-strong digraph one can find a spanning k-arc-strong subdigraph with at most twice the optimal number of arcs in polynomial time by finding k-arc-disjoint out-branchings and k-arc-disjoint in-branchings and taking the union of all these arcs. However, even for tournaments finding a polynomial algorithm to determine a minimum k-arc-strong spanning subdigraph, seems to be very difficult. One reason for this is that although tournaments have a lot of structure, much of this is either lost, or at least difficult to establish as soon as we delete only a relatively small number of arcs from the tournament in question (for example it is an open and apparently very difficult problem to characterize those tournaments that have two arc-disjoint hamiltonian cycles). Hence applying an iterative approach which builds up a minimum spanning k-arc-strong subdigraph from certain spanning strong subdigraphs seems doomed to fail.

Given a k-arc-strong tournament T, we estimate the minimum number of arcs possible in a k-arc-strong spanning subdigraph of T. We give a construction which shows that for each $k \geq 2$ there are tournaments T on n vertices such that every k-arc-strong spanning subdigraph of T contains at least $nk+\frac{k(k-1)}{2}$ arcs. In fact, the tournaments in our construction have the property that every spanning subdigraph with minimum in- and out-degree at least k has $nk+\frac{k(k-1)}{2}$ arcs. This is best possible since it can be shown that every k-arc-strong tournament contains a spanning subdigraph with minimum in- and out-degree at least k and no more than $nk+\frac{k(k-1)}{2}$ arcs. As our main result we prove that every k-arc-strong tournament contains a spanning k-arc-strong subdigraph with no more than $nk+280k^2$ arcs.

Since every k-arc-strong digraph has at least nk arcs this shows that for k-arc-strong tournaments one can get within a function depending only on k of the optimum. The method we use involves iteratively constructing large arc-disjoint strong subdigraphs of T such that the subdigraph D' induced by the union of the arc sets of these subdigraphs contains a large set of vertices X (containing all but possibly a linear function of k vertices from T) and for any two vertices $x, y \in X$, D' contains k arc-disjoint (x, y)-paths.

Our proof uses several new results on digraphs in which the number of non-neighbours of each vertex is bounded by some constant c and digraphs in which some vertices may have more than c non-neighbours but the total number of such vertices is bounded by some other constant.

Finally, we conjecture in (61) that for every k-arc-strong tournament T, the minimum number of arcs in a k-arc-strong spanning subdigraph of T is equal to the minimum number of arcs in a spanning subdigraph of T with the property that every vertex has in- and out-degree at least k. If true, this would imply that there exists a polynomial algorithm to determine the minimum number of arcs in a spanning k-arc-strong subdigraph of a given k-arc-strong tournament.

8. Preserving and increasing the connectivity of graphs and digraphs

The papers (26,40, 48, 46, 51, 55, 59, 66, 70) address this topic.

This collection includes some of my most recent and most complicated works. As this thesis only deals with directed graphs, I will only describe those results related to digraphs. Recall that a *mixed graph* is a graph containing both arcs and edges (alternatively one may think of a mixed graph as a partially oriented graph).

The study of connectivity of graphs and digraphs is a fundamental problem in graph theory with a number of practical applications. In practical applications one is often interested in finding a minimum set of edges (arcs) whose addition to a graph (digraph) G results in a graph (digraph) which satisfies certain edge-connectivity requirements. In general, however, this is a very difficult problem and even seemingly simple special cases are NP-complete. Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathcal{N}$ be a symmetric function (i.e. $r(x, y) = r(y, x)$). In [22] Frank completely solved the problem of deciding the minimum number γ of new undirected edges that have to be added to G in order to obtain a graph G^+ in which the local edge-connectivity $\lambda(x, y; G^+) \geq r(x, y)$ for all pairs of vertices $x, y \in V$. He also described a strongly polynomial algorithm to find a set a γ edges whose addition to G results in a graph G^+ satisfying the condition above. In the same paper Frank solves the analogous problem for directed graphs in the case when the function r is a constant k. Here the goal is to add as few arcs as possible in order to obtain a k-arc-strong digraph. He also proved that if r is not a constant, then the problem becomes NP-complete (even if r is either 0 or 1).

Frank's results (as are most of the other results on edge-connectivity augmentation) are based on the so-called splitting off technique. *Splitting off* a pair of edges (arcs) $e = us, f = st$ means that we replace e and f by a new edge (arc) ut. The resulting graph (digraph) will be denoted by G^{ef}. This operation is defined only if both e and f are undirected (respectively, directed) and then the newly added edge ut is considered undirected (directed). Accordingly, we speak of undirected or directed splittings.

Two theorems of Mader concerning directed and undirected splittings [70, 71], are important tools in the proofs by Frank in [22]. In my paper (26) with Frank and Jackson we follow an analogous line and the basis for the results in that paper is an extension of the existing splitting theorems. When a splitting off operation is performed, the local edge-connectivity never increases. The content of the splitting off theorems is that under certain conditions there is an appropriate pair $\{e = us, f = st\}$ of edges whose splitting preserves all local or global edge-connectivity between nodes distinct from s. Mader showed that if $G = (V + s, E)$ is an undirected graph with a special vertex s of degree different from 3 and not incident to any cut-edge, then there exist a pair of edges $e = us, f = st$ so that $\lambda(x, y; G) = \lambda(x, y; G^{ef})$. Thus local edge-connectivity is preserved between pairs of vertices not involving s. For directed graphs such a result does not hold, but Mader proved that if $D = (V + s, A)$ is a directed graph for which $\lambda(x, y; D) \geq k$ for every $x, y \in V$ and the in-degree of s equals its out-degree, then for every arc $f = st$ there exists an arc $e = us$ so that $\lambda(x, y; D^{ef}) \geq k$ for every $x, y \in V$.

These results, especially Mader's undirected splitting theorem, are very powerful and have several applications in graph theory (as an example, in [70], Mader proved Nash-Williams famous local arc-strong connectivity orientation theorem using this result).

It is interesting to note that with respect to increasing the arc-connectivity eulerian digraphs behave as undirected graphs, namely again there exists a splitting preserving even local-arc-connectivity. This was shown independently by B. Jackson [61] and A. Frank [21].

In (26) we unify all the previous splitting theorems by proving common generalizations of the splitting theorems for undirected and directed eulerian graphs, respectively for directed and directed eulerian graphs. Using these new splitting results for mixed graphs we show how the augmentation problem for mixed graphs can be solved for certain demand functions more general than the uniform one. (By a *mixed graph M* we mean a graph that may have both directed and undirected edges.) When the starting graph is mixed one may wish to add both directed and undirected edges. Unfortunately, we do not have anything to say about this general case. Our results concern only the two extremes when either only directed edges or only undirected edges are allowed to be added to the given mixed graph M. Once again these augmentation results are based on splitting theorems. However, none of the existing splitting theorems were strong enough to be applicable in the mixed case.

As a byproduct of our splitting theorems we obtain a generalization of Edmonds' theorem on arc-disjoint branchings, as well as a new sufficient condition for the existence of k arc-disjoint paths $P_1, \ldots, P_k$ in a digraph D such that P_i starts in s_i and ends in t_i for specified vertices $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(D)$.

For directed graphs the vertex-connectivity augmentation problem (where we wish to add as few new arcs as possible in order to achieve a prescribed vertex-strong connectivity) was solved by A. Frank and T. Jordan [24]. The result by Frank and Jordan also involves a min-max formula for the minimum number γ of new arcs that have to be added to make a given digraph k-strongly connected. For general digraphs one can find the number γ and the desired set of augmenting arcs in polynomial time, but (so far) only using the ellipsoid method. In the case when one has a k-strong digraph and want to make it $(k + 1)$-strong Frank and Jordan have given a more precise description of γ [62]. Using this result I made the observation in with Jackson and Jordán (unpublished) that every tournament can be made k-strong by adding at most $k(k - 1) + 1$ arcs. The best lower bound we know is obtained from the transitive tournament which requires $k(k + 1)/2$ arcs. I conjectured at a meeting in Budapest in 1994 that this is the right number, but we haven't been able to prove this yet. In (40), using a very simple approach, we give a polynomial (combinatorial) algorithm which given a tournament T and a natural number k finds a set of at most $\frac{(4k-2)(4k-3)}{4}$ arcs whose addition to T results in a k-strong digraph.

In (40) we study arc reversals in semicomplete digraphs and augmentations. Let $a_k(D)$ denote the minimum number of new arcs one has to add in order to make the digraph D k-strong and let $r_k(D)$ denote the minimum number of arcs in D that have to be reoriented in order to obtain a k-strong digraph. Clearly $a_k(D) \leq r_k(D)$ for any digraph, since we could add a copy of each reversed arc to the original digraph. On the other hand it is not difficult to construct digraphs for which $r_k(D)$ is finite and $a_k(D) < r_k(D)$. We prove in (40) that if $k \geq 2$ and D is a semicomplete digraph with at least $3k-1$ vertices then $r_k(D) = a_k(D)$. Our proof is constructive and together with the polynomial algorithm for vertex-connectivity augmentation in digraphs [24], it implies a polynomial algorithm to find an optimal set of arcs to reverse, provided that D has at least $3k-1$ vertices. We also give a construction of a semicomplete digraph D with $3k-2$ vertices for which $a_k(D) < r_k(D)$. The example has directed 2-cycles and we believe that for tournaments we can lower the bound $3k-1$ down to $2k+1$ that is every tournament which has any chance of being made k-strong by arc reversals (a k-strong tournament has at least $2k+1$ vertices) can be made so by reversing the same number of arcs as the optimal number needed when we add arcs instead.

By the remarks above we always have $a_k(D) \leq r_k(D)$. I conjectured at the meeting in Budapest in 1994 mentioned above that $r_k(T)$ is always at most $k(k+1)/2$ for every tournament T (again the transitive tournament shows that this would be best possible). So far this conjecture is completely open. In the recent work (66) with Yeo we proved that $r'_k(T) \leq k(k+1)/2$ for every tournament T, where $r'_k(D)$ denotes the minimum number of arcs one needs to reverse in order to obtain a k-arc-strong digraph from D.

9. Generalized colourings

The papers (3,5,11,15,18,24) document my research (which dates back to the period 1987-1992) on this subject.

In this part of my research I study the class NPC of NP-complete problems more directly. This is done through the study of the complexity of the so-called H-colouring problem: Let H be any fixed digraph. An H-colouring of a digraph D (also sometimes called a *homomorphism* $D \to H$*)* is a mapping $f : V(D) \to V(H)$ such that $(f(x), f(y))$ is an arc of H whenever (x, y) is an arc of D. Note that we do not require this mapping to be onto H. The term H-colouring arises from the fact that an n-colouring of an undirected graph G is just a homomorphism $G \to K_n$. The *H-colouring problem* is described as follows:
Instance: A digraph D
Question: Does there exist an $H-$colouring of D?
The H-colouring problem is clearly in NP for any fixed H and is very well suited for studying the class NPC, because for each digraph H we have a new problem in NP (This should be compared for instance with the problem of asking for a hamiltonian cycle in a graph, which corresponds to one simple class of graphs, namely the cycles. In effect what we ask then is: does there exist a homomorphism of C_n onto G, where n is the number of vertices of G?). So in studying the complexity of the H-colouring problem for various classes of digraphs, we learn more about the structure of the class NPC.

For undirected graphs the complexity of the H-colouring problem was completely settled by Hell and Nešetřil in the important paper [51]. Here it is the case that the H-colouring problem is polynomial if and only if H is bipartite. In order to prove this characterization Hell and Nešetřil developed some very powerful tools, which we have used extensively in the papers (3,11, 15) and in refined versions in (24).

For digraphs the situation is much more blurred. There seems to be no nice characterization of those digraphs for which the problem is polynomial (assuming that $P \neq NP$). For example there are oriented trees for which the corresponding colouring problem is NP-complete. This was shown in [48]. In that paper it is also proved that the H-colouring problem is polynomial when H is any oriented path. However if we restrict our attention to digraphs without sources and sinks, then Hell and I have a conjecture in (11) for a characterization of those digraphs for which the problem is polynomial. This conjecture will be described below.

If H' is a subdigraph of H, then a *retraction* of H to H' is a homomorphism $r : H \to H'$ such that $r(h') = h'$ for all vertices in H'. A digraph is a *core* if it does not admit a retraction to a proper subdigraph. We conjecture that if H is a digraph without sources or sinks, then H-colouring is polynomial if and only each component of the core of H is a directed cycle.In (3,11, 15) we verify this conjecture for several classes of digraphs including semicomplete digraphs, bipartite tournaments, digraphs consisting of two disjoint cycles joined by an oriented path and digraphs obtained from an undirected bipartite graph G, by replacing each edge by a directed cycle of length 2 and adding some arcs inside each colour class of G (such digraphs are also called *partitionable* digraphs, see (15)). Our conjecture has been of interest, not only to people who work in the area of homomorphisms, but also to people who work on more general complexity theory. In particular Feder and Vardi [19] discussed it in details in their STOC 93 paper on complexity.

In (24) we take a new perspective on $H-$colouring by defining superhard digraphs. A digraph H is *superhard* if the H^*-colouring problem is $NP-$hard for any digraph H^* which contains H as a, not necessarily induced, subdigraph. This definition is motivated by the fact that in the undirected case any digraph which is hard (i.e. has an $NP-$complete colouring problem) is superhard. This is by no means the case for digraphs. In (24) we identify several classes of superhard digraphs and formulate a conjecture for a characterization of superhard digraphs, namely we conjecture that a connected digraph H is superhard if and only if it does not map to any directed cycle of length greater than 1. We show that this conjecture is equivalent to the conjecture above.

10. Alternating Cycles and paths in edge-coloured multigraphs

The papers (37,42,43) address this topic. The results mentioned here are described in chapter 11 of (1) where we show the connection to directed graphs, in particular to cycles in bipartite digraphs. In (37) we survey results of both theoretical and algorithmic character concerning cycles and paths in edge-coloured multigraphs, and we prove several new results in the area. Describing known results and proving new ones, we show useful connections between problems arising in this area and many problems from different fields of graph theory and algorithms. Let G be a 2-edge-coloured complete graph. M. Bankfalvi and Zs. Bankfalvi [4] obtained necessary and sufficient conditions for G to have a Hamiltonian alternating cycle (that is, one in which the colours of the edges alternate). Generalizing this theorem, P. Das and S.B. Rao [15] characterized those 2-edge-coloured complete graphs G which contain a closed alternating trail visiting each vertex v in G exactly $f(v) > 0$ times. In (43) we solve the more general problem of determining the length of a longest closed alternating trail T_f visiting each vertex v in G at most $f(v) > 0$ times. Our result is a generalization of a theorem by R. Saad [80] that determines the length of a longest alternating cycle in G. We prove the existence of a polynomial algorithm for finding the desired trail T_f. In particular, this gives a solution to a problem by Y. Manoussakis (cf. [80]). Finally, in (42) we prove that if an edge-coloured complete graph contains a 2-factor such that no cycle in this has two consecutive edges of the same colour, then the graph has a hamiltonian path such that no two consecutive edges have the same colour. Our proof is constructive and implies polynomial algorithm to find such a hamiltonian path in an edge-coloured complete graph, or detect that no such path exists.

11. Miscellaneous results

11.1 Sufficient conditions for hamiltonicity

In (30) we show that it is possible to obtain sufficient conditions for hamiltonian cycles in digraphs by combining local structure and degree conditions. Traditionally the conditions require a high degree sum for every pair of non-adjacent vertices, or even for every pair x, y for which there is no arc from x to y (e.g. in Woodall's condition [96]). Our result is that for the well-known conditions of Woodall and Ghouila-Houri it is only necessary to require high degree sum for vertices x, y which are not adjacent, but should have been so if the digraph had been a locally semicomplete digraph (that is there exist some $z \in V(D) - \{x, y\}$ such that $z \to x, z \to y$, or $x \to z, y \to z$). This result may be the starting point of further research in this area. It seems much more natural only to require high degree sum only for certain subsets of the vertices. That way the results will also be applicable to digraphs which are not so dense. In particular, in our case there is no restriction on the degrees when the actual digraph is locally semicomplete. In (30) we conjecture an extension of Meyniels theorem based on imposing the degree condition $d(x) + d(y) \geq 2n - 1$ only on pairs of non-adjacent vertices with a common in-neighbour. In (50) we derive some support for that conjecture. In particular, we prove that the condition above guarantees a cycle factor and a hamiltonian path. Furthermore, we show that if we replace $2n - 1$ by $\frac{5}{2}n - 4$ above, then the digraph will be hamiltonian.

11.2 Euler tours traversing specified vertices in given orders.

In (17) we consider an eulerian digraph G and a set $x_1, x_2, \ldots, x_k$ of specified vertices called terminals. We consider the question of whether or not G admits an eulerian trail visiting the vertices $x_1, x_2, \ldots, x_k$

(i) in a prescribed order,
(ii) in a unique order (that is, all eulerian trails visit the terminals in that order), and
(iii) in the order $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}$ for every permutation $\pi \in S_k$.

The problem (ii) was solved by Ibaraki and Poljak [59]. In (17) we show that an instance $(G, x_1, x_2, \ldots, x_k)$ which is infeasible for problem (iii) and is minimal with respect to edge contractions must have restricted degrees on the vertices. We also formulate topological obstructions for problem (i), which arise from certain embeddings of G on a surface.

12. Concluding remarks

I hope that after reading this report the readers will agree that I have contributed significantly to the development of the area of directed graphs in at least the following subareas:

– Generalizations of tournaments
– Connectivity of digraphs
– Algorithms for paths and cycles in digraphs
– Structural aspects of digraphs, in particular paths and cycles.

My research has stimulated the work of many other researchers. In particular, 3 Ph.D. theses [32, 56, 86] have been written based on classes of digraphs introduced by me. In addition, the theses [45, 73, 98] are strongly influenced by my research. Furthermore, through my book with Gutin I have not only documented my own research and that of others on digraphs, but also provided the research community with a rich source of information that is likely to stimulate research on digraphs for many years to come.

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April 9, 2001

Jørgen Bang-Jensen

13. Terminology

Here we shall define some of the terms used in the previous sections. We assume that the reader is familiar with the standard terminology of undirected graphs and refer to [5],[8]. For a complete account of terminology of directed graphs see my book (1).

– *Digraphs:*
  A *digraph* D consists of a pair $V(D), E(D)$, where $V(D)$ is a finite set of *vertices* and $E(D)$ is a set of ordered pairs xy of vertices called *arcs*. In our definition of a digraph we do not allow multiple arcs in the same direction between two vertices. An *oriented* graph is a digraph with no cycle of length 2. If x and y are distinct vertices of a digraph D and there is an arc from x to y in D, then we say that x *dominates* y. For a given vertex x of a digraph D the *inset* (or, the *in-neighbours* of x), denoted I_x, respectively the *outset* (or, the *out-neighbours* of x), denoted O_x, of x consists of those vertices of $D - x$ which dominate, respectively are dominated by x. The in-neighbours (resp. out-neighbours) of any subdigraph is similarly defined.
– *Induced subdigraphs:*
  For any subset A of $V(D) \cup E(D)$, $D - A$ denotes the subgraph obtained by deleting all vertices of A and their incident arcs and then deleting the arcs of A still present. The subgraph *induced* by a vertex set A of D is defined as $D - (V(D) \setminus A)$ and is denoted by $D(A)$.
– *Decomposable and extended digraphs*
  Let D be a digraph on n vertices and let $S_1, S_2, \ldots, S_n$ be distinct digraphs. The digraph $D[S_1, S_2, \ldots, S_n]$ is the digraph obtained from D by replacing the ith vertex of D by the digraph S_i in such a way that for every arc $i{\to}j$ in D, $D[S_1, S_2, \ldots, S_n]$ contains all possible arcs from $V(S_i)$ to $V(S_j)$ all the original arcs of an S_i are also in $D[S_1, S_2, \ldots, S_n]$. In the case when no S_i contains an arc, i.e. the underlying graphs are independent sets, we call the digraph $D[S_1, S_2, \ldots, S_n]$ an *extension* of D. If D' is a digraph and there exist digraphs D (on n vertices) and $S_1, S_2, \ldots, S_n$ such that $D' = D[S_1, S_2, \ldots, S_n]$ (or more precisely D' is isomorphic to $D[S_1, S_2, \ldots, S_n]$), then we say that D' is *decomposable* with decomposition $D, S_1, S_2, \ldots, S_n$.
– *Various classes of digraphs:*
  A *semicomplete* digraph is a digraph with no non-adjacent vertices. A *tournament* is an oriented graph with no non-adjacent vertices. Thus tournaments are a special subclass of the semicomplete digraphs. A *local tournament* (*locally semicomplete digraph*) is a digraph D that satisfies the following condition: for every vertex x, $D(O_x)$ and $D(I_x)$ are tournaments (semicomplete digraphs). A digraph is *quasi-transitive* if x and z are

adjacent whenever there is a directed path of length two between them. A *bipartite tournament* is an orientation of a complete bipartite graph. A *arc-local tournament* is a digraph for which every vertex of I_x (O_x) is adjacent to every vertex of I_y (O_y) for every arc $x \to y$. A *Multipartite tournament* is an orientation of a complete k-partite graph for some $k \geq 2$. A *Semicomplete multipartite digraph* is a digraph whose underlying graph is a complete k-partite graph for some $k \geq 2$, i.e., the difference from multipartite tournaments is that semicomplete multipartite digraphs may have cycles of length two.

– *Paths and cycles:*
  A *path* is a digraph with vertex set $x_1, x_2, ..., x_n$ and arc set
  $x_1 \to x_2, x_2 \to x_3, ..., x_{n-1} \to x_n$, such that all the vertices and arcs shown are distinct. We call such a path an (x_1, x_n)-path. A path that contains all the vertices of D is a *hamiltonian path.* A hamiltonian path that starts in a vertex x and ends in the vertex y is called an $(x, y)-$hamiltonian path. A digraph D is *hamiltonian connected* if D has an $(x, y)-$hamiltonian path for any choice of distinct vertices $x, y \in V(D)$.
  A *cycle* is defined analogously. A *k-cycle* is a cycle of length k. In this survey the words path and cycle always refer to directed paths and cycles.
– *Disjoint paths:*
  Two paths P, Q are *vertex-disjoint* or just disjoint if no vertex appears on both paths. P, Q are *arc-disjoint* if no arc appears on both paths. A digraph D is *k-linked* if for every choice of $2k$ distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of D there exists k disjoint paths $P_1, \ldots, P_k$ such that P_i starts in x_i and ends in y_i.
  A *factor* is a spanning collection of vertex-disjoint cycles. If P is a path in a digraph D such that $D - P$ has a factor, then we say that P is a path with a *cofactor*.
  A collection F of pairwise vertex-disjoint paths and cycles of a digraph D is called a *k-path-cycle factor* of D if F covers $V(D)$ and has exactly $k \geq 0$ paths. F is called a *k-path factor* if it is contains only paths. We shall call a 0-path-cycle factor a *cycle factor.*
  A k-path-cycle F *starts* at a vertex x if one the paths of F starts at x. The *path-covering number* of a digraph D ($pc(D)$) is the minimum integer k such that D has k-path factor. Similarly $pc_x(D)$ is the minimum number of paths in a path factor that starts in x.
– *Branchings:*
  An *out-branching* rooted a v is a spanning tree which is directed in such a way that each vertex $x \neq v$ has one arc coming in. An *in-branching* is defined analogously.
– *Strong components:*
  We say that a digraph D is *connected* if the underlying undirected graph of D is connected. A *strong component* D' of a digraph D is a maximal subdigraph , such that for any two vertices $x, y \in D'$, D' contains an $(x, y)-$path and a $(y, x)-$path. A digraph D is *strong* if it has only one strong component. If a digraph D is not strong, then we can label its strong components $D_1, D_2, ..., D_k, k \geq 2$, such that no vertex of D_i
  dominates any vertex of D_j if $j < i$. For general digraphs this labeling is not unique.
– *k-strong and k-arc-strong digraphs:*
  A digraph D is *k-strong* if for any set A of at most k-1 vertices, $D - A$ is strong. D is

k-arc-strong if for any set A of at most k-1 arcs, $D - A$ is strong. We use $\lambda(x, y; D)$ to denote the number of arc-disjoint paths from x to y in the digraph D.

14. List of Publications by Jørgen Bang-Jensen

1. **Books and theses:**
   Jørgen Bang-Jensen and Gregory Gutin, Digraphs: Theory, Algorithms and Applications, Springer Verlag, London (2000) xxii+ 754pp.
   see also http://www.imada.sdu.dk/Research/Digraphs/
2. Jørgen Bang-Jensen, Path problems and generalized colouring problems for tournaments and other classes of digraphs: characterizations, algorithms and complexity, Ph.D. Thesis, Odense University, May 1988.
   Research Papers:
3. Jørgen Bang-Jensen, Pavol Hell and Gary MacGillivray, The complexity of colouring by semicomplete digraphs, *Siam Journal on Discrete Mathematics* 1 (1988) 281-298
4. Jørgen Bang-Jensen, On the 2-linkage problem for semicomplete digraphs. Graph Theory in the memory of G.A. Dirac, **Annals of Discrete Mathematics, Vol 41** (1989) 23-38.
5. Jørgen Bang-Jensen, On the complexity of generalized colourings by directed graphs.In conference proceedings from Twente workshop on Graphs and Combinatorial Optimization May 24-26, 1989. Memorandum no. 787 Universiteit Twente, faculteit der togepaste wiskunde, U.Faigle, C.Hoede (eds.), 13-20.
6. Jørgen Bang-Jensen and Pavol Hell, The effect of two directed cycles on the complexity of colourings by directed graphs, *Discrete Applied Mathematics* **26** (1990) 1-23.
7. Jørgen Bang-Jensen, Locally semicomplete digraphs: a generalization of tournaments, *Journal of Graph Theory* **14** (1990) 371-390.
8. a. Jørgen Bang-Jensen, Pavol Hell and Jing Huang, Local tournaments and proper circular-arc graphs, *Springer Lecture Notes in Computer Science* **450** (1990) 101 - 108.
   b. Jørgen Bang-Jensen, Pavol Hell and Jing Huang, Local tournaments and proper circular-arc graphs, Simon Fraser University Technical Report CSS/LCCR TR90-11, (1990) **Expanded version of 6a.**.
9. Jørgen Bang-Jensen, Edge disjoint In- and Out-branchings in Tournaments and related path problems, *Journal of Combinatorial Theory Series B* **51** (1991) 1-23.
10. Jørgen Bang-Jensen, A note on a special case of the 2-path problem for semicomplete digraphs. In **Graph Theory, Combinatorics, and Applications** Eds. Alavi, Chartrand, Oellermann and Schwenk, Wiley (1991) 77-86.
11. Jørgen Bang-Jensen and Gary MacGillivray, Further effects of two directed cycles on the complexity of H-colouring, *Journal of Combinatorial Mathematics and Combinatorial Computing* 10 (1991) 33-50.
12. Jørgen Bang-Jensen, Yannis Manoussakis and Carsten Thomassen, A polynomial algorithm for hamiltonian connectedness in semicomplete digraphs, *J.Algorithms* 13 (1992) 114-127.
13. Jørgen Bang-Jensen and Carsten Thomassen , A polynomial algorithm for the 2-path problem for semicomplete digraphs, *SIAM Journal on Discrete Mathematics* 5 (1992) 366-376.
14. Jørgen Bang-Jensen, On the structure of locally semicomplete digraphs, *Discrete Mathematics* 100 (1992) 243-265.
15. Jørgen Bang-Jensen, Pavol Hell and Gary MacGillivray, On the complexity of colouring by supergraphs of bipartite graphs, *Discrete Mathematics* 109 (1992) 27-44.
16. Jørgen Bang-Jensen and Pavol Hell, Fast algorithms for finding hamiltonian paths and cycles in in-tournament digraphs, *Discrete Applied Mathematics* 41 (1993) 75-79.
17. Jørgen Bang-Jensen, Svatopluk Poljak, Eulerian trails through a set of terminals in specific unique and all orders, in **Comtempory Mathematics** 147 (1993) 247-258.
18. Jørgen Bang-Jensen and Gary MacGillivray, On the complexity of colouring by digraphs With at most one directed cycle, *Ars Combinatorica* vol 25A (1993) 175-186.
19. Jørgen Bang-Jensen, Jing Huang and Erich Prisner, In-tournament digraphs, *Journal of Combinatorial Theory B* **59** (1993) 267-287.
20. Jørgen Bang-Jensen and Yannis Manoussakis, Cycles through k vertices in bipartite tournaments, *Combinatorica* **14** (2) (1994) 1-4.
21. Jørgen Bang-Jensen and Pavol Hell, Chordal proper circular arc graphs, *Discrete Mathematics* **128** (1994) 395-398.
22. Jørgen Bang-Jensen, Pavol Hell and Jing Huang, Optimal recognition of local tournaments, *Congressus Nummerantium* **100** (1994) 141-146.
23. Jørgen Bang-Jensen, Gregory Gutin and Jing Huang, Weakly hamiltonian-connected ordinary-multipartite tournaments, Discrete Mathematics **138** (1995) 63-74.

24. Jørgen Bang-Jensen, Pavol Hell and Gary MacGillivray, Hereditarily hard colouring problems, *Discrete Mathematics* **138** (1995) 75-92.
25. Jørgen Bang-Jensen and Yannis Manoussakis, Weakly hamiltonian connected bipartite tournaments *Journal of Combinatorial Theory B* **63** (1995) 261-280.
26. Jørgen Bang-Jensen, Andras Frank and Bill Jackson, Preserving and increasing local edge-connectivity in mixed graphs, *SIAM J. Disc. Math* **8** (1995) 155-178.
27. Jørgen Bang-Jensen and Jing Huang, Quasi-transitive digraphs, *Journal of Graph Theory* **20** (1995) 141-161.
28. Jørgen Bang-Jensen, Digraphs With the path merging property, *Journal of Graph Theory* **20** (1995) 255-265.
29. Jørgen Bang-Jensen, Yubao Guo and Lutz Volkmann, Weakly Hamiltonian-connected locally semicomplete digraphs, *Journal of Graph Theory* **21** (1996) 163-172.
30. Jørgen Bang-Jensen, Gregory Gutin and Hao Li, sufficient conditions for a digraph to be hamiltonian, *J. Graph Theory* **22** (1996) 181-187.
31. Jørgen Bang-Jensen, Gregory Gutin and Jing Huang, A sufficient condition for hamiltonian cycles in complete multipartite digraphs, *Discrete Mathematics* **161** (1996) 1-12.
32. Jørgen Bang-Jensen, Gregory Gutin and Anders Yeo, k-strong and k-cyclic digraphs, *Discrete Mathematics* **162** (1996) 1-11.
33. Jørgen Bang-Jensen, Gregory Gutin, Paths, trees and cycles in Semicomplete digraphs-A survey, **Surveys in Graph Theory, eds. Chartrand and Jacobson**, *Congr. Numer.* **115** (1996) 131-170.
34. Jørgen Bang-Jensen, Gregory Gutin, Vertex heaviest paths and cycles in quasi-transitive digraphs, *Discrete Mathematics* **163** (1997) 217-223.
35. Jørgen Bang-Jensen, Mohammed El Haddad, Yannis Manoussakis and Teresa Przytycka, Parallel algorithms for the hamiltonian cycle and hamiltonian path problems in semicomplete bipartite digraphs, *Algorithmica* **17** (1997) 67-87.
36. Jørgen Bang-Jensen, Gregory Gutin, Paths and cycles in extended and decomposable digraphs, *Discrete Mathematics*, **164** (1997) 5-19.
37. Jørgen Bang-Jensen, Gregory Gutin, Alternating cycles and paths in edge-coloured multigraphs, *Discrete Mathematics* **165-166** (1997) 39-60.
38. Jørgen Bang-Jensen, Yubao Guo, Gregory Gutin and Lutz Volkmann, A classification of locally semicomplete digraphs, *Discrete mathematics* **167-168** (1997) 99-112.
39. Jørgen Bang-Jensen, Gregory Gutin and Anders Yeo, Hamiltonian cycles avoiding all arcs in prescribed cliques in tournaments,*Combinatorics, Probability and Computing* **6** (1997) 255-261.
40. Jørgen Bang-Jensen, Tibor Jordan, Adding and reversing arcs in semicomplete digraphs, *Combinatorics, Probability and Computing* **7** (1998) 17-25.
41. Jørgen Bang-Jensen and Jing Huang, Kings in Quasi-transitive digraphs, *Discrete mathematics* **185** (1998) 19-27.
42. Jørgen Bang-Jensen, Gregory Gutin and Anders Yeo, Properly coloured Hamiltonian paths in edge-coloured complete graphs, *Disc. Appl. Math* **82** (1998) 247-250.
43. Jørgen Bang-Jensen, Gregory Gutin, Alternating cycles and trails in 2-edge-coloured complete multigraphs, *Discrete Mathematics* **188** (1998) 61-72.
44. Jørgen Bang-Jensen, Gregory Gutin and Anders Yeo, A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs, *Journal of Graph Theory* **29** (1998) 111-132.
45. Jørgen Bang-Jensen, Gregory Gutin, Generalizations of tournaments: A survey, *Journal of Graph Theory* **28** (1998) 171-202.
46. a Jørgen Bang-Jensen and Tibor Jordan, Edge-connectivity augmentation perserving simplicity, *Siam J. Disc. Math.* **11** (1998) 603-623.
    b Jørgen Bang-Jensen and Tibor Jordan, Edge-connectivity augmentation preserving simplicity, *Proceedings of 38th IEEE FOCS* (1997) 486-495.
47. Jørgen Bang-Jensen, Linkages in locally semicomplete digraphs and quasi-transitive digraphs, *Discrete Mathematics*, **196** (1999) 13-27.
48. Jørgen Bang-Jensen and Bill Jackson, Augmenting hypergraphs by edges of size two, *Mathematical programming Series B*, **84** (1999) 467-481.
49. Jørgen Bang-Jensen and Gregory Gutin, On the complexity of Hamiltonian path and cycle problems in certain classes of digraphs, *Discrete Applied Mathematics*, **95** (1999) 41-60.
50. Jørgen Bang-Jensen, Yubao Guo and Anders Yeo, A new sufficient condition for a digraph to be Hamiltonian, *Discrete Applied Mathematics*, **95** (1999) 61-72.
51. a Jørgen Bang-Jensen, Harold Gabow, Zoltán Szigeti and Tibor Jordán, Edge-connectivity augmentation with partition constraints, *Siam J. Disc. Math.*, **12** (1999) 160-207.
    b Jørgen Bang-Jensen, Harold Gabow, Zoltán Szigeti and Tibor Jordán, Edge-connectivity augmentation with partition constraints, *Proc. 9th Annual ACM-SIAM Symp. on Discrete Algorithms*, 1998, 306-315.
52. Jørgen Bang-Jensen and Yubao Guo, A note on vertex pancyclic oriented graphs, *J. Graph Theory*, **31** (1999) 313-318.
53. Jørgen Bang-Jensen, Yubao Guo and Anders Yeo, Complementary cycles containing prescribed vertices in Tournaments, *Discrete Mathematics* **214** (2000) 77-87.
54. Jørgen Bang-Jensen, Jing Huang and Anders Yeo , Convex-round and concave-round graphs, *Siam J. Disc. Math.* **13** (2000) 179-193.
55. Jørgen Bang-Jensen and Tibor Jordán, Splitting off edges]within a specified subset preserving the edge-connectivity of the graph, *J. Algorithms* **37** (2000) 326-343.

56. Jørgen Bang-Jensen and Anders Yeo, The minimum strong spanning subdigraph problem for extended semicomplete digraphs and semicomplete bipartite digraphs, *J. Algorithms*, to appear.
57. Jørgen Bang-Jensen, Jing Huang and Anders Yeo, Strongly connected subgraphs with the minimum number of arcs in quasi-transitive digraphs, *Siam J. Disc. Math*, submitted.
58. Jørgen Bang-Jensen, Jing Huang, Gary MacGillivray and Anders Yeo, Domination in convex bipartite and round graphs, *Information Processing Letters*, submitted.
59. Jørgen Bang-Jensen and Tibor Jordán, Splitting off edges between specified subsets without decreasing the edge-connectivity, *Discrete Mathematics* (2000) submitted.
60. Jørgen Bang-Jensen and Jing Huang, Convex-round graphs are circular-perfect, *Journal of Graph Theory* (2000), submitted.
61. Jørgen Bang-Jensen, Jing Huang and Anders Yeo, Spanning k-arc-strong subdigraphs Jørgen Bang-Jensen, few arcs in k-arc-strong tournaments, *Journal of Graph Theory* (2001), submitted.
62. Jørgen Bang-Jensen and Anders Yeo, Decomposing k-arc-strong tournaments into arc-disjoint spanning subdigraphs, *Combinatorica* (2001), submitted.

Extended abstracts and contributions to problem collections:

63. Jørgen Bang-Jensen, Path and cycle problems for generalizations of tournaments, presented at 1995 Prague midsummer combinatorial workshop,*KAM series in Discrete Mathematics-Combinatorics Operations Research-Optimization* **95-309** (1995).
64. Jørgen Bang-Jensen, Problems concerning global connectivity of directed graphs, *Electronic Notes in Discrete Mathematics*, Volume 5 (2000), 4 pages.
65. Jørgen Bang-Jensen and Anders Yeo, Making a tournament k-arc-strong by reversing arcs, *Electronic Notes in Discrete Mathematics* submitted.

In preparation:

66. Jørgen Bang-Jensen and Anders Yeo, Reversing arcs to increase arc-connectivity of tournaments.

Editorial contributions:

67. Jørgen Bang-Jensen and Bjarne Toft, Unsolved problems presented at the Julius Petersen Graph Theory Conference, *Discrete Mathematics* 101 (1992) 351-360.
68. Jørgen Bang-Jensen, L.D. Andersen, T.R. Jensen, L.K.Jørgensen, G.Sabidussi, C.Thomassen, B. Toft and P.D. Vestergaard, The Julius Petersen Graph Theory Centennial, **Topics in Discrete Mathematics** 6 (1992), North Holland.

Manuscripts and Preprints:

69. Jørgen Bang-Jensen, Arc-local Tournaments: a generalization of tournaments and bipartite tournaments. Preprint no 2 (1993) Department of Mathematics and Computer Science, Odense University.
70. Jørgen Bang-Jensen, Bill Jackson, Minimal augmentation and vertex splitting in mixed graphs, Preprint no 5 (1992) Department of Mathematics and Computer Science, Odense University.
71. Jørgen Bang-Jensen, Luke O'Connor, Cycle coloring is NP-complete.
72. Jørgen Bang-Jensen, Gregory Gutin, Finding maximum vertex weight paths and cycles in Φ-decomposable digraphs, using flows in networks, Preprint no 51 (1993), Department of Mathematics and Computer Science, Odense University.
73. Jørgen Bang-Jensen, A reformulation of Huang's structure theorem on local tournament digraphs with some consequences, Preprint no 13(1994), Department of Mathematics and Computer Science, Odense University.
74. Jørgen Bang-Jensen, Mohamed El Haddad, Parallel algorithms for hamiltonian paths specified (unordered) endvertices in semicomplete bipartite digraphs and extended tournament, preprint no 8 (1996), Department of Mathematics and Computer Science, Odense University.

References

1. A. V. Aho, M. R. Garey and J. D. Ullman. The transitive reduction of a directed graph, *Siam J. Computing* **1(2)** (1972) 131-137.
2. B.Alspach and C.Tabib, A note on the number of 4-circuits in a tournament, in *Annals of Discrete Mathematics* **12** (1982) 13-19.
3. E. Bampis, Y, Manousakis and I. Milis, An RNC algorithm for the alternating Hamiltonian cycle problem in 2-edge-coloured complete graphs. Preprint no. 894, L.R.I., Universite Paris XI (1994).
4. M. Bankfalvi and Zs. Bankfalvi, Alternating hamiltonian circuits in two-coloured complete graphs, in **Theory of Graphs (Proc. Colloq. Tihany 1968)**, Academic Press New York (1968) 11-18.
5. L.W.Beineke and R.Wilson eds., *Selected Topics in Graph Theory 3*, Academic Press, London (1988).
6. A. Benkour, Y. Manoussakis, V. Paschos and R. Saad, On the complexity of finding alternating hamiltonian and eulerian cycles in edge-coloured graphs, *Lecture Notes in Comp. Sci.* **557** Springer Verlag Berlin 190-198.
7. B. Bhattacharya, P. Hell and J. Huang, A linear algorithm for maximum weight cliques in proper circular arc graphs, *Siam J. Disc. Math* **9** (1996) 274-289.
8. J. A. Bondy, U. S. R. Murty, **Graph Theory with applications**, MacMillan Press (1976).
9. M.F. Bridgland and K.B. Reid, Stability of Kings in tournaments, **Progress in Graph Theory** (J.A. Bondy and U.S.R. Murty, eds.), Academic, N. Y. (1984) 117 - 128.
10. Y. H. Bu and K. M. Zhang, Arc-pancyclicity of local tournaments, *Ars Combinatoria* **47** (1997) 242-257.
11. Y. H. Bu and K. M. Zhang, Completely strong path-connectivity of local tournaments, submitted.
12. G.-R. Cai and Y.-G. Sun, *The minimum augmentation of any graph to a K-edge-connected graph*, Networks, 19 (1989), pp. 151-172.
13. J. Cheriyan and R. Thurimella, Approximating minimum-size k-connected spanning subgraphs via matching (extended abstract), 37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996) IEEE Comput. Soc. Press Los Alamitos, CA (1996) 292-301.
14. A. G. Chetwynd and A.J.W. Hilton, Alternating hamiltonian cycles in two coloured complete bipartite graphs, *J. Graph Theory* **16** (1992) 153-158.
15. P. Das and S. B. Rao, Alternating eulerian trails with prescribed degrees in two edge-coloured complete graphs, *Dicrete Math.* **43** (1983) 9-20.
16. X. Deng, P. Hell and J. Huang, Linaer-time representation algorithms for proper circular-arc graphs and proper interval graphs, *Siam J. Computing* **25** (1996) 390-403.
17. J.Edmonds, Edge-disjoint branchings, in *Combinatorial Algorithms*, Algoritmic Press, New York (1973), 91-96.
18. K. Eswaran and R.E. Tarjan, *Augmentation Problems*, SIAM J. Computing 5, 653–665 (1976).
19. T. Feder and M. Y. Vardi, Monotone monadic SNP and constraint satisfaction, in proceedings of 25th STOC, San Diego, California (1993) 612-622.
20. P. Fraisse and C. Thomassen, Hamiltonian cycles avoiding prescribed arcs in tournaments, *Graphs and Combinatorics* **3** (1987) 239-250.
21. A. Frank, On connectivity properties of eulerian digraphs, *Annals of Discrete Mathematics* **41** (1989) 179-194.
22. A. Frank, Augmenting graphs to meet edge-connectivity requirements, *SIAM J. Disc. Math.* **5** (1992) 25-53.
23. A. Frank, Connectivity augmentation problems in network design, in: Mathematical Programming: State of the Art 1994, (Eds: J.R. Birge and K.G. Murty), The University of Michigan, Ann Arbor, MI, 34-63, 1994.
24. A. Frank and T. Jordan, Minimal edge-coverings of pairs of sets, *Journal of Combinatorial Theory* B **65** (1995) 73-110.
25. M.R.Garey and D.S.Johnson, **Computers and Intractability**, W.H.Freedman, San Francisco, 1979.
26. H.N. Gabow, *Applications of a poset representation for edge-connectivity and graph rigidity*, Proc. 32nd IEEE Symposium on Foundations of Computer Scienc e, San Juan, Puerto Rico (1991), 812-821.
27. H.N. Gabow, *Efficient splitting off algorithms for graphs*, Proceedings of the 26th Annual ACM Symposium on the Theory of Computing, 696-705, 1994.
28. A. Ghouilà-Houri, Caractérisation des graphes non orientès dont on peut orienter les arrêtes de maniere à obtenir le graphe d'un relation d'ordre. *C. R. Acad. Sci. Paris* **254** (1962) 1370 - 1371.
29. P. Gibbons, R. Karp, V. Ramachandran, D. Soroker and R. Tarjan, Transitive compaction in parallel via branchings, *J. Algorithms* **12** 110-125.

30. M.C. Golumbic, **Algorithmic Graph Theory and Perfect Graphs**, Academic Press, N.Y. (1980)
31. M. Grötschel, C.L. Monma and M. Stoer, Design of survivable networks, in: Handbook in Operations Research and Management Science Vol. 7, Network Models (M.O. Ball, T.L. Magnanti , C.L. Monma and G.L. Nemhauser, eds.) North-Holland, Amsterdam, 617-672, 1994.
32. Y. Guo, Locally semicomplete digraphs, Ph. D-thesis, RTWH Aachen (1995).
33. Y. Guo, Strongly hamiltonian-connected locally semicomplete digraphs, *J. Graph Theory* **22** (1996) 65-73.
34. Y. Guo, Path-connectivity in local tournaments. 15th British Combinatorial Conference (Stirling, 1995) *Discrete Math.* **167/168** (1997), 353–372.
35. Y. Guo, Spanning local tournaments in locally semicomplete digraphs. 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995). *Discrete Appl. Math.* **79** (1997), 119–125.
36. Y. Guo and L. Volkmann, Connectivity properties of locally semicomplete digraphs, *J. Graph Theory* 18 (1994) 269-280.
37. Y. Guo and L. Volkmann, On complementary cycles in locally semicomplete digraphs, *Discrete Mathematics* **135** (1994) 121-127.
38. Y. Guo and L. Volkmann, Locally semicomplete digraphs that are complementary m-pancyclic, *J. Graph Theory* **21** (1996) 121-136.
39. Y. Guo and L. Volkmann, Bypaths in tournaments. 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995), *Discrete Appl. Math.* **79** (1997), 127–135.
40. G. Gutin, A criterion for complete bipartite digraphs to be hamiltonian, *Vestsi Acad. Navuk BSSR Ser. Fiz.-Mat Navuk* **1** (1984) 99-100 (In Russian).
41. G. Gutin, A characterization of complete n-partite digraphs that have a hamiltonian path, *kibernetica* **1** (1988) 107-108 (In Russian).
42. G. Gutin, Characterizations of vertex pancyclic and pancyclic ordinary complete multipartite digraphs, *Discrete Mathematics* **141** (1995) 153-162.
43. G. Gutin, Efficient algorithms for finding the longest cycles in certain multipartite digraphs, Tech. Report no 266 Department of Comp. Sci. Tel-Aviv Univeristy (1992).
44. G. Gutin, Finding a longest path in a complete multipartite digraph, *SIAM J. Disc. Math.* **6** (1993) 270-273.
45. G. Gutin, Paths and cycles in digraphs Ph. D thesis, Tel-Aviv University (1993).
46. G. Gutin, Polynomial algorithms for finding hamiltonian paths and cycles in quasi-transitive digraphs, *Austral Asian J. Combinatorics* **10** (1994) 231-236.
47. G. Gutin, The radii of n-partite tournaments, *Math. Notes*J**40** (1986) 414-417.
48. W.Gutjahr,E.Welzl and G.Woeginger, Polynomial graph-colourings, Tech. Rept. B-88-06, Freie Universität Berlin 1988.
49. R. Häggkvist and Y. Manoussakis, Cycles and Paths in bipartite tournaments with spanning configurations, *Combinatorica* 9 (1989) 33-38.
50. P. Hell and J. Huang, Lexicographic orientation and representation algorithms for comparability graphs, proper circular-arc graphs and proper interval graphs, *J. Graph Theory* **20** (1995) 361-374.
51. P.Hell and J.Nešetřil, On the complexity of H-coloring, *Journal of Combinatorial Theory* B, 48 (1990) 92-110.
52. A. J. Hoffmann, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, *Proc. Symposia on Applied Math.* **10** (1960) 113-127.
53. H. T. Hsu, An algorithm for finding a minimal equivalent graph of a digraph, *J. Assoc. Computing Machinery* **22** (1975) 11-16.
54. T-S. Hsu and M-Y. Kao, Optimal augmentation for bipartite componentwise biconnectivity in linear time, Algorithms and Computation (Proc. ISAAC '96), Springer Lecture Notes in Computer Science 1178, 213-222, 1996.
55. J. Huang, On the structure of local tournaments, *J. Combin. Theory Ser. B* **63** 200-221.
56. J. Huang, Tournament-like oriented graphs, Ph. D-thesis, Simon Fraser University (1992).
57. J. Huang, Which digraphs are round?, Australasian J. Combinatorics, **19** (1999) 203-208.
58. J. Huang and W. Li, Toppling kings in a tournament by introducing new kings, J. Graph Theory, Vol. 11 (1987) 7 - 11.
59. T. Ibaraki and S. Poljak, Weak three-linking in Eulerian digraphs, *SIAM J. Disc. Math.* **4** (1991) 84-98.
60. B. Jackson and O. Ordaz, A Chvatal-Erdös condition for (1,1)-factors in digraphs, *Disc. Math.* **57** (1985) 199-201.
61. B. Jackson, Some remarks on arc-connectivity, vertex splitting, and orientation in graphs and digraphs, *J. Graph Theory* **12** (1988) 429-436.
62. T. Jordán, Connectivity augmentation problems in graphs Ph. D-thesis, Eötvos University, Budapest (1995).
63. T. Jordán, Two NP-complete augmentation problems, Preprint no 8 (1997) Department of Mathematics and Computer Science, Odense University.
64. G. Kant and H.L. Bodlaender, Planar graph augmentation problems, Algorithms and Data Structures (Proc. WADS '91), Springer Lecture Notes in Computer Science 519, 286-298, 1991.

65. S. Khuller, B. Raghavachari and N. Young, Approximating the minimum equivalent digraph, *SIAM J. Computing* **24** (1995) 859-872.
66. S. Khuller, B. Raghavachari and N. Young, On strongly connected digraphs with bounded cycle length, *Disc. Applied Math.* **69** (1996) 281-289.
67. M. Las Vergnas, Sur les circuits dans les sommes complettes de graphes orientes, *Cahiers du centre d'etudes de recherche operationelle*, **15** (1973), Paris.
68. L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam, 1979.
69. L. Lovász, On two min-max theorems in graph theory, *J. Combin. Theory Ser B.* **21** (1976) 26-30.
70. W. Mader, A reduction method for edge-connectivity in graphs, *Annals of Disc. Math.* **3** (1978) 145-164.
71. W. Mader, Konstruktion aller n-fach kantenzusammenhangenden Digraphen, *european J. Combin.* **3** (1982) 63-67.
72. Y. Manoussakis and Z. Tuza, Polynomial algorithms for finding paths and cycles in bipartite tournaments, *SIAM J. Disc. Math.* **3** (1990) 537-543.
73. G. MacGillivray, The complexity of generalized colouring, Ph.D. Thesis, Department of Mathematics, Simon Fraser University (1990)
74. J.W.Moon, Topics on tournaments, Holt, Rinehard and Winston, New York (1968).
75. K. Mulmuley, U. V. Vazirani, V.B. Vazirani, Matching is as easy as matrix inversion, *Combinatorica* **7** (1987) 105-113.
76. H. Nagamochi and T. Ibaraki, Deterministic $\tilde{O}(nm)$ time edge-splitting algorithm in undirected graphs, STOC 28 (1996) 64-73.
77. D. Naor, D. Gusfield and C. Martel, *A Fast Algorithm for Optimally Increasing the Edge-Connectivity*, SIAM J. Computing 26, pp. 1139-1165 (1997). Preliminary version in Proc. 31st Annual IEEE Symposium on Foundations of Comp. Sci., 698–707 (1990).
78. V. Petrovic and C. Thomassen, Kings in k-partite tournaments, *Discrete Mathematics* **98** (1991) 237-238.
79. K.B. Reid, Every vertex a king, Discrete Mathematics 38 (1982) 93 - 98.
80. R. Saad, Sur quelqes problemes de complexite dans les graphes, Ph. D-thesis, L.R.I. Universite Paris XI (1992).
81. K. Simon, Finding a minimal transitive reduction in a strongly connected digraph within linear time. In: *Graph Theoretic concepts in computer science (Kerkrade, 1989).* Springer Verlag Berlin (1990) 244-259.
82. D. J. Skrien, A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular arc graphs, and nested interval graphs, *J. Graph Theory* 6 (1982) 309-316.
83. Z. M. Song, Complementary cycles of all lengths in tournaments, *J. Combin. Theory* **B 57** (1993) 18-25.
84. D.Soroker, Fast parallel algorithms for finding hamiltonian paths and cycles in tournaments, *Journal of Algorithms* 9 (1988), 276-286.
85. E. Speckenmeyer, On feedback problems in digraphs, *Lecture notes in Comp. Sci.* **411** (1989) 218-232.
86. Meike Tewes, In-tournaments and semicomplete multipartite digraphs, PhD thesis, RWTH Aachen 1998.
87. C.Thomassen, Connectivity in tournaments, in **Graph Theory and Combinatorics, B. Bollobas editor**, Academic Press, london (1984) 305-313.
88. C.Thomassen, Hamiltonian connected tournaments, *Journal of Combinatorial Theory, Ser. B* , 28 (1980), 142-163.
89. C. Thomassen, Highly connected non 2-linked digraphs. *Combinatorica* **11** (1991) 393-395.
90. F. Tian, Z. S. Wu and C. Q. Zhang, Cycles of each length in tournaments, *J. Combin. Theory* **B 33** (1982) 245-255.
91. A. Tucker, Matrix characterizations of circular arc graphs, *Pacific J. Math.* 39 (1971) 535 - 545.
92. J. Urrutia and F. Gavril, An algorithm for fraternal orientation of graphs, *Information Processing Letters* **41** (1992) 271-274.
93. A. Vetta, Approximating the minimum strongly connected subgraph via a matching lower bound, Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms (2001) 417-426.
94. T. Watanabe and M. Yamakado, A linear time algorithm for smallest augmentation to 3-edge-connect a graph, IEICE Trans. Fundamentals, Vol. E76-A , No.4, April 1993, 518-530.
95. T. Watanabe and A. Nakamura, Edge-Connectivity Augmentation Problems, *Journal of Computer and System Sciences* 35, 96–144 (1987).
96. D. R. Woodall, Sufficient conditions for circuits in graphs, *Proc. London Math. Soc.* **24** (1972) 739-755.
97. A. Yeo, One diregular subgraphs in semicomplete multipartite digraphs, *J. Graph Theory* **24** (1997) 1-11.
98. A. Yeo, Semicomplete multipartite digraphs, Ph.D. thesis, Department of Mathematics and Computer Science, Odense University (1998).
99. C. Q. Zhang, Hamiltonian paths in multipartite oriented graphs, *Annals of Disc. Math.* **41** (1989) 499-514.
100. C. Q. Zhang and C. Zhao, On locally semicomplete digraphs, *J. Combin. Math and Combin. Comput.* bf 20 (1996) 186-192.
101. K. M. Zhang, Completely strong path-connected tournaments, *J. Combin. Theory* **B 33** (1982) 166-177.
102. H. Zhou and X. Zhu, Multiplicativity of acyclic local tournaments, *Combinatorica* **17** (1997) 135-145.