

# MATEMATIKKENS HISTORIE

## EULER (1707-1783)

f. i BASEL 15. april

1722 Master v. Univ. i Basel

1733 Position i Sct. Petersburg  
(assistent til D. Bernoulli)

1741 Tog til Berlin efter henv.  
af Frederik den store

1766 vender tilbage til Rusland  
(Katharine den store)

→ Blind

(LDDRAG AF !!!)

EULERS PUBLIKATIONER

(1748) INTRODUCTIO IN ANALYSIN  
INFINITORUM  
(Introduction to Analysis  
of the Infinite. To bind)

(1755) INSTITUTIONES CALCULI  
DIFFERENTIALIS  
(FOUNDATION OF Differential  
Calculus. To bind)

(1768-1770) INSTITUTIONES CALCULI  
INTEGRALIS

# EULER : 'FOUNDATIONS OF DIFFERENTIAL CALCULUS' :

7. Let us represent each of these series of differences by the following scheme, in order that we can more easily see their respective relationships:

## Arithmetic Progression:

$$x, x + \omega, x + 2\omega, x + 3\omega, x + 4\omega, x + 5\omega, \dots$$

## Values of the Function:

$$y, y^I, y^{II}, y^{III}, y^{IV}, y^V, \dots$$

## First Differences:

$$\Delta y, \Delta y^I, \Delta y^{II}, \Delta y^{III}, \Delta y^{IV}, \dots$$

## Second Differences:

$$\Delta\Delta y, \Delta\Delta y^I, \Delta\Delta y^{II}, \Delta\Delta y^{III}, \dots$$

## Third Differences:

$$\Delta^3 y, \Delta^3 y^I, \Delta^3 y^{II}, \dots$$

## Fourth Differences:

$$\Delta^4 y, \Delta^4 y^I, \dots$$

HER BETYDER :

$$\Delta y = y^I - y \quad \Delta y^I = y^{II} - y^I \quad osv.$$

$$\Delta\Delta y = \Delta y^I - \Delta y, \quad \Delta\Delta y^I = \Delta y^{II} - \Delta y^I$$

⋮

# FOUNDATIONS OF DIFFERENTIAL CALCULUS

## EXEMPLER PĀ DIFFERENCER:

**Example 1.** Find the differences of all orders of  $x^2$ .

Since here  $y = x^2$ , we have  $y^I = (x + \omega)^2$ , so that

$$\Delta y = 2\omega x + \omega^2,$$

and this is the first difference. Now, since  $\omega$  is a constant, we have  $\Delta\Delta y = 2\omega^2$  and  $\Delta^3 y = 0$ ,  $\Delta^4 y = 0, \dots$

**Example 4.** Find the differences of all orders of  $x^n$ .

Let  $y = x^n$ . Since  $y^I = (x + \omega)^n$ ,  $y^{II} = (x + 2\omega)^n$ ,  $y^{III} = (x + 3\omega)^n, \dots$ , the expanded powers are as follows:

$$\begin{aligned} y &= x^n, \\ y^I &= x^n + \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\omega^3 x^{n-3} \\ &\quad + \dots, \\ y^{II} &= x^n + \frac{n}{1}2\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2}4\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}8\omega^3 x^{n-3} \\ &\quad + \dots, \\ y^{III} &= x^n + \frac{n}{1}3\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2}9\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}27\omega^3 x^{n-3} \\ &\quad + \dots, \\ y^{IV} &= x^n + \frac{n}{1}4\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2}16\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}64\omega^3 x^{n-3} \\ &\quad + \dots. \end{aligned}$$

21. We do not exclude from this form of expression even the differences of transcendental functions, as will clearly appear from the following examples.

**Example 1.** Find the first difference of the natural logarithm of  $x$ .

Let  $y = \ln x$ . Since  $y^I = \ln(x + \omega)$ , we have

$$\Delta y = y^I - y = \ln(x + \omega) - \ln x = \ln\left(1 + \frac{\omega}{x}\right).$$

Elsewhere<sup>2</sup> we have shown how this kind of logarithm can be expressed in an infinite series. We use this to obtain

$$\Delta y = \Delta \ln x = \frac{\omega}{x} - \frac{\omega^2}{2x^2} + \frac{\omega^3}{3x^3} - \frac{\omega^4}{4x^4} + \dots$$

## FOUNDATION OF DIFFERENTIAL CALCULUS

114. The analysis of the infinite, which we begin to treat now, is nothing but a special case of the method of differences, explained in the first chapter, wherein the differences are infinitely small, while previously the differences were assumed to be finite. Hence, this case, in which the whole of analysis of the infinite is contained, should be distinguished from the method of differences. We use special names and notation for the infinitely small differences. With Leibniz we call infinitely small differences by the name *differentials*. From the discussion in the first chapter on the different orders of differences, we can easily understand the meaning of first, second, third, and so forth, differentials of any function. Instead of the symbol  $\Delta$ , by which we previously indicated a difference, now we will use the symbol  $d$ , so that  $dy$  signifies the first differential of  $y$ ,  $d^2y$  the second differential,  $d^3y$  the third differential, and so forth.

: COM NEWTON, ENGLÄNDERE OM DERES  
BEGREBER OG NOTATION, BLA.:

116. It might be uncivil to argue with the English about the use of words and a definition, and we might easily be defeated in a judgment about the purity of Latin and the adequacy of expression, but there is no doubt that we have won the prize from the English when it is a question of notation.

For example, the tenth differential, or fluxion, is very inconveniently represented with ten dots, while our notation,  $d^{10}y$ , is very easily understood. There are cases where differentials of even much higher order, or even those of indefinite

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✓ 118. Since up to this time we have used the letter  $\omega$  to denote the difference or the increment by which the variable  $x$  is understood to increase, now we understand  $\omega$  to be infinitely small, so that  $\omega$  is the differential of  $x$ , and for this reason we use our method of writing  $\omega = dx$ . From now on,  $dx$  will be the infinitely small difference by which  $x$  is understood to increase. In like manner the differential of  $y$  we express as  $dy$ . If  $y$  is any function of  $x$ , the differential  $dy$  will indicate the increment that  $y$  receives when  $x$  changes to  $x + dx$ . Hence, if we substitute  $x + dx$  for  $x$  in the function  $y$  and we let  $y^I$  be the result, then  $dy = y^I - y$ , and this is understood to be the first differential, that is, the differential of the first order. Later we will consider the other differentials.

# FOUNDATION OF DIFFERENTIAL CALCULUS

152. Since the differential of the variable  $x$  is equal to  $dx$ , when  $x$  is incremented,  $x$  becomes equal to  $x + dx$ . Hence, if  $y$  is some function of  $x$ , and if we substitute  $x + dx$  for  $x$ , we obtain  $y^I$ . The difference  $y^I - y$  gives the differential of  $y$ . Now if we let  $y = x^n$ , then

$$y^I = (x + dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1 \cdot 2}x^{n-2}dx^2 + \dots,$$

and so

$$dy = y^I - y = nx^{n-1}dx + \frac{n(n-1)}{1 \cdot 2}x^{n-2}dx^2 + \dots$$

In this expression the second term and all succeeding terms vanish in the presence of the first term. Hence,  $nx^{n-1}dx$  is the differential of  $x^n$ , or

$$d.x^n = nx^{n-1}dx.$$

180. We are investigating the differential of the hyperbolic logarithm of  $x$  and we let  $y = \ln x$ , so that we have to define the value of  $dy$ . We substitute  $x + dx$  for  $x$  so that  $y$  is transformed into  $y^I = y + dy$ . From this we have

$$y + dy = \ln(x + dx), \quad dy = \ln(x + dx) - \ln x = \ln\left(1 + \frac{dx}{x}\right).$$

But we have seen before<sup>1</sup> that the hyperbolic logarithm of this kind of expression  $1 + z$  can be expressed in an infinite series as follows:

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

When we substitute  $dx/x$  for  $z$  we obtain

$$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \dots$$

Since all of the terms of this series vanish in the presence of the first term, we have

$$d \ln x = dy = \frac{dx}{x}.$$

It follows that the differential of any logarithm whatsoever that has the ratio to the hyperbolic logarithm of  $n : 1$ , has the form  $n dx/x$ .