A Note on Lattices and Fixed Points

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1 Introduction

In this paper, we define complete lattices, and we discuss the existence and construction of fixed points for monotone functions on these lattices. The aim is to be able to solve equational systems, where the variables take on finite set-values. Being directed by this specific goal, we will not necessarily state the most general variants of the lattice results. More often, we restrict ourselves to the results necessary to solve equational systems.

More general theorems on lattice-theoretical fixed points can be found in [1].

2 Preliminaries

In this section, we define the most basic mathematical notions.

Definition 1 If U is a set and $\sqsubseteq$ a binary relation on U, then $\sqsubseteq$ is a *partial order* if:

- $\forall x \in U\colon\ x \sqsubseteq x$ (reflexitivity)
- $\forall x, y, z \in U\colon\ (x \sqsubseteq y) \wedge (y \sqsubseteq z) \Rightarrow (x \sqsubseteq z)$ (transitivity)
- $\forall x, y \in U\colon\ (x \sqsubseteq y) \wedge (y \sqsubseteq x) \Rightarrow (x = y)$ (antisymmetry)

□

Example 1 Consider the set of integers $\mathcal{Z}$ with the usual ordering $\leq$. Define the set U by $U = \{(x, y) \mid x, y \in \mathcal{Z}\}$, and define $\sqsubseteq$ by $(x, y) \sqsubseteq (u, v)$ if and only if $x \leq u$ and $y \leq v$. Then $\sqsubseteq$ is a partial order.

Definition 2 Assume that $\sqsubseteq$ is a partial order on U, and that $\sqcup$ is an operation on U such that for all $A \subseteq U$, if $\sqcup A$ is defined, then

- $\forall x \in A\colon\ x \sqsubseteq \sqcup A$
- $\forall y \in U\colon\ (\forall x \in A\colon\ x \sqsubseteq y) \Rightarrow (\sqcup A \sqsubseteq y)$

If defined, $\sqcup A$ is referred to as the *least upper bound* of A. □

The first condition states that $\sqcup A$ is in fact an upper bound, and the second that it is the least. Similarly, we define the following:

Definition 3 Assume that $\sqsubseteq$ is a partial order on U, and that $\sqcap$ is an operation on U such that for all $A \subseteq U$, if $\sqcap A$ is defined, then

- $\forall x \in A\colon\ \sqcap A \sqsubseteq x$
- $\forall y \in U\colon\ (\forall x \in A\colon\ y \sqsubseteq x) \Rightarrow (y \sqsubseteq \sqcap A)$

If defined, $\sqcap A$ is referred to as the *greatest lower bound* of A. $\square$

If $A = \{x_1, \ldots, x_n\}$, then we may write $\sqcup A$ as $x_1 \sqcup \cdots \sqcup x_n$ and $\sqcap A$ as $x_1 \sqcap \cdots \sqcap x_n$.

Lattices, Functions, and Fixed Points

In this section, we define lattices and complete lattices. Then we consider monotone functions on these lattices, and we prove results about the fixed points of these functions.

Definition 4 If $\sqsubseteq$ is a partial order on a nonempty set U, and the operations $\sqcup$ and $\sqcap$ are defined for sets containing exactly two elements, then the pair $(U, \sqsubseteq)$ is called a *lattice*.

If the operations $\sqcup$ and $\sqcap$ are defined for arbitrary subsets of U, then the pair $(U, \sqsubseteq)$ is called a *complete* lattice. $\square$

Two elements of a complete lattice $(U, \sqsubseteq)$ are particularly interesting: the element $\top = \sqcup U$ (top) and $\perp = \sqcap U$ (bottom). Clearly, $\forall x \in U\colon\ \perp \sqsubseteq x \sqsubseteq \top$.

Example 2 Consider the partial order defined in example 1. Notice that $(U, \sqsubseteq)$ is indeed a lattice, but it is not complete.

Now, we turn our attention to monotone functions on lattices.

Definition 5 Let U_1 and U_2 be sets with partial orders $\sqsubseteq_1$ and $\sqsubseteq_2$, respectively. A function $f\colon\ U_1 \to U_2$ is called *increasing* if $\forall x, y \in U_1\colon\ (x \sqsubseteq_1 y) \Rightarrow (f(x) \sqsubseteq_2 f(y))$. Similarly, f is called *decreasing* if $\forall x, y \in U_1\colon\ (x \sqsubseteq_1 y) \Rightarrow (f(y) \sqsubseteq_2 f(x))$. $\square$

All results in the rest of this paper are stated for increasing functions and minimal fixed points. Dual results exist, but we shall not mention them here.

Definition 6 If U is a set and $f\colon\ U \to U$ is a function, then $u \in U$ is called a *fixed point* for f if $f(u) = u$.

A fixed point $u \in U$ for f is called *minimal* if for all other elements $v \in U$, which are fixed points for f, we have that $v \not\sqsubseteq u$. If a function f has exactly one minimal fixed point, then this fixed point is called the *least* fixed point for f. $\square$

We can now prove that an increasing function on a complete lattice must have a least fixed point.

Lemma 1 If $(U, \sqsubseteq)$ is a complete lattice and $f\colon\ U \to U$ is an increasing function, then f has exactly one minimal fixed point, the least fixed point for f.

Proof Let $A = \{x \in U \mid f(x) \sqsubseteq x\}$. Notice that A contains all the fixed points for f. Now,

$$\begin{array}{ll} & \forall x \in A\colon\ \sqcap A \sqsubseteq x, \text{ as } \sqcap A \text{ is a lower bound} \\ \Downarrow & \\ & \forall x \in A\colon\ f(\sqcap A) \sqsubseteq f(x), \text{ as } f \text{ is increasing} \\ \Downarrow & \\ & \forall x \in A\colon\ f(\sqcap A) \sqsubseteq x, \text{ as } f(x) \sqsubseteq x \text{ for all } x \in A \text{ (transitivity)} \\ \Downarrow & \\ & f(\sqcap A) \sqsubseteq \sqcap A, \text{ since } f(\sqcap A) \text{ is a lower bound and } \sqcap A \text{ is the greatest} \\ \Downarrow & \\ & f(f(\sqcap A)) \sqsubseteq f(\sqcap A), \text{ as } f \text{ is increasing} \\ \Downarrow & \\ & f(\sqcap A) \in A, \text{ by definition of } A \\ \Downarrow & \\ & \sqcap A \sqsubseteq f(\sqcap A), \text{ as } \sqcap A \text{ is a lower bound} \end{array}$$

From the two inequalities $f(\sqcap A) \sqsubseteq \sqcap A$ and $\sqcap A \sqsubseteq f(\sqcap A)$, we conclude that $f(\sqcap A) = \sqcap A$ (antisymmetry). So, $\sqcap A$ is a fixed point. As $\sqcap A$ by definition is a lower bound for all other fixed points, it must be the least. □

Knowing that there exists a least fixed point, it would be useful to have a simple method for computing it. It turns out that there is a simple iterative method for this.

Lemma 2 Let $(U, \sqsubseteq)$ be a complete lattice and $f\colon U \to U$ an increasing function. If U is finite, then the least fixed point for f can be found as $f^k(\perp)$ for some $k \in \mathbb{N}$.

Proof Let $u = \sqcup\{f^i(\perp) \mid i \in \mathbb{N}\}$. From $\perp \sqsubseteq f(\perp)$, it follows by simple induction (using the monotonicity of f) that for all i, $f^i(\perp) \sqsubseteq f^{i+1}(\perp)$. So, we have for all i that $\sqcup\{\perp, f(\perp), \ldots, f^i(\perp)\} = f^i(\perp)$. If there are infinitely many i's such that $f^i(\perp) \sqsubset f^{i+1}(\perp)$, then U must be infinite. As this is not the case, there are only finitely many such i's. So, there exists a k such that for all $i \geq k$, we have that $f^i(\perp) = f^{i+1}(\perp)$. We conclude that $u = f^k(\perp)$.

First, we observe that u is indeed a fixed point as

$$u = f^k(\perp) = f(f^k(\perp)) = f(u)$$

Now assume that u' is also a fixed point for f. We obtain that $f(\perp) \sqsubseteq f(u') = u'$, as f is increasing. By simple induction, it follows that $f^k(\perp) \sqsubseteq u'$. But then $u \sqsubseteq u'$, so u is the least fixed point for f. □

In the next section, we will need *product* lattices. They are defined as follows:

Definition 7 Assume that $(U_1, \sqsubseteq_1), \ldots, (U_n, \sqsubseteq_n)$ are all complete lattices, and that for all i, $(U_i, \sqsubseteq_i)$ has least upper bound and greatest lower bound operators $\sqcup_i$ and $\sqcap_i$. Then the product lattice $(U, \sqsubseteq)$ of $(U_1, \sqsubseteq_1), \ldots, (U_n, \sqsubseteq_n)$ is defined by

- $U = U_1 \times \cdots \times U_n$
- $(x_1, \ldots, x_n) \sqsubseteq (y_1, \ldots, y_n) \;\Leftrightarrow\; (x_1 \sqsubseteq_1 y_1) \wedge \cdots \wedge (x_n \sqsubseteq_n y_n)$
- $(x_1, \ldots, x_n) \sqcup (y_1, \ldots, y_n) = ((x_1 \sqcup_1 y_1), \ldots, (x_n \sqcup_n y_n))$
- $(x_1, \ldots, x_n) \sqcap (y_1, \ldots, y_n) = ((x_1 \sqcap_1 y_1), \ldots, (x_n \sqcap_n y_n))$ □

Proposition 1 If $(U, \sqsubseteq)$ is a product lattice, then it is a complete lattice.

Proof Easy. $\square$

3 Equational Systems

In this section, we define equational systems over complete lattices, and we develop a method for finding the least solution to such systems.

Definition 8 An *equational system* over a product lattice $(U, \sqsubseteq)$, where $U = U_1 \times \cdots \times U_n$, consists of n equations

$$\begin{aligned} M_1 &= f_1(M_1, \ldots, M_n) \\ M_2 &= f_2(M_1, \ldots, M_n) \\ &\;\;\vdots \\ M_n &= f_n(M_1, \ldots, M_n) \end{aligned}$$

where the M_j's are variables and for all $i \in \{1, \ldots, n\}$, $f_i : \ U \rightarrow U_i$ is an increasing function.

A *solution* $\mathcal{L}$ to an equational system assigns to each variable M some value $\mathcal{L}(M) \in U$ such that all the equalities hold. $\square$

In the following, we let $\overline{x}$ denote $(x_1, \ldots, x_n)$.

Lemma 3 Let $\mathcal{S}$ be an equational system over the product lattice $(U, \sqsubseteq)$. Then there exists a unique least solution to $\mathcal{S}$. Furthermore, it can be found as the least fixed point of $F\colon\ U \rightarrow U$ defined by $F(\overline{x}) = (f_1(\overline{x}), \ldots, f_n(\overline{x}))$.

Proof By proposition 1, $(U, \sqsubseteq)$ is a complete lattice, and as $f_1, \ldots, f_n$ are increasing functions, so is F. Clearly, any solution to $\mathcal{S}$ must be a fixed point for F. Likewise, any fixed point for F must be a solution to $\mathcal{S}$. The result then follows from lemma 1. $\square$

One application for the results above is of particular interest to us. That is when the set U is the power set of some base set B.

Proposition 2 If B is a set, then $(2^B, \subseteq)$ with least upper bound and greatest lower bound operators $\cup$ and $\cap$, respectively, is a complete lattice.

Proof Easy. Recall that the union and the intersection of arbitrary collections of sets are always defined. $\square$

In the following, we let $\perp_i$ denote the bottom element of the complete lattice $(U_i, \sqsubseteq_i)$.

Theorem 1 Let $\mathcal{S}$ be an equational system over the finite product lattice $(U, \sqsubseteq)$. Then there exists a unique least solution to $\mathcal{S}$. Furthermore, it can be found as $F^k(\perp_1, \ldots, \perp_n)$, for some $k \in I\!N$ and some function F.

Proof Using the right-hand side functions of the equations in $\mathcal{S}$, we define $F\colon U \to U$ by $F(\overline{x}) = (f_1(\overline{x}), \ldots, f_n(\overline{x}))$. From lemma 3, it follows that the least fixed point for F is the least solution to $\mathcal{S}$. As $(\perp_1, \ldots, \perp_n)$ is clearly the bottom element of U, it follows from lemma 2 that there exists a $k \in I\!N$ such that this fixed point equals $F^k(\perp_1, \ldots, \perp_n)$. □

Corollary 1 The least solution to equational systems over finite product lattices with n set-valued variables can be solved by iterating from the n-tuple $(\emptyset, \ldots, \emptyset)$.

Proof Directly from theorem 1 and proposition 2. □

4 Concluding Remarks

Notice that when computing solutions by iterating a function from the bottom element, there are three costs to consider.

- the number of iterations to find the fixed point
- the cost of computing each element of the iteration
- the cost of checking whether or not a fixed point has been reached

Occasionally, applications are such that a finite and complete lattice cannot be established. In those cases, one might consider a related result described briefly below. The idea is to require less of the structure, but then more of the functions.

Assume that $\sqsubseteq$ is a partial order on U. A subset A of U is called a *chain* (or sometimes a *directed set*) if $\forall x, y \in A\colon\ (x \sqsubseteq y) \vee (y \sqsubseteq x)$.

The pair $(U, \sqsubseteq)$ is called a *complete partial order* (or sometimes a *chain complete partial order*) if $\sqcup A$ is defined for all chains A.

Assume that $(U_1, \sqsubseteq_1)$ and $(U_2, \sqsubseteq_2)$ are complete partial orders with least upper bound operators $\sqcup_1$ and $\sqcup_2$, respectively. An increasing function $f\colon\ U_1 \to U_2$ is *continuous* if for all nonempty chains A, we have that $f(\sqcup_1 A) = \sqcup_2\{f(x) \mid x \in A\}$. Notice that the image of a chain is again a chain, so that the least upper bound is defined.

If f is a continuous function on a complete partial order $(U, \sqsubseteq)$, then the least fixed point of f exists, and it equals $\sqcup\{f_i(\perp) \mid i \geq 0\}$. Notice that $\perp = \sqcup\emptyset$. Again, if U is not finite, then the least fixed point might not belong to $\{f^i(\perp) \mid i \geq 0\}$, so we cannot necessarily compute it by iteration in finite time.

References

[1] Alfred Tarski. A Lattice-Theoretical Fixpoint Theorem and its Applications. *Pacific J. Math*, 5:285–309, 1955.